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https://doi.org/10.1007/s40324-018-0161-5

# Improved variational iteration method for solving a class of nonlinear Fredholm integral equations

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Received: 1 December 2017 / Accepted: 3 May 2018


**Abstract** In this paper, an efficient numerical method which is a combination of the variational iteration method and the spectral collocation method is developed for solving a class of nonlinear Fredholm integral equations (NFIEs). This method is easy to implement, requiring no tedious computational work and possesses the spectral accuracy. In addition, it does not require calculating Adomian's polynomials and Lagrange's multiplier values. Several numerical examples are included to demonstrate the validity and efficiency of the proposed method. The obtained results have been compared with the exact solutions so that the high accuracy of the results are clear.

**Keywords** Variational iteration method · Spectral collocation method · Nonlinear Fredholm integral equation

**Mathematics Subject Classification** 45G10 · 45B05 · 65M70

## 1 Introduction

Integral equations play a crucial role in many branches of science and engineering such as biological models, mathematical economics, continuum mechanics, potential theory, geophysics, electricity and magnetism, fluid dynamics, antenna synthesis problem communication theory, radiation, etc. [4, 32, 33]. Analytical solutions of integral equations, either do not exist or it's hard to compute. Eventual an exact solution is computable, the required calculations may be tedious, or the resulting solution may be difficult to interpret. Due to this, it is required to obtain an efficient numerical solution. There are numerous studies

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Published online: 30 May 2018

in literature concerning the numerical solution of Fredholm integral equations such as the modified homotopy perturbation method [13], the Toeplitz matrix method [2], the Bernstein polynomials [25], the Harmonic wavelet method [8], the Sinc-collocation method [14], a Chebyshev polynomials [22], the triangular function Method [24], the wavelet meshless method [23]. The topic of variational iteration method (VIM) was proffered originally by the Chinese mathematician J. H. He, who modified the Lagrange multiplier method into an iteration scheme [20]. The VIM is an efficient method for solving linear and nonlinear ordinary differential equations, partial differential equations and integral equations. Applications of the method have been enlarged due to its flexibility, convenience and accuracy. Moreover the method gives rapidly convergent successive approximations of the exact solution if such a solution exists; otherwise a few approximations can be used for numerical purposes. Meanwhile, the variational iteration method has been modified by many authors [1, 27]. For more applications of the variational iteration method [10, 15–19].

The aim of this paper is to propose an effective algorithm, requiring no tedious computational work, based on the VIM and the spectral collocation technique for obtaining a highly accurate numerical solution for the following NFIE in the form:

$$u(t) = f(t) + \int_0^T k(t,s)G(u(s))ds, \quad t \in [0, T], \tag{1}$$

where the kerel $k(t,s)$, $f(t)$ and $G(u(s))$ are smooth functions. The existence and uniqueness of the solution for Eq. (1) are presented in Refs. [3, 11, 21, 31].

The contents of this paper are organized as follows: Sect. 1 is the introduction; In Sect. 2 some preliminaries consisting of the standard variational iteration method (VIM) is briefly. Section 3, is devoted to the study of the main properties of the Chebyshev polynomials and spectral variational iteration method (SVIM). In Sect. 4, a number of examples is proposed to illustrate the validity of suggested method and finally, a brief conclusion is presented in Sect. 5.

## 2 Variational iteration method

The VIM gives rapidly convergent by using successive approximations of the exact solution if such a solution exists, otherwise the approximations can be used for numerical purposes. To elucidation the basic idea of the VIM, we consider Eq. (1) as below:

$$L[u(t)] + N[u(t)] = f(t), \tag{2}$$

where L with the property L$v \equiv 0$ when $v \equiv 0$ denotes the auxiliary linear operator with respect to u, N is a nonlinear continuous operator with respect to u and f(t) is the source term.

According to [12], we construct the following family of the explicit iterative processes for (2) as:

$$L[u_{k+1}(t) - u_k(t)] = -A[u_k(t)], \tag{3}$$

where

$$A[u_k(t)] = L[u_k(t)] + N[u_k(t)] - f(t) = u_k - \int_0^T k(t,s)G(u_k(s))dx - f(t), \tag{4}$$

and the subscript $k$ denotes the $k$th iteration. Next the successive approximations $\mathrm{u_k(t)},\ \mathrm{k} \geq 0$ of the VIM iterative will be readily obtained by using any selective function $u_0(t)$. The zeroth approximation $u_0(t)$ may be freely chosen with possible unknown constants, or it can be found from solving its corresponding linear equation ($L[u_0(t)] = 0$ or $L[u_0(t)] = f(t)$). Consequently, the exact solution can be obtained by using:

$$u(t) = \lim_{k\to\infty} u_k(t). \tag{5}$$

The variational iteration formula (3), makes a recurrence sequence $\{\mathrm{u_k(t)}\}$ which outlined depends on the proper selection of the initial approximation $u_0(t)$. Obviously, the limit of the sequence will be the solution of Eq. (1) if the sequence is convergent. The iteration process (3) under certain conditions is convergent. Here we assume that for every positive integer $\mathrm{k},\ \mathrm{u_k} \in \mathrm{C[0, T]}$ and $\{\mathrm{u_k}\}$ is uniformly convergent.

