



The capacity of some classes of polyhedra

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Abstract

K. Borsuk in 1979, in Topological Conference in Moscow, introduced the concept of the capacity of a compactum. In this paper, we compute the capacity of the product of two spheres of the same or different dimensions and the capacity of lense spaces. Also, we present an upper bound for the capacity of a \mathbb{Z}_n -complex, i.e., a connected finite 2-dimensional CW-complex with finite cyclic fundamental group \mathbb{Z}_n .

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1. Introduction and motivation

In this paper, every polyhedron and CW-complex is assumed to be finite and connected. Also, every map between two CW-complexes is assumed to be cellular. Since every polyhedron is homotopy equivalent to a finite CW-complex of the same dimension, and conversely, we use the terms “polyhedron” and “finite CW-complex” interchangeably. We assume that the reader is familiar with the basic notions and facts of homotopy theory.

K. Borsuk in [5], introduced the concept of the capacity of a compactum as follows: the capacity $C(X)$ of a compactum X is the cardinality of the set of all shapes of compacta A which are shape dominated by X (for more details, see [15]).

For polyhedra, the notions shape and shape domination in the above definition can be replaced by the notions homotopy type and homotopy domination, respectively. Indeed, by some known results in shape theory one can conclude that for any polyhedron P , there is a one to one functorial correspondence between the shapes of compacta shape dominated by P and the homotopy types of CW-complexes (not necessarily finite) homotopy dominated by P (see [14]).

M. Mather in [16] proved that every polyhedron dominates only countably many different homotopy types (hence shapes). Note that the capacity of a polyhedron is a homotopy invariant, i.e., for polyhedra X and Y with the same homotopy type, $C(X) = C(Y)$. This property can be useful for distinguishing two polyhedra up to homotopy equivalence. Hence it seems interesting to find polyhedra with finite capacity and compute the capacity of some of their well-known spaces. Borsuk in [5] asked a question: “Is it true that the capacity of every finite polyhedron is finite?”. D. Kolodziejczyk in [12] gave a negative

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answer to this question. Also, she investigated some conditions for polyhedra to have finite capacity ([13,14]). For instance, a polyhedron Q with finite fundamental group $\pi_1(Q)$ and a polyhedron P with abelian fundamental group $\pi_1(P)$ and finitely generated homology groups $H_i(\tilde{P})$, for $i \geq 2$ where \tilde{P} is the universal cover of P , have finite capacities.

Borsuk in [5] mentioned that the capacities of $\bigvee_k S^1$ and S^n are equal to $k + 1$ and 2 , respectively. The authors in [17] computed the capacities of Moore spaces $M(A, n)$ and Eilenberg-MacLane spaces $K(G, n)$. In fact, we showed that the capacities of a Moore space $M(A, n)$ and an Eilenberg-MacLane space $K(G, n)$ are equal to the number of direct summands of A and semidirect factors of G up to isomorphism, respectively. Also, we computed the capacity of the wedge of finitely many Moore spaces of different degrees and the capacity of the product of finitely many Eilenberg-MacLane spaces of different homotopy types. In [18], we showed that the capacity of $\bigvee_{n \in I} (\bigvee_{i_n} S^n)$ is equal to $\prod_{n \in I} (i_n + 1)$, where $\bigvee_{i_n} S^n$ denotes the wedge of i_n copies of S^n , I is a finite subset of \mathbb{N} and $i_n \in \mathbb{N}$. In fact, we proved that every space homotopy dominated by $\bigvee_{n \in I} (\bigvee_{i_n} S^n)$ has the homotopy type of $\bigvee_{n \in I} (\bigvee_{j_n} S^n)$, where $0 \leq j_n \leq i_n$.

M. Abbasi et al. in [1] computed the capacity of 2-dimensional manifolds. They showed that the capacities of a compact orientable surface of genus $g \geq 0$ and a compact non-orientable surface of genus $g > 0$ are equal to $g + 2$ and $\lfloor \frac{g}{2} \rfloor + 2$, respectively. In [18], we proved the capacity of a 2-dimensional CW-complex P with free fundamental group $\pi_1(P)$ is finite and is equal to $(\text{rank } \pi_1(P) + 1) \times (\text{rank } H_2(P) + 1)$.

