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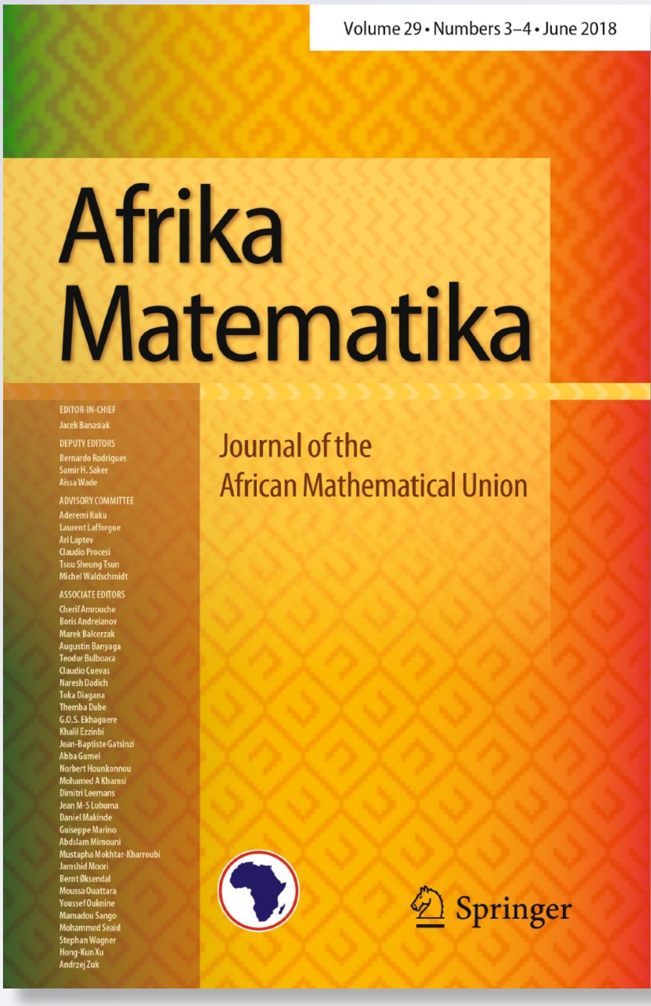

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On the structure of isoclinism classes of the non-commuting graphs

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Abstract In this paper, we introduce the new concept isoclinism of two non-commuting graphs. We describe it with this hope to determine the properties of the graph with large number of vertices and edges more easier by use of its smaller correspondence graph in its isoclinic class. In 1939, Hekster classified the groups by n-isoclinism which was weaker than isomorphism, where n is a positive integer. The abelian groups are in the same class by group-isoclinism, although their intrinsic properties are not the same. The notion of isoclinic groups is the inspiration to define the isoclinism of two graphs. The isoclinism of two graphs is a pair of significant special isomorphism between the quotient graphs of the given graphs. We observe that all complete 3-partite non-commuting graphs are in the same isoclinic class and the non-commuting graph associated to the dihedral group of order 8, D_8 is the representative of this class.

Keywords Isoclinism · Non-commuting graph · Quotient graph

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1 Introduction

In order to classify solvable groups, Hall introduced the concept of isoclinism [3]. He extended it to the notion of $\mathcal{V}$-isologism with respect to a given variety $\mathcal{V}$. In 1939, Hekster considered the variety of nilpotent groups of class at most n and arose the concept of n-isoclinism. From the standpoint of group theory, isomorphic groups have the same properties and need not be distinguished. Hekster classified the groups by a new notion which was weaker than isomorphism. It is a more general concept which divides the class of all groups into families (see [4] for more details). Let us recall the definition of two n-isoclinic groups.