**Theorem 2.1** *If the sequence* $\{\mathrm{u_k(t)}\}$ *converges, where* $\mathrm{u_k(t)}$ *is produced by the variational iteration formulation of* (3), *then it must be the exact solution of* (1).

*Proof* The proof is similar to the given [12].

## 3 Chebyshev polynomials and spectral variational iteration method

In this section, we give the SVIM and apply here it to problem (1). At first, we describe the following properties of shifted Chebyshev polynomials.

### 3.1 Properties of shifted Chebyshev polynomials

This section is devoted to introducing Chebyshev polynomials (of the first kind), and expressing some basic properties of them. The Chebyshev polynomials, $T_n(x)$, $n$=0, 1,…, are the eigenfunctions of the singular Sturm–Liouville problem $(1 - x^2)T_n''(x) - xT_n'(x) + n^2T_n(x) = 0$ and are of high interest recently [5, 6].

Let $T_j(x),\ \mathrm{x} \in [-1, 1]$ be the standard Chebyshev polynomial of degree j. Then we have:

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_{j+1} = 2xT_j(x) - T_{j-1}(x), \quad j \geq 1. \tag{6}$$

In order to use these polynomials on the interval [0, T], we define the so-called shifted Chebyshev polynomials by introducing the change of variable $x = \frac{2t}{T} - 1$. Let the shifted Chebyshev polynomials $T_j\left(\frac{2t}{T} - 1\right)$ be denoted by $\mathrm{T_{T,j}(t)}$. We infer from the aforementioned that:

$$\mathrm{T_{T,j+1}(t)} = 2\left(\frac{2\mathrm{t}}{\mathrm{T}} - 1\right)\mathrm{T_{T,j}(t)} - \mathrm{T_{T,j-1}(t)} \quad \mathrm{j} = 1, 2, \ldots, \tag{7}$$

where $\mathrm{T_{T,0}(t)} = 1$ and $T_{T,1}(t) = \frac{2t}{T} - 1$.

Assume $\mathrm{w(t) = t\,(T - t)}$. The orthogonality condition is:

$$\int_0^{\mathrm{T}} \mathrm{T_{T,j}(t)T_{T,k}(t)w^{\frac{-1}{2}}\,dt} = h_j \delta_{\mathrm{j,k}}, \tag{8}$$

where $h_j = \frac{b_j}{2}\pi$, $b_0 = 2$, $b_j = 1$, $j \geq 1$ and $\delta_{j,k}$ is the Kronecker delta function. A function u(t), square integrable in (0, T), may be expressed in terms of shifted Chebyshev polynomials as follows:

$$u(t) = \sum_{j=0}^{\infty} c_j T_{T,j}(t), \tag{9}$$

where the coefficients $c_j$ are given by:

$$c_j = \frac{1}{h_j} \int_0^T u(t) T_{T,j}(t) w^{\frac{-1}{2}} dt, \quad j = 0, 1, 2, \ldots \tag{10}$$

In practice, only the first (N + 1)-terms shifted Chebyshev polynomials are considered. Hence, u(t) can be expanded in the form:

$$u^N(t) = \sum_{j=0}^{N} c_j T_{T,j}(\mathrm{t}). \tag{11}$$

**Theorem 3.1.1** *Let $u(t) \in H^k(-1, 1)$ (Sobolev space) and $u^N(t) = \sum_{j=0}^{N} c_j T_{T,j}(t)$, then:*

$$\left\| u(t) - u^N(t) \right\|_{L_w^2[-1,1]} \leq C_0 M^{-K} \, \|u(t)\|_{H_w^k(-1,1)}$$

*where $C_0$ is a positive constant, which depends on selected norm and independent with $u(t)$ and $M$* [7].

*We now deal with the shifted Chebyshev–Gauss–Lobatto (SC–GL) interpolation. We denote:*

$$x_j^N = -cos \frac{\pi j}{N}, \quad 0 \leq j \leq N, \tag{12}$$

*which are the standard Chebyshev–Gauss–Lobatto points on the interval* $[-1, 1]$ *and*

$$t_{T,j}^N = \frac{T}{2}(x_j^N + 1) = \frac{T}{2}\left(1 - cos \frac{\pi j}{N}\right), \quad 0 \leq j \leq N. \tag{13}$$

*The nodes of the Shifted Chebyshev–Gauss–Lobatto interpolation on the interval* [0, T] *are the zeros of* $T_{T,N-1}(t)$*, which are denoted by* $t_{T,j}^N$, $0 \leq j \leq N$.