We outline the main results of the paper. In Section 3, we compute the capacity of the product of two spheres of the same or different dimensions. Then, in Section 4, we compute the capacity of lens spaces which are a class of closed orientable 3-manifolds. Also, by a similar method to computation of capacity of a lens space, we show that the capacity of a real projective n -space is equal to 2. Note that this result was proved by Y. Kodama et al. in [11] in a different manner. Finally, in Section 5, we find an upper bound for the capacity of a \mathbb{Z}_n -complex, a 2-dimensional CW-complex with finite cyclic fundamental group \mathbb{Z}_n . In fact, we show that every space homotopy dominated by a \mathbb{Z}_n -complex P where $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_m^{\alpha_m}$ (for mutually distinct primes p_i and positive integers α_i) has the homotopy type of a \mathbb{Z}_m -complex where $m = p_{i_1}^{\alpha_{i_1}} \cdots p_{i_j}^{\alpha_{i_j}}$ for $i_1, \dots, i_j \in \{1, \dots, m\}$.

2. Preliminaries

We recall here some facts that we will use throughout the paper.

Definition 2.1. [21]. Let P be a CW-complex. For each $n \geq 1$, the condition \mathcal{D}_n on P is defined as follows:

\mathcal{D}_n : $H_i(\tilde{P}) = 0$ for $i > n$, and $H^{n+1}(P; \mathcal{B}) = 0$ for all coefficient bundles \mathcal{B} (for more details, see [19]). Note that \tilde{P} denotes the universal covering space of P .

Definition 2.2. [8]. A Moore space of degree n ($n \geq 2$) is a simply connected CW-complex X with a single non-vanishing homology group of degree n , that is $\tilde{H}_i(X, \mathbb{Z}) = 0$ for $i \neq n$. A Moore space of degree n is denoted by $M(A, n)$ where $A \cong \tilde{H}_n(X, \mathbb{Z})$.

As an example, the n -sphere S^n for $n \geq 2$ is a Moore space of degree n , $S^n = M(\mathbb{Z}, n)$.

Theorem 2.3. [8]. The homotopy type of a Moore space $M(A, n)$ is uniquely determined by A and n for $n \geq 2$.

Theorem 2.4. [8, Theorem 4.32]. (Hurewicz theorem) If a topological space X is $(n - 1)$ -connected, $n \geq 2$, then $\tilde{H}_i(X) = 0$ for $i < n$, $h_n^X : \pi_n(X) \cong H_n(X)$ and $h_{n+1}^X : \pi_{n+1}(X) \rightarrow H_{n+1}(X)$ is epimorphism, where $h_i^X : \pi_i(X) \rightarrow H_i(X)$ denotes the i -th Hurewicz map.

Theorem 2.5. [10, page 91]. For all $2 \leq r \leq p + q + \min\{p, q\} - 3$, we have

$$\pi_r(S^p \vee S^q) \cong \pi_r(S^p) \oplus \pi_r(S^q) \oplus \pi_r(S^{p+q-1}).$$

Theorem 2.6. [3, Proposition 2.6.15]. *A simply connected space X is homotopy equivalent to a wedge of Moore spaces if and only if h_n^X is split surjective for all $n > 1$.*

Recall that an epimorphism $f : G \rightarrow H$ is called split surjective if the following short exact sequence is split

$$0 \rightarrow \text{Ker } f \hookrightarrow G \xrightarrow{f} H \rightarrow 0.$$

Theorem 2.7. [4]. *Let X be a topological space which is homotopy dominated by a closed (compact without boundary) connected topological n -dimensional manifold M . If $H^n(X; \mathbb{Z}_2) \neq 0$, then X has the homotopy type of M .*

Lemma 2.8. [8, Example 2.43]. *Let X be the Eilenberg-MacLane space $K(\mathbb{Z}_m, 1)$. Then $H_n(X) \cong \mathbb{Z}_m$ for odd n and $H_n(X) \cong 0$ for even $n > 0$.*

3. The capacity of product of two spheres

In this section, we compute the capacity of product of two spheres of the same or different dimensions.

Lemma 3.1. *The capacity of $S^1 \times S^n$ is equal to 4, for $n \geq 2$.*

Proof. Put $P = S^1 \times S^n$. By the Kunnetth formula (see, for example, [8, Theorem 3B.6]), we know that

$$H_i(P) \cong \begin{cases} \mathbb{Z}, & i = 0, 1, n, n+1 \\ 0, & \text{otherwise.} \end{cases}$$

Suppose that X is homotopy dominated by P and \tilde{X} denotes the universal covering space of X . Then $\pi_1(X)$ and $H_i(X)$ are isomorphic to a direct suumand of $\pi_1(P) \cong \mathbb{Z}$ and of $H_i(P)$, respectively, for all $i \geq 2$.

First, let $H_{n+1}(X) \cong \mathbb{Z}$. By the universal coefficient theorem for cohomology (see, for example, [8, Theorem 3.2]), we have

$$H^{n+1}(X; \mathbb{Z}_2) \cong \text{Hom}(H_{n+1}(X), \mathbb{Z}_2) \cong \mathbb{Z}_2 \neq 0.$$

Then, since P is a closed compact connected $n+1$ -dimensional topological manifold, X and P have the same homotopy type by Theorem 2.7.