Definition 1.1 Let G_1 and G_2 be two groups. Then the pair (φ, ψ) is called an n-isoclinism from G_1 to G_2 whenever,

(i) φ is an isomorphism from $\dfrac{G_1}{Z_n(G_1)}$ to $\dfrac{G_2}{Z_n(G_2)}$, where $Z_n(G_1)$ and $Z_n(G_2)$ are the n-th term of the upper central series of G_1 and G_2, respectively.
(ii) ψ is an isomorphism from $\gamma_{n+1}(G_1)$ to $\gamma_{n+1}(G_2)$, with the law $[g_{11}, \ldots, g_{1n}, g_{1(n+1)}] \mapsto [g_{21}, \ldots, g_{2n}, g_{2(n+1)}]$ in which $g_{2j} \in \varphi(g_{1j} Z_n(G_1))$, $1 \leq j \leq n+1$. If there is such a pair (φ, ψ) with the above properties, then we say that G_1 and G_2 are n-isoclinic and denoted by $G_1 \widetilde{_n} G_2$.

The essence of group n-isoclinism was the key to define n-isoclinism of the graphs, in the third section. For this aim we consider a pair of significant special isomorphism between the quotient graphs of the given graphs. If there is such as pair, then we call two graphs are n-isoclinic.

Let G be a group. The non-commuting graph associated to the group G was considered by Paul Erdős for the first time. It is a graph with vertex set whole elements of the group G and two vertices are adjacent if and only if they do not commute. In this paper we consider the induced subgraph of the non-commuting graph, with non-central elements of G as the vertex set, unless mentioned otherwise. It is denoted by Γ_G. There are large amount of research on the non-commuting graph one may see [1,11].

The distance between two vertices u and v, denoted by $d(u, v)$, is the length of a shortest path between u and v in the graph. For any graph Γ and positive integer n, the n-th derived graph of Γ is a graph with the same vertices as Γ, two vertices in the derived graph being connected by an edge if and only if the distance between them is exactly n in the main graph. Let us denote it by $\Gamma^{(n)}$. The 1st derived graph is the graph itself and we denote $\Gamma^{(2)}$ by Γ'. The center of the graph Γ is the set of all vertices z for which the greatest distance $d(z, x)$ to other vertices x is minimal. We denote this set of vertices by $Z(\Gamma)$. Note that two phrases central vertices of the graph and central elements of the group are different concepts.

In Sect. 2, we become familiar with some properties of central vertices of a connected graph in particular the non-commuting graph. We observe that the existence of a central vertex v for the non-commuting graph associated to the group G implies that the center of the group, $Z(G) = 1$ and the centralizer of v, $C_G(v) = \{1, v\}$. Moreover, an ascending chain of induced subgraphs is presented.

In Sect. 3, we present the notion of isoclinism of two non-commuting graphs. For this end, we require to define an appropriate quotients. According to the definition of the quotient graphs we need a suitable equivalent relation over the vertices of the graph. We consider two equivalence relation R and T_G over the vertices of the non-commuting graph associated to the group G. The twin equivalence relation R was studied vastly before (see [6,9]). Further, the equivalence relation T_G is the belonging of vertices in the same central factor of the group G. In the third section after defining the non-commuting graph isoclinism, we observe that two isoclinic groups have isoclinic non-commuting graphs. Hence, if one consider class of all non-commuting graphs, then it can be classified into some families via graph isoclinism. We perceive that each class contains a graph as a representative of the whole family with the least size and all other graphs in this pack has similar properties some how. We call this representative a stem. Finally, we prove that all complete 3-partite non-commuting graphs are in the same isoclinic class and we can consider $\Gamma_{D_8} \cong \Gamma_{Q_8}$ as the stem of this family, where D_8 and Q_8 are dihedral group and quaternion group of order 8, respectively. Furthermore, we have two distinct isoclinic classes of the graphs for all complete 4-partite non-commuting graphs. By similar method, we deduce that there are 4, 3 and 2 distinct classes of isoclinism for complete 5-partite, 6-partite and 8-partite non-commuting graphs, respectively.

The reader can use the references [2,7] for notations in graph theory and group theory.

2 Preliminaries

Let us start with the following result about the central vertices of a connected graph.