*Let* $P_N(T)$ *be the set of polynomials of degree at most N owing to the property of the standard Chebyshev–Gauss–Lobatto formula, it expresses that for any* $\emptyset \varepsilon P_{(2N-1)}(0, T)$

$$\int_0^T \emptyset(t) w^{\frac{-1}{2}}(t) dt = \int_{-1}^{1} \emptyset\left(\frac{T}{2}(x+1)\right)(1 - x^2)^{\frac{-1}{2}} dx$$

$$= \frac{\pi}{\tilde{C}_j N} \sum_{j=0}^{N} \emptyset\left(\frac{T}{2}(x_j^N + 1)\right) = \frac{\pi}{\tilde{C}_j N} \sum_{j=0}^{N} \emptyset(t_{T,j}^N). \tag{14}$$

*where* $\tilde{C}_0 = \tilde{C}_N = 2$ *and* $\tilde{C}_j = 1$ *for* $j = 1, 2, \ldots, N - 1$.

*Next, we introduce the following discrete inner product and the discrete norm,*

$$(u, v)_{T,N} = \frac{\pi}{\tilde{C}_j N} \sum_{j=0}^{N} u(t_{T,i}^N) v(t_{T,i}^N), \quad \|v\|_{T,N} = (v, v)_{T,N}^{\frac{1}{2}}. \tag{15}$$

*where* $\tilde{C}_0 = \tilde{C}_N = 2$ *and* $\tilde{C}_j = 1$ *for* $j = 1, 2, \ldots, N - 1$.

### 3.2 Spectral variational iteration method

In general, the application of the VIM to solve the nonlinear Fredholm integral equation leads to the calculation of terms that are not needed and more time is consumed in repeated calculations for series solutions. Furthermore its successive iterations may be very complex, so that the resulting integrations in its iterative relation may be impossible to perform analytically, Next to overcome these shortcomings a new spectral VIM is proposed. As will be shown in this article later, the new method will be inherit the strengths of the VIM and will be very simple to implement and save time and calculations. To this end, in view of (3), we derive the following stable technique for solving (1),

$$L\left[u_{k+1}^N(t_{T,j}^N) - u_k^N(t_{T,j}^N)\right] = -A\left[u_k^N(t_{T,j}^N)\right], \quad 0 \le j \le N. \tag{16}$$

As pointed out before, this is an explicit approach and under certain conditions has a unique solution. Here, in order to directly calculate the unknown $u^N(\mathrm{t}_{\mathrm{T,k}}^{\mathrm{N}})$, we give a simple implementation by expanding $u^N(t)$ by the shifted Chebyshev polynomials, which lead to stable algorithm.

The polynomial:

$$p(t) \cong u^N(t) = \sum_{j=0}^{N} u_j^N T_{T,j}(t), \tag{17}$$

interpolates the points $(\mathrm{t}_{\mathrm{T,j}}^{\mathrm{N}}, u_j^N)$, $0 \le j \le N$, that is, $p(\boldsymbol{t}_T^N) = \boldsymbol{u}^N$, The value of the integral at the nodes is defined by $\int_0^{\mathrm{T}} \mathrm{k(t, s)}u(\mathrm{s}) = F \cdot (\boldsymbol{u}^N)$, where F is the Fredhlom integral operator. We remark that the Fredhlom integral operator $F \cdot (\boldsymbol{u}^N)$ becomes after discretization with shifted Chebyshev polynomials:

$$\begin{aligned} F \cdot (\boldsymbol{u}^N) &= \int_0^{\mathrm{T}} \mathrm{k(t, s)} \sum_{j=0}^{N} u_j^N T_{T,j}(s)ds = \sum_{j=0}^{N} u_j^N \int_0^{\mathrm{T}} \mathrm{k(t, s)} T_{T,j}(s)\mathrm{ds} \\ &= \sum_{j=0}^{N} u_j^N \cdot \mathrm{I_j}(t) = \sum_{j=0}^{N} u_j^N \cdot \sum_{i=0}^{N} k_{ij} T_{T,i}(t) = \sum_{i=0}^{N} \left[\sum_{j=0}^{N} k_{ij} u_j^N\right] T_{T,i}(t). \end{aligned}$$

Consequently, if $\boldsymbol{u}_j^N = (u_0^N, u_1^N, \ldots, u_N^N)^T$ are the coefficients of $\boldsymbol{u}^N$, then K $\boldsymbol{u}_j^N$ are the coefficients of $F \cdot (\boldsymbol{u}^N)$, given by the matrix $K = (k_{ij})_{i,j=0,1,\ldots,N}$.