Second, let $H_{n+1}(X) = 0$. Then we just have the following cases:

Case One. $\pi_1(X) = 1$ and $H_i(X) = 0$ for all $i \geq 2$. Then by the Whitehead Theorem (see [8, Corollary 4.33]), X and $\{*\}$ have the same homotopy type.

Case Two. $\pi_1(X) = 1$, $H_n(X) \cong \mathbb{Z}$ and $H_i(X) = 0$ for all $i \neq n$. Then X is the Moore space $M(\mathbb{Z}, n)$ and so, X has the homotopy type of S^n .

Case Three. $\pi_1(X) \cong \mathbb{Z}$. We know that $\tilde{P} = \mathbb{R} \times S^n$ is the universal covering space of $P = S^1 \times S^n$ and \tilde{X} is homotopy dominated by \tilde{P} . Since the capacity of a compactum is a homotopy invariant, so $C(\tilde{P}) = C(S^n) = 2$ and so \tilde{X} has the homotopy type of $\{*\}$ or \tilde{P} . If \tilde{X} and \tilde{P} have the same homotopy type, then the domination map $d_X : P \rightarrow X$ induces isomorphisms

$$d_{X*} : \pi_1(P) \rightarrow \pi_1(X), \quad \tilde{d}_{X*} : H_i(\tilde{P}) \rightarrow H_i(\tilde{X})$$

for all $i \geq 2$ (note that an epimorphism between isomorphic Hopfian groups is an isomorphism). Then by the Whitehead Theorem (see, for example, [10, Theorem 3.7, p. 113]), d_X is a homotopy equivalence. Thus X and P have the same homotopy type which is a contradiction because $H_{n+1}(X) = 0$. Hence \tilde{X} has the homotopy type of $\{*\}$ and so, X is the Eilenberg-MacLane space $K(\mathbb{Z}, 1)$. This shows that X has the homotopy type of S^1 . \square

Lemma 3.2. *The capacity of $S^n \times S^n$ is equal to 3, for all $n \geq 1$.*

Proof. For the case $n = 1$, $S^1 \times S^1$ is a product of Eilenberg-MacLane spaces and by [17, Proposition 4.6], $C(S^1 \times S^1) = 3$.

Let $n \geq 2$ and $P = S^n \times S^n$. By the Kunneth formula, we know that

$$H_i(P) \cong \begin{cases} \mathbb{Z}, & i = 0, 2n \\ \mathbb{Z} \times \mathbb{Z}, & i = n \\ 0 & \text{otherwise.} \end{cases}$$

Suppose X is homotopy dominated by P . Then $H_i(X)$ is isomorphic to a direct summand of $H_i(P)$ for all $i \geq 2$.

First, let $H_{2n}(X) \cong \mathbb{Z}$. By the universal coefficient theorem for cohomology, we have $H^{2n}(X; \mathbb{Z}_2) \neq 0$. Then, since P is a closed compact connected $2n$ -dimensional topological manifold, X and P have the same homotopy type by Theorem 2.7.

Second, let $H_{2n}(X) = 0$. Then we just have the following cases:

Case One. $H_i(X) = 0$ for all $i \geq 1$. Then by the Whitehead Theorem, X and $\{*\}$ have the same homotopy type.

Case Two. $H_n(X) \cong \mathbb{Z}$ and $H_i(X) = 0$ for all $i \neq n$. Then X is the Moore space $M(\mathbb{Z}, n)$ and so, X and S^n have the same homotopy type.

Case Three. $H_n(X) \cong \mathbb{Z} \times \mathbb{Z}$ and $H_i(X) = 0$ for all $i \neq n$. Then X has the homotopy type of $S^n \vee S^n$. But $S^n \vee S^n$ is not homotopy dominated by $P = S^n \times S^n$ because $\pi_{2n-1}(S^n \vee S^n) \cong \pi_{2n-1}(S^n) \times \pi_{2n-1}(S^n) \times \mathbb{Z}$ (by Theorem 2.5) is not isomorphic to a direct summand of $\pi_{2n-1}(S^n \times S^n) \cong \pi_{2n-1}(S^n) \times \pi_{2n-1}(S^n)$. Thus this case does not occur. \square

Lemma 3.3. *Let X be a simply connected CW-complex with $H_n(X) \cong H_{n+1}(X) \cong \mathbb{Z}$ and $H_i(X) = 0$ for all $i \neq n, n + 1$. Then X has the homotopy type of $S^n \vee S^{n+1}$.*

Proof. First, we show that X is $(n - 1)$ -connected. This is obvious for $n = 2$ by the hypothesis. So let $n \geq 3$. Since X is simply connected, $\pi_2(X) \cong H_2(X)$ by Theorem 2.4. But $H_2(X) = 0$ by the hypothesis. Hence $\pi_2(X) = 0$ and so, X is a 2-connected space. If $n = 3$, we are done. Otherwise, by a similar argument, one can easily see that X is $(n - 1)$ -connected.