Lemma 2.1 *Let Γ be a connected graph. Then $Z(\Gamma)$ is the set of all vertices that join to all other vertices.*

Proof Let $z \in Z(\Gamma)$. Since Γ is a connected graph, z is adjacent to at least one another vertex. Thus by definition of a central vertex it is adjacent to all other vertices. □

Let us denote the set of all vertices of the connected graph Γ for which their distances to other vertices is at most n, by $Z_n(\Gamma)$. By diameter vertex of the non-commuting graph we mean a vertex which is not central vertex. Therefore a diameter vertex of the non-commuting graph is a vertex for which there is at least a vertex which is not adjacent to. Since Abdollahi et al. [1] proved that the diameter of any non-commuting graph is 2, the central vertices are adjacent to all other vertices of non-commuting graph. Now if $v \in Z(\Gamma_G)$ and $1 \neq z \in Z(G)$, then vz is a vertices which is not adjacent to v which is a contradiction so $Z(G) = 1$ and we have the following result.

Lemma 2.2 *Let Γ_G be the non-commuting graph associated to the non-abelian group G. If $v \in Z(\Gamma_G)$, then $Z(G) = 1$ and consequently $Z(\Gamma_G) = \{v \in V(\Gamma_G) : C_G(v) = \{1, v\}\}$.*

The induced subgraph of Γ_G with vertex set $Z(\Gamma_G)$ is a complete graph $K_{|Z(\Gamma_G)|}$. The 2nd derived graph of the non-commuting graph is the graph with vertex set $G \backslash Z(G)$ such that two vertices are adjacent if their distance is exactly 2. Thus those vertices of the non-commuting graph are adjacent in $\Gamma_G^{(2)} = \Gamma'_G$ which do commute. Let us denote the induced subgraph of Γ over the vertex set $Y \subseteq V(\Gamma)$ by Γ_Y. If Γ is a graph such that $\mathrm{diam}(\Gamma) = n$, then $\Gamma_{Z_n(\Gamma)} = \Gamma$. Consequently, for every graph Γ, there exists an ascending chain of induced subgraph,

$$\Gamma_{Z(\Gamma)} \subseteq \Gamma_{Z_2(\Gamma)} \subseteq \ldots \subseteq \Gamma_{Z_i(\Gamma)} \ldots .$$

Let us call it the central series of Γ. It is clear that for all connected graphs, central series terminate to the graph itself after finite steps. Therefore, if $\mathrm{diam}(\Gamma) \leq n$, then we call Γ a connected graph of connectivity class at most n.

Obviously, if Γ is a connected graph of connectivity class at most n, then $\Gamma^{(n+1)}$ is an empty graph. The non-commuting graph associated to a group G is a connected graph of connectivity class 2.

Twin vertices in graphs correspond to vertices sharing the same neighborhood. For each vertex $v \in V(\Gamma)$, let $N_i(v) = \{u \in V(\Gamma) : d(u, v) = i\}$. Two distinct vertices u and v are n-twins, if $N_n(v)\backslash\{u\} = N_n(u)\backslash\{v\}$. We define $u \equiv_{R_n} v$ if and only if $u = v$ or u and v are n-twins (see [6]). If two vertices are 1-twin, then simply we call them twin. This relation is an equivalence relation. The equivalence class of v and the twin graph of Γ are denoted by v^* and Γ^*, respectively.

A quotient graph Σ of a graph Γ is a graph whose vertices are blocks of a partition of the vertices of Γ, where block B is adjacent to block C if some vertex in B is adjacent to some vertex in C with respect to the edge set of Γ. In other words, if Γ has edge set $E(\Gamma)$ and vertex set $V(\Gamma)$ and λ is the equivalence relation induced by the partition, then the quotient graph has vertex set $V(\Gamma)/\lambda$ and edge set $\{([u]_\lambda, [v]_\lambda) | (u, v) \in E(\Gamma)\}$.