The matrix K can be calculated starting from the physical values:

$$\mathrm{I_j}(t_s) = \int_0^{\mathrm{T}} \mathrm{K(t_s, s)} T_{T,j}(s)\mathrm{ds} = \sum_{i=0}^{N} w_i \mathrm{K(t_s, t_i)} T_{T,j}(\mathrm{t_i}), \quad s, j = 0, 1, \ldots, N.$$

For more details, the interested reader may observe [9].

Generally, in order to solve problem (1) using a spectral collocation scheme, the interpolating polynomial $p(t)$ is required to satisfy the equation at the interior nodes SC-GL. The value of the interpolating polynomial at the interior nodes SC–GL are $p(\boldsymbol{t}_{T,m}^N) = (\boldsymbol{u}^N)_m = I_{m,:}\boldsymbol{u}^N$, (m = 0 : N) where $I_{m,:}$ denotes the $m$ row of the identity matrix of order N + 1. For the interpolating polynomial to satisfy the NFIEs of (1) at each interior node, the collocation equation:

$$p(\boldsymbol{t}_{T,m}^N) = f(\boldsymbol{t}_{T,m}^N) + \int_0^T k(\boldsymbol{t}_T^N, s)G(p(s))ds. \tag{18}$$

Should be satisfied. Substituting the above matrix relation, the collocation Eq. (18) can be written as:

$$I_{m,:}\boldsymbol{u}^N = \boldsymbol{f}_m + I_{m,:}(F \cdot G(\boldsymbol{u}^N)), \tag{19}$$

where $\boldsymbol{f}_m = \{f(t_{(T,0)}^N), \ldots, f(t_{T,N}^N)\}$. Now in view of (3) and the definitions of L and A, by substituting the integration matrix relation, we will have the following explicit VIM for solving (1) which is called the spectral VIM (SVIM):

$$\boldsymbol{u}_{k+1}^N = \boldsymbol{u}_k^N - I_{m,:}^{-1}(I_{m,:}\boldsymbol{u}_k^N - \boldsymbol{f}_m - I_{m,:}(F \cdot G(\boldsymbol{u}^N)). \tag{20}$$

If we define $\mathbf{L} = I_{m,:}, \mathbf{f} = \mathbf{f}_m$ and $\boldsymbol{N}\boldsymbol{u}_k^N = I_{m,:}(F \cdot G(\boldsymbol{u}^N))$, then we will have the following explicit iterative relation for finding the solution vector $\boldsymbol{u}_{k+1}^N$:

$$\boldsymbol{u}_{k+1}^N = \boldsymbol{u}_k^N - \mathbf{L}^{-1}(\boldsymbol{L}\boldsymbol{u}_k^N - \mathbf{f} - \boldsymbol{N}\boldsymbol{u}_k^N). \tag{21}$$

Here the vector $\boldsymbol{u}_{k+1}^N$ is defined as $\boldsymbol{u}_{k+1}^N = \{u_{k+1}^N(t_{(T,0)}^N), \ldots, y_{k+1}^N(t_{(T,N)}^N)\}$.

Here, we may determine the initial approximation by solving the linear system $\mathbf{L}\boldsymbol{u}_0^N = \mathbf{f}$. Thus, starting from the initial approximation $\boldsymbol{u}_0^N$, we can use the recurrence formula (20) to successively obtain directly $\boldsymbol{u}_{k+1}^N$ for $k \geq 0$.

## 4 Numerical examples

In what follows, to illustrate the performance of the SVIM method in solving nonlinear Fredholm integral equations and justify the accuracy and efficiency of the presented method, we consider four examples. We mention that All the computations were performed using software Matlab with machine precision, and terminated when the current iterate satisfies $\|\boldsymbol{u}_{k+1}^N - \boldsymbol{u}_k^N\|_\infty \leq 10^{-16}$, where $\boldsymbol{u}_k^N$ is the solution vector of the $k$th SVIM iteration. Moreover to study the convergence behavior of the SVIM method, we applied the following laws:

a. The $L_2$-error norm of the solution which is defined by:

$$L_2 = \sqrt{\sum_{j=o}^{N} (\boldsymbol{u}^N(\mathrm{t}_{\mathrm{T,j}}^{\mathrm{N}}) - \boldsymbol{u}_k^N(\mathrm{t}_{\mathrm{T,j}}^{\mathrm{N}}))^2}, \quad 0 \leq j \leq N.$$

b. The $L_\infty$-error norm of the solution which is defined by:

$$L_\infty = max_j \left| \boldsymbol{u}^N(\mathrm{t}_{\mathrm{T,j}}^{\mathrm{N}}) - \boldsymbol{u}_k^N(\mathrm{t}_{\mathrm{T,j}}^{\mathrm{N}}) \right|, \quad 0 \leq j \leq N,$$

where $\boldsymbol{u}^N(\mathrm{t}_{\mathrm{T,j}}^{\mathrm{N}})$and $\boldsymbol{u}_k^N(\mathrm{t}_{\mathrm{T,j}}^{\mathrm{N}})$ are exact and numerical solution, respectively, and $\{\mathrm{t}_{\mathrm{T,j}}^{\mathrm{N}}\}$, $0 \leq j \leq N$, are collocation node points. The following algorithm, based on the method presented in Sect. 3.2 has been used to solve examples.