Now, by Theorem 2.4, $h_n^X : \pi_n(X) \cong H_n(X)$ and $h_{n+1}^X : \pi_{n+1}(X) \rightarrow H_{n+1}(X)$ is an epimorphism. Clearly, h_n^X is a split surjective homomorphism. Also, since

$$0 \rightarrow \ker h_{n+1} \hookrightarrow \pi_{n+1}(X) \xrightarrow{h_{n+1}^X} H_{n+1}(X) \rightarrow 0$$

is a short exact sequence and $H_{n+1}(X) \cong \mathbb{Z}$ is a projective \mathbb{Z} -module, h_{n+1}^X is also split surjective. On the other hand, since $H_i(X) = 0$ for all $i \neq n, n + 1$, h_i^X is also split surjective. Thus by Theorem 2.6, the space X is homotopy equivalent to a wedge of Moore spaces. By the hypothesis $H_n(X) \cong H_{n+1}(X) \cong \mathbb{Z}$, so X has the homotopy type of $S^n \vee S^{n+1}$. \square

Lemma 3.4. *The capacity of $S^n \times S^{n+1}$ is equal to 4, for all $n \geq 2$.*

Proof. Put $P = S^n \times S^{n+1}$. By the Kunneth formula, we have

$$H_i(P) \cong \begin{cases} \mathbb{Z}, & i = 0, n, n + 1, 2n + 1, \\ 0, & \text{otherwise.} \end{cases}$$

Suppose that X is homotopy dominated by P . Then $H_i(X)$ is isomorphic to a direct summand of $H_i(P)$ for all $i \geq 2$.

First, let $H_{2n+1}(X) \cong \mathbb{Z}$. By the universal coefficient theorem for cohomology, we have $H^{2n+1}(X; \mathbb{Z}_2) \neq 0$. Then, since P is a closed compact connected $(2n + 1)$ -dimensional topological manifold, X and P have the same homotopy type by Theorem 2.7.

Second, let $H_{2n+1}(X) = 0$. We just have the following cases:

Case One. $H_i(X) = 0$ for all i . Then by the Whitehead Theorem, X and $\{*\}$ have the same homotopy type.

Case Two. $H_n(X) \cong \mathbb{Z}$ and $H_i(X) = 0$ for all $i \neq n$. Then X is the Moore space $M(\mathbb{Z}, n)$ and so, X and S^n have the same homotopy type.

Case Three. $H_{n+1}(X) \cong \mathbb{Z}$ and $H_i(X) = 0$ for all $i \neq 3$. Then X is the Moore space $M(\mathbb{Z}, n+1)$ and so, X and S^{n+1} have the same homotopy type.

Case Four. $H_n(X) \cong H_{n+1}(X) \cong \mathbb{Z}$ and $H_i(X) = 0$ for all $i \neq 2, 3$. Then by Lemma 3.3, X and $S^n \vee S^{n+1}$ have the same homotopy type. But $S^n \vee S^{n+1}$ is not homotopy dominated by $S^n \times S^{n+1}$ because $\pi_{2n}(S^n \vee S^{n+1})$ is not isomorphic to a subgroup of $\pi_{2n}(S^n \times S^{n+1})$. Thus this case does not happen. \square

Lemma 3.5. *The capacity of $S^n \times S^m$ is equal to 4, where $2 < n+1 < m$.*

Proof. Put $P = S^n \times S^m$. By the Kunneth formula, we know that

$$H_i(P) \cong \begin{cases} \mathbb{Z}, & i = n, m, n+m, \\ 0, & \text{otherwise.} \end{cases}$$

Then $H_i(X)$ is isomorphic to a direct summand of $H_i(P)$ for all $i \geq 2$.

First, let $H_{n+m}(X) \cong \mathbb{Z}$. By the universal coefficient theorem for cohomology, $H^{n+m}(X; \mathbb{Z}_2) \neq 0$. Then, since P is a closed compact connected $(n+m)$ -dimensional topological manifold, X and P have the same homotopy type by Theorem 2.7.

Second, let $H_{n+m}(X) = 0$. We have the following cases:

Case One. $H_i(X) = 0$ for all i ; Then by the Whitehead Theorem, X and $\{*\}$ have the same homotopy type.

Case Two. $H_n(X) \cong \mathbb{Z}$ and $H_i(X) = 0$ for all $i \neq n$; Then X is the Moore space $M(\mathbb{Z}, n)$, and so X and S^n have the same homotopy type.