The twin graph Γ^* is a graph with vertex set $V(\Gamma^*) = \{v^* : v \in V(\Gamma)\}$, where $u^*v^* \in E(\Gamma^*)$ if and only if $uv \in E(\Gamma)$. Twin vertices is one of the known forms of symmetries in the graphs. Moreover, Γ^* can be considered as the quotient graph of Γ with respect to the equivalence relation R, let us denote it by Γ/R or Γ^*. For the details about twin non-commuting graph one can refer to [9]. The twin vertices of a vertex $v \in \Gamma_G$ is defined by $v^* = \{s \in G\backslash Z(G) : C_G(s) \cup \{v\} = C_G(v) \cup \{s\}\}$.

Proposition 2.3 *Let Γ_G be the non-commuting graph.*

(i) *All central vertices of the non-commuting graph are twins. In other words, all of them made a vertex in the twin non-commuting graph.*
(ii) *The central vertices of the non-commuting graph Γ_G are isolated vertices of $\Gamma^{'}_G$.*

Proof The first part follows by the above argument and Lemma 2.2 and the second part is clear by the definition of central vertices. □

Theorem 2.4 *If $\Gamma^{'}_G \cong \Gamma^{'}_H$, then Γ_X is isomorphic to the induced subgraph of Γ_Y, where X, Y are H or G.*

Proof Since $\Gamma^{'}_G \cong \Gamma^{'}_H$, the number of isolated vertices of $\Gamma^{'}_G$ and $\Gamma^{'}_H$, i.e. the number of central vertices of the non-commuting graph of Γ_H and Γ_G are equal. The central vertices form a complete graph in the non-commuting graphs. The other vertices are not connected directly. The inductor of the connection of them is at lease one central vertex. Consequently the diversity of Γ_G and Γ_H occurs in the difference of the number of these central vertices which are used in joining diameter vertices. Hence the assertion is clear. □

The tensor product $\Gamma \times \Delta$ of graphs Γ and Δ is a graph such that the vertex set of $\Gamma \times \Delta$ is the Cartesian product $V(\Gamma) \times V(\Delta)$ and any two vertices (u_1, u_2) and (v_1, v_2) are adjacent in $\Gamma \times \Delta$ if and only if u_i is adjacent with v_i for $i = 1, 2$.

Theorem 2.5 *Let H be the subgroup of G and Γ_H the induced subgraph of the non-commuting graph Γ_G. Then $\frac{\Gamma_H \times K_{|Z(\Gamma_G)|}}{\lambda} \cong \Gamma_H$, where λ is the equivalence relation on the vertices of the graph $\Gamma_H \times K_{|Z(\Gamma_G)|}$ such that $(u_1, u_2) \ \lambda \ (v_1, v_2)$ if and only if $u_1 = v_1$.*

Proof Define the map $\theta : V\left(\frac{\Gamma_H \times K_{|Z(\Gamma_G)|}}{\lambda}\right) \longrightarrow V(\Gamma_H)$, where $\theta([(u, v)]_\lambda) = u$. This map is a bijection and since the adjacency of vertices of the graph $\frac{\Gamma_H \times K_{|Z(\Gamma_G)|}}{\lambda}$ depends on the adjacency of vertices of the graph Γ_H, θ preserves edges. □

Recall that the intersection of two graphs Γ and Δ is the graph $\Gamma \cap \Delta$ with vertex set $V(\Gamma) \cap V(\Delta)$ and edge set $E(\Gamma) \cap E(\Delta)$. The following corollary is the direct result of Theorem 2.5.

Corollary 2.6 *If* Γ_H *is induced subgraph of* Γ_G *and* $\Gamma_G \cong \dfrac{\Gamma_H \times K_{|Z(\Gamma_G)|}}{\lambda}$, *then*

(i) $\Gamma'_G \cong \Gamma'_H$.
(ii) $\Gamma_H \cap K_{|Z(\Gamma_G)|} \cong K_{|Z(\Gamma_H)|}$.
(iii) $\Gamma'_G \cap K_{|Z(\Gamma_G)|} \cong \Gamma'_H \cap K_{|Z(\Gamma_H)|}$.