**Table 1** The $L_\infty$, $L_2$ errors used for Example 4.1

| N | 6 | 8 | 10 | 12 | 14 | 16 |
|---|---|---|---|---|---|---|
| $L_2$ | 2.2525$E$−03 | 8.5824$E$−05 | 5.5129$E$−06 | 3.1698E−07 | 1.3148$E$−08 | 4.2019$E$−10 |
| $L_\infty$ | 1.0912$E$−03 | 3.6381$E$−05 | 2.1034$E$−06 | 1.1087E−07 | 4.2710$E$−09 | 1.2796E−10 |

**Algorithm 4.1**

**Begin**

1. Input N, tspan (the desired range), max iteration and max error(desired tolerance) for NFIE.

2. Construct the interpolation polynomial $p(t)$ by the shifted Chebyshev polynomials from (17).

3. Choose L[u(t)], N[u(t)] and f(t) for NFIE.

4. Determine the initial approximation by solving the linear system $\mathbf{L}\boldsymbol{u}_0^N = \mathbf{f}$.

5.Set it=0 and unorm=1.

6. While (it< max iteration ) and (unorm > max error ) do

7. Compute the values of L[u(t)], N[u(t)] and f(t) at the interior nodes SC-GL(collocation points)

8. Substituting the above matrix at the collocation equation (18) .

9. Using the explicit iterative relation (20) to obtaining the solution vector $\mathbf{u}_{k+1}^{N}$.

10. Evaluate the value of unorm and Update the data of $\boldsymbol{u}_k^N$

end while

**End**

***Example 4.1*** Consider the following NFIE [30]:

$$u(t) = \sin(\pi t) + \frac{1}{5}\int_0^1 \cos(\pi s)\sin(\pi s)u^3(s)ds.$$

The exact solution of this problem is $u(t) = \sin(\pi t) + \frac{1}{3}(20 - \sqrt{391})\cos(\pi t)$.

Table 1, illustrates the $L_\infty$, $L_2$ errors for different values of $N$. Figure 1 shows that the absolute error for $N = 16$ and the behavior of the exact and numerical solutions (our method) of this example when N=16 is presented in Fig. 2. As expected, the exponential rate of convergence is observed for the above nonlinear problem.

***Example 4.2*** Consider the following NFIE [29]:

$$u(t) = -t^2 - \frac{t}{3}(2\sqrt{2} - 1) + 2 + \int_0^1 st\sqrt{u(s)}ds,$$

with the exact solution $u(t) = 2 - t^2$.