Case Three. $H_m(X) \cong \mathbb{Z}$ and $H_i(X) = 0$ for all $i \neq m$; Then X is the Moore space $M(\mathbb{Z}, m)$, and so X and S^m have the same homotopy type.

Case Four. $H_n(X) \cong H_m(X) \cong \mathbb{Z}$ and $H_i(X) = 0$ for all $i \neq n, m$. We know that S^n and S^m have CW decompositions $\{a, e^n\}$ and $\{b, e^m\}$ respectively, where a and b are 0-cells. Hence $P = S^n \times S^m$ has a CW decomposition $\{a \times b, a \times e^m, e^n \times b, e^n \times e^m\}$. One can consider $S^n \vee S^m$ as the subspace $S^n \times \{b\} \cup \{a\} \times S^m$ of $P = S^n \times S^m$. Then $S^n \vee S^m$ has a CW decomposition $\{a \times b, a \times e^m, e^n \times b\}$. Since $2 < n+1 < m$, then $H_n(P) = H_n(X) = C_n(X)$ and $H_m(P) = H_m(X) = C_m(X)$. Hence the homomorphism $i_* : H_i(S^n \vee S^m) \rightarrow H_i(P)$ is identity for $i = n, m$, where $i : S^n \vee S^m \hookrightarrow P$ is the inclusion map. Now, consider the map

$$h = i \circ d_X : S^n \vee S^m \rightarrow X,$$

where $d_X : P \rightarrow X$ is the domination map. It is easy to see that $(d_X)_* : H_i(P) \rightarrow H_n(X)$ is an isomorphism for $i = n, m$ (epimorphism between two isomorphic Hopfian groups). Hence, the map $h_* : H_i(S^n \vee S^m) \rightarrow H_i(X)$ is an isomorphism for all $i \geq 2$ and so by the Whitehead Theorem, $h : S^n \vee S^m \rightarrow X$ is a homotopy equivalence. This shows that $S^n \vee S^m$ is homotopy dominated by P which is a contradiction because by Theorem 2.5, $\pi_{n+m-1}(S^n \vee S^m)$ is not isomorphic to a subgroup of $\pi_{n+m-1}(S^n \times S^m)$. Thus this case does not happen. \square

Finally by Lemmas 3.1, 3.2, 3.4 and 3.5, we can conclude the following theorem which is the main result of this section.

Theorem 3.6. *For $n, m \geq 1$, the capacity of $S^n \times S^m$ is equal to 4 if $n \neq m$ and it is equal to 3 if $n = m$.*

4. The capacity of lens spaces

In this section, we use reference [9] for expressing the classification of closed (compact and without boundary) orientable 3-manifolds.

By Kneser's Theorem, every compact orientable 3-manifold M factors as a connected sum of primes, $M = P_1 \# \cdots \# P_n$, and this decomposition is unique up to insertion or deletion of S^3 summands. On the other hand, primes 3-manifold that are closed and orientable can be grouped broadly into three classes:

- (1) **Infinite cyclic fundamental group.** Only one such manifold (which is closed and orientable) is $S^1 \times S^2$. The capacity of such manifold is 4 by Theorem 3.6.
- (2) **Infinite noncyclic fundamental group.** Such a manifold M is an Eilenberg-MacLane space $K(\pi, 1)$. It has been shown that π is a finitely generated torsion-free group. Recall that a group homomorphism $f : G \rightarrow H$ is an r -homomorphism if there exists a homomorphism $g : H \rightarrow G$ such that $f \circ g = id_H$. Then H is called an r -image of G . By [17, Proposition 4.4], the capacity of an Eilenberg-MacLane space $K(\pi, 1)$ is equal to the number of r -images of π up to isomorphism. In particular, if π is a nilpotent group, then π has only finitely many r -images up to isomorphism by [14, Corollary 1].
- (3) **Finite fundamental group.** For such a manifold M , the universal covering space \tilde{M} is simply connected and closed, hence a homotopy sphere. All these manifolds (which are spherical 3-manifolds) have the form $M = S^3/\Gamma$ for Γ a finite subgroup of $SO(4)$ acting freely on S^3 by rotations. Thus S^3 is the universal covering space of M and $\Gamma = \pi_1(M)$. The spherical manifolds with cyclic fundamental group $\pi_1(M) = \Gamma$ are the lens spaces which are defined as follows:

Let p and q be relatively prime integers. Regard S^3 as all $(z_0, z_1) \in \mathbb{C}^2$ with $|z_0|^2 + |z_1|^2 = 1$. Let $\zeta = e^{2\pi i/p}$ be a primitive p th root of unity; define $h : S^3 \rightarrow S^3$ by $h(z_0, z_1) = (\zeta z_0, \zeta^q z_1)$, and define an equivalence relation on S^3 by $(z_0, z_1) \sim (z'_0, z'_1)$ if there exists an integer m with $h^m(z_0, z_1) = (z'_0, z'_1)$. The quotient space S^3/\sim is called a lens space and is denoted by $L(p, q)$. Note that $\pi_1(L(p, q)) \cong \mathbb{Z}_p$. In the next theorem, we compute the capacity of lens spaces.