In general $\Gamma_H \cap K_{|Z(\Gamma_G)|}$ is isomorphic with induced subgraph of $K_{|Z(\Gamma_H)|}$.

3 Graph isoclinism

Shortly after the notion of isoclinism of the groups was defined, Hall generalized this to what he called $\mathcal{V}$-isologiam, where $\mathcal{V}$ is some variety of groups. Isologism is so to speak isoclinism with respect to a certain variety. In this way for each variety an equivalence relation on the class of all groups arises. The larger the variety, the weaker this equivalence relation is (see [5] for more details). Let us start with the following definition.

Definition 3.1 Let Γ_G and Γ_H be two non-commuting graphs. The pair of graph isomorphisms (α, β) which are presented as follows,

(i) $\alpha : \dfrac{\Gamma'_G}{T_G} \longrightarrow \dfrac{\Gamma'_H}{T_H}$,

(ii) $\beta : \dfrac{\frac{\Gamma_G}{T_G}}{R} \longrightarrow \dfrac{\frac{\Gamma_H}{T_H}}{R}$,

is called an isoclinism between Γ_G and Γ_H, where T_X is an equivalence relation on the vertices of the graph Γ_X which is partition the vertices by this rule

(a) If $|Z(G)| \neq |Z(H)|$, then $x\ T_X\ y \Longleftrightarrow x, y \in Z(X)r$, for some $r \in X$,
(b) If $|Z(G)| = |Z(H)|$, then $x\ T_X\ y \Longleftrightarrow x = y$,

for $x, y \in V(\Gamma_X)$ and $X = G$ or H. Moreover, R is the twin equivalence relation. If there is such a pair (α, β) for two non-commuting graphs, then we call two graphs are isoclinic. We denote two isoclinic graphs by $\Gamma_G \sim \Gamma_H$.

Let $Z_n(X)$ be the n-th term of upper central series for the group X and T^n_X be the equivalent relation which is defined by $x\ T^n_X\ y \Longleftrightarrow x, y \in Z_n(X)r$, for some r in the group X and $n > 1$ a positive integer. In the Definition 3.1, if we replace the equivalent relations T_X and R by T^n_X and R_n, then Γ_G and Γ_H are called n-isoclinic and is denoted by $\Gamma_G \ \widetilde{n}\ \Gamma_H$.

From the definition it is straightforward that two non-commuting graphs associating to two groups whose order of centers are equal are isoclinic, if their derived and twin graphs are isomorphic.

With out any ambiguity, the equivalence relation T_X and R can optional be chosen. For instance it may depend on the condition under which two vertices join in an arbitrary graph. Thus by imposing different equivalence relations we obtain distinct equivalent classes and a new concept similar to the $\mathcal{V}$-isologism in the group theory.

In the following two results consider the same notations as in the Definition 3.1.

Lemma 3.2 *Let Γ_G be the non-commuting graph associated to the group G.*

(i) *If v_1, v_2 are two twin vertices, then $[v_1^*]_{T_G} = [v_2^*]_{T_G}$.*
(ii) *Let $[v_1]_{T_G} = [v_2]_{T_G}$. Then v_1 and v_2 are twins.*

Proof (i) Suppose $v_1^* = v_2^*$. By definition $v_1^* = \{w \in G\backslash Z(G) : C_G(v_1) \cup \{w\} = C_G(w) \cup \{v_1\}\} = v_2^*$. Now we can classify this set by the equivalence relation T_G, as follows $v_1^* = \{[w_i] : [w_i] = \{x_i z_j : z_j \in Z(G)\}\}$, where $G = \cup_{i \in I} Z(G)x_i$, $x_i \in G$. Hence the assertion is clear.
(ii) By the hypothesis $C_G(v_1) = C_G(v_2)$. □

In the text the notation $[v]_{T_G}$ is the representation of the coset $vZ(G)$ and in the graph $\frac{\Gamma_G}{T_G}$ we just consider one element of this set as a vertex. Thus it is enough to consider the element v instead of $[v]_{T_G}$. Moreover, $C_G([v]_{T_G})$ means the centralizer of the element v in the group G. Example 3.3 is useful to clarify the quotients of the graph.