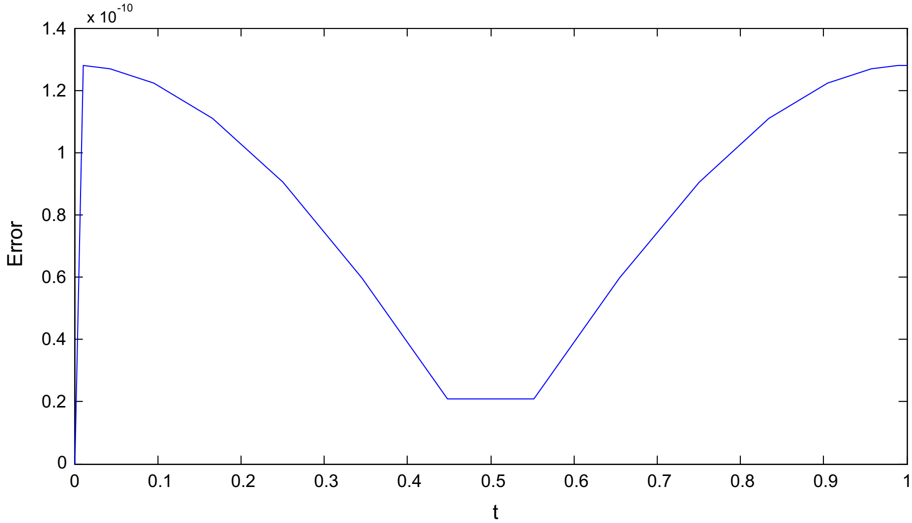


**Fig. 1** Absolute error of the spectral PIM by N = 16

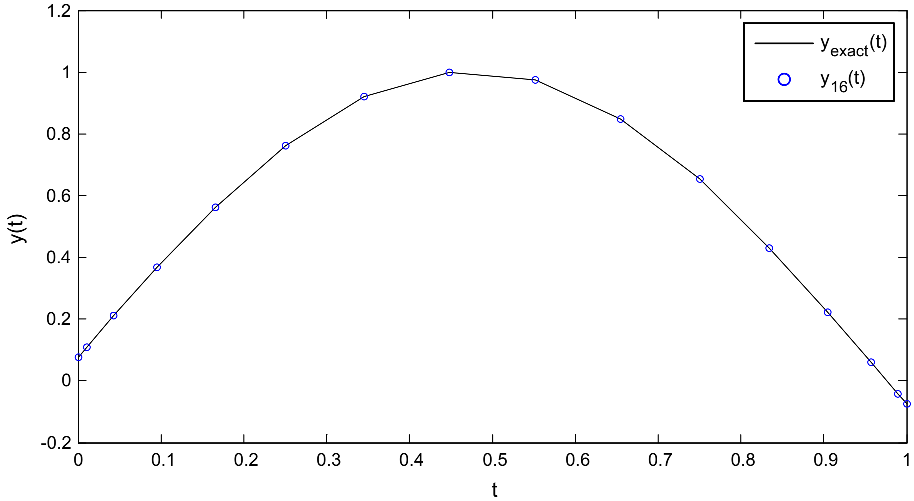


**Fig. 2** Numerical solution of the spectral PIM by N = 16

For different values of N, the errors in Table 2 are given with $L_\infty$, $L_2$ errors. Figure 3 shows that the absolute error for $N = 16$ and the behavior of the exact and numerical solutions (our method) of this example when N = 16 is presented in Fig. 4. The exponential rate of convergence is observed for the NFIE.

***Example 4.3*** Consider the following NFIE [28]:

$$u(t) = 1 + \mathrm{t} + \left(1 - \frac{3}{2}\ln 3 + \frac{\sqrt{3}}{6}\pi\right) t^2 + \int_0^1 2t^2 s ln(u(s)) ds,$$

with the exact solution $u(t) = 1 + t + t^2$.

**Table 2** The $L_\infty$, $L_2$ errors used for Example 4.2

| N | 6 | 8 | 10 | 12 | 14 | 16 |
|---|---|---|---|---|---|---|
| $L_2$ | $6.8857E-06$ | $1.3955E-07$ | $4.1414E-09$ | $1.5694E-10$ | $6.9528E-12$ | $3.4258E-13$ |
| $L_\infty$ | $4.4680E-06$ | $7.8940E-08$ | $2.1038E-09$ | $7.2979E-11$ | $2.9991E-12$ | $1.3833E-13$ |