Theorem 4.1. *The capacity of the lens space $L(p, q)$ is equal to 2, where p and q are relatively prime integers.*

Proof. Suppose X is homotopy dominated by $P = L(p, q)$ with domination map $d_X : P \rightarrow X$ and \tilde{X} denotes the universal covering space of X . Then $\pi_1(X)$ and $H_i(X)$ are isomorphic to a direct summand of $\pi_1(P) \cong \mathbb{Z}$ and of $H_i(P)$, respectively, for all $i \geq 2$. We know that

$$H_i(P) \cong \begin{cases} \mathbb{Z}, & i = 0, 3 \\ \pi_1(P) \cong \mathbb{Z}_p, & i = 1 \\ 0, & \text{otherwise.} \end{cases}$$

We have the following cases:

Case One. $\pi_1(X) = 1$ and $H_i(X) = 0$ for all $i \geq 2$. Then by the Whitehead Theorem X and $\{*\}$ have the same homotopy type.

Case Two. $\pi_1(X) = 1$ and $H_3(X) \cong \mathbb{Z}$ and $H_i(X) = 0$ for all $i \neq 3$. Then $H^3(X; \mathbb{Z}_2) \neq 0$ and so, X and $L(p, q)$ have the same homotopy type (by Theorem 2.7) which is a contradiction because $\pi_1(X) = 1$. Thus this case does not occur.

Case Three. $\pi_1(X) \cong \mathbb{Z}_p$. Then $(d_X)_* : \pi_1(P) \rightarrow \pi_1(X)$ is an isomorphism. Since \tilde{X} is homotopy dominated by $\tilde{P} = S^3$ and $C(S^3) = 2$, then \tilde{X} has the homotopy type of $\{*\}$ or S^3 . If \tilde{X} has the homotopy type of $\{*\}$, then X is the Eilenberg-MacLane space $K(\mathbb{Z}_p, 1)$. By Lemma 2.8, $H_i(K(\mathbb{Z}_p, 1))$ is nonzero for infinitely many values of i which is a contradiction with the fact that $H_i(X)$ is a summand of $H_i(P)$, for all $i \geq 0$. So, \tilde{X} has

the homotopy type of S^3 . Also, the homomorphism $(d_X)_* : H_i(P) \rightarrow H_i(X)$ which is induced by the domination map d_X is an epimorphism between two isomorphic Hopfian groups and so, is an isomorphism for all $i \geq 2$. Now by the Whitehead Theorem, we obtain that $d_X : P \rightarrow X$ is a homotopy equivalence. Hence X has the homotopy type of P . \square

By a similar argument to the previous theorem, we can compute the capacity of a real projective n -space. Note that this result was proved by Y. Kodama et al. in [11] in a different manner.

Theorem 4.2. *The capacity of $\mathbb{R}P^n$ is equal to 2, for all $n \geq 2$*

Proof. Suppose that X is homotopy dominated by $P = \mathbb{R}P^n$ with the domination map $d_X : P \rightarrow X$ and \tilde{X} denotes the universal covering space of X . Then $\pi_1(X)$ and $H_i(X)$ are isomorphic to a direct summand of $\pi_1(P) \cong \mathbb{Z}_2$ and of $H_i(P)$, respectively, for all $i \geq 2$. We have the following cases:

Case One. $\pi_1(X) \cong \mathbb{Z}_2$. Since \tilde{X} is homotopy dominated by $\tilde{P} = S^n$ and $C(S^n) = 2$, \tilde{X} has the homotopy type of $\{*\}$ or S^n . If \tilde{X} has the homotopy type of $\{*\}$, then X is the Eilenberg-MacLane space $K(\mathbb{Z}_2, 1)$. By Lemma 2.8, $H_i(K(\mathbb{Z}_2, 1))$ is nonzero for infinitely many values of i . But this is a contradiction with the fact that $H_i(X)$ is a summand of $H_i(P)$, for all $i \geq 0$. Hence \tilde{X} has the homotopy type of S^n and so, $H_i(\tilde{X}) \cong H_i(S^n)$ for $i \geq 0$. Also, the homomorphism $(\tilde{d}_X)_* : H_i(S^n) \rightarrow H_i(\tilde{X})$ which is induced by the domination map \tilde{d}_X is an epimorphism between two isomorphic Hopfian groups and so, it is an isomorphism for all $i \geq 2$. Thus by the Whitehead Theorem, $d_X : P \rightarrow X$ is a homotopy equivalence and so, X and P have the same homotopy type.

Case Two. $\pi_1(X) = 1$. First, let n be even. We know that

$$H_i(P) = \begin{cases} \mathbb{Z}, & i = 0 \\ \mathbb{Z}_2, & i \text{ is odd and } 0 < i < n \\ 0, & \text{otherwise.} \end{cases}$$

Since X is 1-connected, by Theorem 2.4 $\pi_2(X) \cong H_2(X)$. But $H_2(X) = 0$ and so, X is 2-connected. Now again by Theorem 2.4, $\pi_3(X) \cong H_3(X)$. If $H_3(X) \neq 0$, then $H_3(X) \cong \mathbb{Z}_2$. Hence $\pi_3(X) \cong \mathbb{Z}_2$. But this is a contradiction since $\pi_3(X)$ is a summand of $\pi_3(P) \cong \pi_3(S^n) = 0$. Therefore, $H_3(X) = 0$ and so, $\pi_3(X) = 0$. Then X is 3-connected and by Theorem 2.4, $\pi_4(X) \cong H_4(X)$. Therefore, X is 4-connected because $H_4(X)$ is a summand of $H_4(P) = 0$. Again, by Theorem 2.4 $\pi_5(X) \cong H_5(X)$. By continuing this process, we have $H_i(X) = 0$ for all $1 \leq i \leq n-1$. On the other hand, since $H_i(X)$ is a summand of $H_i(P)$, we have $H_i(X) = 0$ for all $i \geq n$. Thus X is a simply connected CW-complex with trivial homology groups and hence by the Whitehead Theorem, X has the homotopy type of $\{*\}$.

Second, let n be odd. We know that

$$H_i(P) = \begin{cases} \mathbb{Z}, & i = 0, n \\ \mathbb{Z}_2, & i \text{ is odd and } 0 < i < n \\ 0, & \text{otherwise.} \end{cases}$$

Similar to the previous argument, we obtain $H_i(X) = 0$ for all $i \neq n$. If $H_n(X) \neq 0$, then $H_n(X) \cong \mathbb{Z}$ and so X has the homotopy type of S^n . But S^n can not be homotopy dominated by P . Because if S^n is homotopy dominated by P , since $H^n(S^n; \mathbb{Z}_2) \neq 0$, then by Theorem 2.7, S^n has the homotopy type of P which is a contradiction. Therefore $H_n(X) = 0$ and so by the Whitehead Theorem, X has the homotopy type of $\{*\}$. \square

5. An upper bound for the capacity of a \mathbb{Z}_n -complex

In this section, we present an upper bound for the capacity of a \mathbb{Z}_n -complex, i.e., a 2-dimensional CW-complex with finite cyclic fundamental group \mathbb{Z}_n .

Definition 5.1. [7]. A π -complex is a 2-dimensional CW-complex with the fundamental group π . The set $HT(\pi)$ denotes the set of all π -complexes.

One can consider $HT(\pi)$ as a graph whose edges connect the type of each π -complex X to the type of its sum $X \vee S^2$ with the 2-sphere S^2 . These graphs are actually trees; they clearly contain no circuits, and they are connected because any two π -complexes have the same type once each is summed with an appropriate number of copies of the 2-sphere S^2 (see [7]).

Definition 5.2. [7]. A root is the homotopy type of a 2-dimensional CW-complex that does not admit a factorization involving an S^2 summand.

Definition 5.3. [7]. A junction is the homotopy type of a 2-dimensional CW-complex that admits two or more inequivalent factorizations involving an S^2 summand

Indeed, the roots generate the rest of the types in the tree $HT(\pi)$ under the operation of forming sum with S^2 and the junctions determine the shape of the tree (see [7]).

Lemma 5.4. [2, p. 78]. *The tree $HT(\mathbb{Z}_n)$ has exactly one root given by the pseudo projective plane $\mathbb{P}_n = S^1 \cup_f e^2$ which is obtained by attaching a 2-cell e^2 to S^1 via the map $f : S^1 \rightarrow S^1$ of degree n .*

Lemma 5.5. [21, Complement, p. 64]. *If X satisfies \mathcal{D}_2 , it is equivalent to a 3-dimensional CW-complex.*

Lemma 5.6. [6, Corollary 2, p. 412]. *If X is an $(n + 1)$ -complex dominated by an n -complex, then there exists a wedge of k copies of n -spheres W , where k is equal to the number of $(n + 1)$ -cells in X , such that $X \vee W \simeq X^{(n)}$, the n -skeleton of X .*