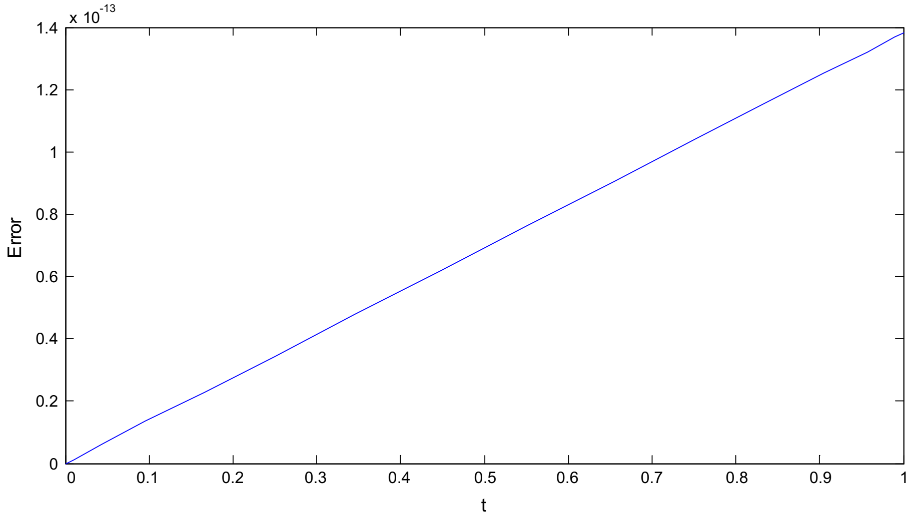


**Fig. 3** Absolute error of the spectral PIM by N = 16

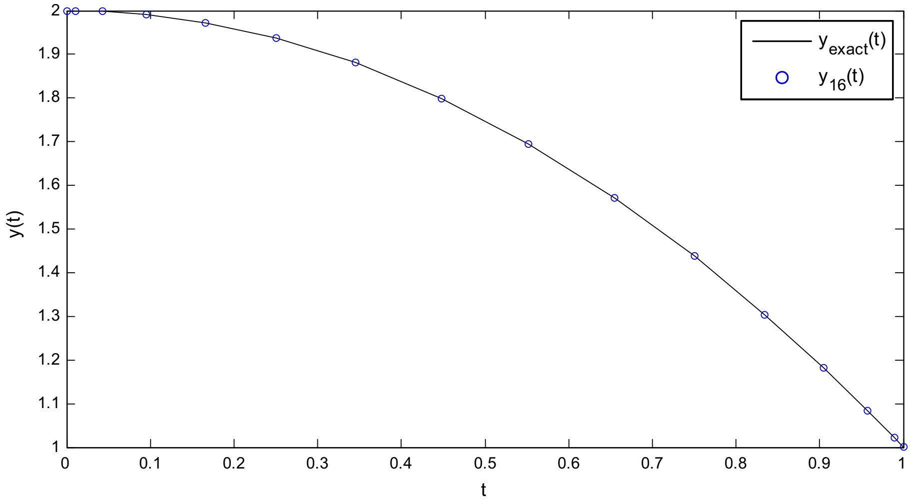


**Fig. 4** Numerical solution of the spectral PIM by N = 16

The $L_\infty$, $L_2$ errors are given in Table 3. Figure 5 shows that the absolute error for N = 16. Figure 6 depict the exact and numerical solutions for N = 16. Again, we can see spectral accuracy for large values of N.

**Table 3** The $L_\infty$, $L_2$ errors used for Example 4.3

| N | 6 | 8 | 10 | 12 | 14 | 16 |
|---|---|---|---|---|---|---|
| $L_2$ | $4.5781E{-}06$ | 1.9484E−08 | 6.5911E−10 | $2.6549E{-}12$ | 2.2998E−13 | 9.6786E−16 |
| $L_\infty$ | $3.3504E{-}06$ | $1.2540E{-}08$ | $3.8304E{-}10$ | $1.4175E{-}12$ | 1.1457E−13 | 4.4408E−16 |

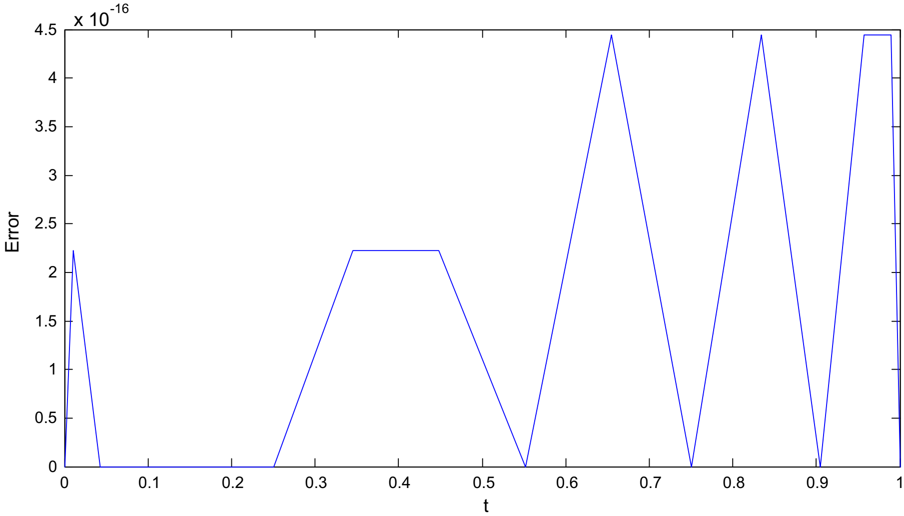


**Fig. 5** Absolute error of the spectral PIM by N = 16

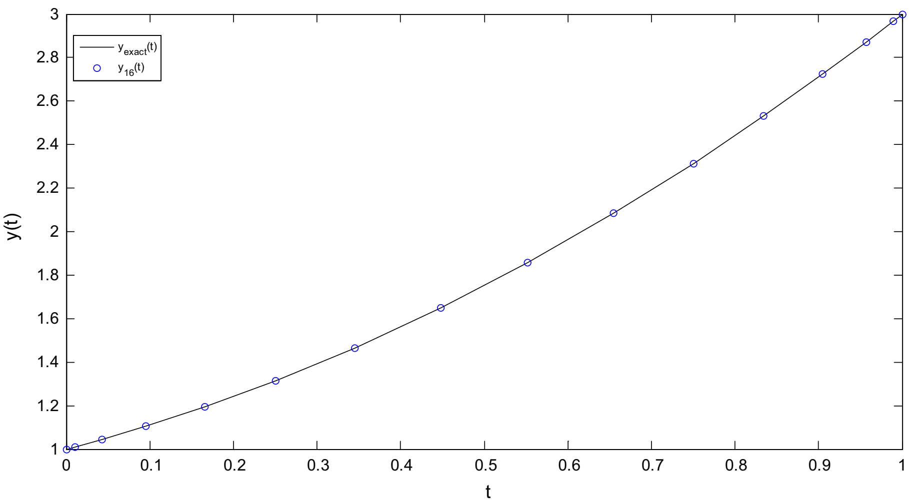


**Fig. 6** Numerical solution of the spectral PIM by N = 16

**Table 4** The $L_\infty$, $L_2$ errors used for Example 4.4

| N | 6 | 8 | 10 | 12 | 14 | 16 |
|---|---|---|---|---|---|---|
| $L_2$ | 2.4981$E$−05 | 4.0865$E$−07 | 1.1040$E$−10 | 6.9346$E$−11 | 7.2906E−13 | 3.1165E−15 |
| $L_\infty$ | 1.6210E−05 | 2.3116E−07 | 5.6087E−11 | 3.2245E−11 | 3.1441$E$−13 | 1.3322$E$−15 |