Lemma 5.7. [7]. *Let X be a \mathbb{Z}_n -complex. Then X has the homotopy type of the wedge $\mathbb{P}_n \vee S^2 \vee \dots \vee S^2$ of the pseudo projective plane \mathbb{P}_q and rank $H_2(X)$ copies of 2-sphere S^2 .*

Lemma 5.8. [7]. *The following are equivalent statements for a finitely presented group π .*

- (1) *The tree $HT(\pi)$ of homotopy types of π -complexes has a single root.*
- (2) *For π -complexes, there is a cancellation law for S^2 -summands, i.e.,*

$$X \vee S^2 \simeq Y \vee S^2 \quad \text{implies} \quad X \simeq Y.$$

We recall the following old problem concerning CW complexes.

If X is a CW-complex homotopy dominated by an n -dimensional CW-complex, then is X homotopy equivalent to an n -dimensional CW-complex?

C.T.C. Wall in [21] showed that the answer is yes if $n > 2$. The Stallings-Swan Theorem [20] answers the problem affirmatively for $n = 1$. But the answer for the case $n = 2$ is still unknown. J.M. Cohen in [6] showed that if X is dominated by a 2-dimensional complex, then there is a wedge of 2-spheres W such that $X \vee W$ is of the homotopy type of a 2-dimensional complex (for more details, see [6]). In the next theorem, we give a positive answer to the above question for \mathbb{Z}_n -complexes. In addition, we determine a space which is homotopy dominated by a \mathbb{Z}_n -complex up to homotopy equivalent and using this result, we present an upper bound for the capacity of a \mathbb{Z}_n -complex.

Theorem 5.9. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_m^{\alpha_m}$ where p_i 's are mutually distinct primes and α_i 's are positive integers. Then every space homotopy dominated by a \mathbb{Z}_n -complex has the homotopy type of a \mathbb{Z}_m -complex where $m = p_{i_1}^{\alpha_{i_1}} \dots p_{i_j}^{\alpha_{i_j}}$ for $i_1, \dots, i_j \in \{1, \dots, m\}$.*

Proof. Suppose P is a \mathbb{Z}_n -complex and X is homotopy dominated by P . By Lemma 5.5, we can suppose that X is a 3-dimensional complex. It is easy to see that $\pi_1(X)$ is isomorphic to a direct summand of $\pi_1(P)$. So we can suppose that $\pi_1(X) = \mathbb{Z}_m$ where $m = p_{i_1}^{\alpha_{i_1}} \cdots p_{i_j}^{\alpha_{i_j}}$ for $i_1, \dots, i_j \in \{1, \dots, m\}$. By Lemma 5.6, there exists a wedge of k copies of 2-spheres W such that $X \vee W \simeq X^{(2)}$. Since $\pi_1(X^{(2)}) = \pi_1(X) = \mathbb{Z}_m$, so by Lemma 5.7 we have

$$X \vee W \simeq X^{(2)} \simeq \mathbb{P}_m \vee S^2 \vee \cdots \vee S^2$$

of rank $H_2(X^{(2)})$ copies of the 2-sphere S^2 .

By the hypothesis, X is homotopy dominated by P which is a finite 2-dimensional polyhedron. Therefore, $H_2(X)$ is isomorphic to a subgroup of the free abelian group $H_2(P)$. Then $H_2(X)$ is also a free abelian group of finite rank. On the other hand, since $X \vee W \simeq X^{(2)}$, $H_2(X) \oplus H_2(W) \cong H_2(X^{(2)})$. Hence, we obtain

$$\text{rank } H_2(X) = \text{rank } H_2(X^{(2)}) - k.$$

Now by Lemmas 5.4 and 5.8, $X \simeq \mathbb{P}_m \vee S^2 \vee \cdots \vee S^2$ with k copies of S^2 , where k is the rank of $H_2(X)$. \square

Corollary 5.10. *Let P be a \mathbb{Z}_n -complex where $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_m^{\alpha_m}$ for mutually distinct primes p_i and positive integers α_i . Then $C(P) \leq 2^m \times (\text{rank } H_2(P) + 1)$.*

Proof. By Theorem 5.9, every space homotopy dominated by P has the homotopy type of $\mathbb{P}_m \vee S^2 \vee \cdots \vee S^2$, where $m = p_{i_1}^{\alpha_{i_1}} \cdots p_{i_j}^{\alpha_{i_j}}$ for $i_1, \dots, i_j \in \{1, \dots, m\}$, with k copies of the 2-sphere S^2 for every $0 \leq k \leq \text{rank } H_2(P)$. Thus the proof is complete. \square

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