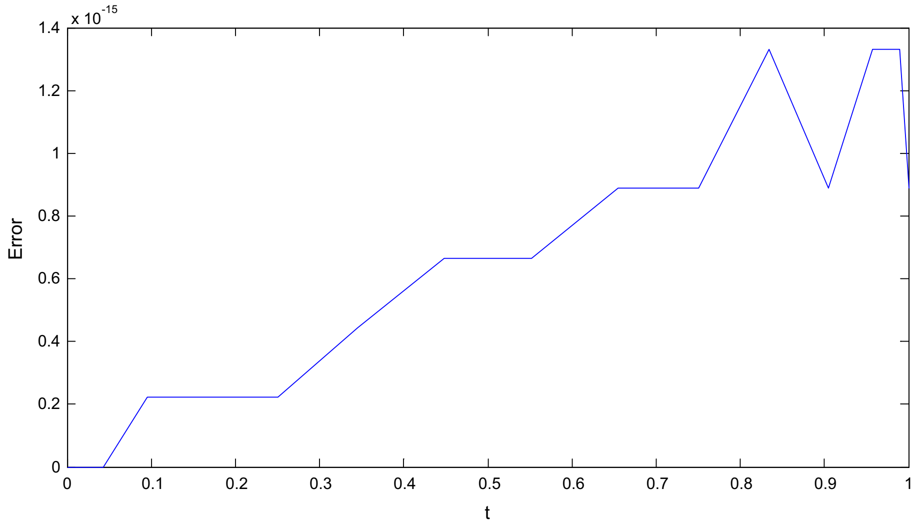


**Fig. 7** Absolute error of the spectral PIM by N = 16

***Example 4.4*** Consider the NFIE [26]:

$$u(t) = e^t - \frac{1}{2}t(\cos(1) - \cos(e)) + \frac{1}{2}\int_0^1 te^s \sin(u(s))ds,$$

Which has the exact solution $u(t) = e^t$.

In Table 4, the $L_\infty$, $L_2$errors for different values of N are demonstrated. The other results are presented in Figs. 7 and 8 with N = 16 similar to examples 1, 2 and 3.

## 5 Conclusion

This article introduced an efficient numerical method based on a hybrid of spectral and variational iteration method for solving a class of nonlinear Fredholm integral equations (NFIEs). This new method is easy and straightforward to implement and accurate when applied to the nonlinear Fredholm integral equations. The effectiveness of this approach is demonstrated by solving several problems. The SVIM numerical results were compared with exact results and excellent agreement was obtained.

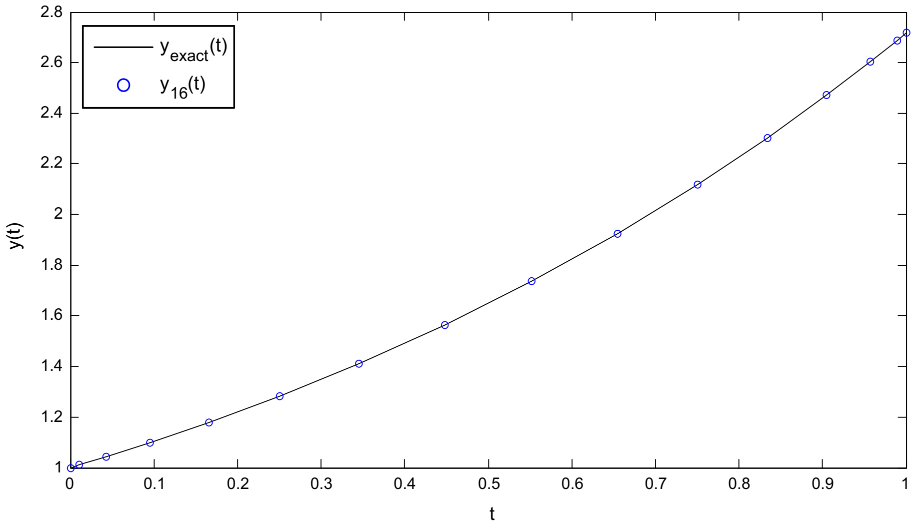


**Fig. 8** Numerical solution of the spectral PIM by $N = 16$

**Acknowledgements** The authors are very grateful to both reviewers for carefully reading the paper and for their comments and suggestions which helped to improve the paper.

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