Example 3.3 Let $G = S_3 \times \mathbb{Z}_2$, where S_3 and $\mathbb{Z}_2$ are symmetric group of order 6 and cyclic group of order 2, respectively. The vertices of the non-commuting graph $\Gamma_{S_3 \times \mathbb{Z}_2}$ are non-central elements of the group $S_3 \times \mathbb{Z}_2$. Moreover,

$$V\left(\frac{\Gamma_{S_3\times\mathbb{Z}_2}}{T_G}\right) = \{[((1\ 2), 0)]_{T_G}, [((1\ 3), 0)]_{T_G}, [((2\ 3), 0)]_{T_G}, \\ [((1\ 2\ 3), 0)]_{T_G}, [((1\ 3\ 2), 0)]_{T_G}\}$$

and

$$V\left(\frac{\frac{\Gamma_{S_3\times\mathbb{Z}_2}}{T_G}}{R}\right) = \{[((1\ 2), 0)]^*_{T_G}, [((1\ 2\ 3), 0)]^*_{T_G}\}.$$

Although, $V\left(\frac{\Gamma_{S_3\times\mathbb{Z}_2}}{R}\right) = \{((1\,2), 0)^*, ((1\,3), 0)^*, ((2\,3), 0)^*, ((1\,2\,3), 0)^*\} = V\left(\frac{\frac{\Gamma_{S_3\times\mathbb{Z}_2}}{R}}{T_G}\right)$.

Example 3.3 implies that the isomorphism $\frac{\frac{\Gamma_G}{T_G}}{R} \cong \frac{\frac{\Gamma_G}{R}}{T_G}$ does not hold in general.

Lemma 3.4 *Suppose (φ, ψ) is the isoclinism between two groups G and H, where $\varphi : \frac{G}{Z(G)} \longrightarrow \frac{H}{Z(H)}$ such that $\varphi(g_i Z(G)) = h_i Z(H)$ and the isomorphism $\psi : G' \longrightarrow H'$ which inspire by φ, $1 \le i \le [G : Z(G)]$. Then*

(i) *For any diameter vertex of the graph Γ_G, there is a diameter vertex for the non-commuting graph of Γ_H.*
(ii) *The equality $C_G(g_i) = C_G(g_j)$ is equivalent to $C_H(h_i) = C_H(h_j)$.*
(iii) *$\frac{C_G(g_i)}{Z(G)}$ and $\frac{C_H(h_i)}{Z(H)}$ are isomorphic.*

Proof (i) Suppose x is a diameter vertex of Γ_G. Therefore by definition of the diameter vertex of the non-commuting graph, there is at least a vertex y such that $[x, y] = 1$. Since $x, y \in G$ so $x \in g_1 Z(G)$ and $y \in g_2 Z(G)$. Thus $\psi([x, y]) = \psi([g_1, g_2]) = [h_1, h_2] = 1$. This fact implies that $h_1 \in V(\Gamma_H)$ exists such that it is a diameter vertex.

(ii) It is clear that,

$$\begin{aligned} x \in C_H(h_i) &\Longleftrightarrow [x, h_i] = 1 \\ &\Longleftrightarrow 1 = [x, h_i] = \psi([y, g_i]) \\ &\Longleftrightarrow [y, g_i] = 1 \\ &\Longleftrightarrow y \in C_G(g_i) = C_G(g_j) \\ &\Longleftrightarrow 1 = \psi([y, g_j]) = [x, h_j] \\ &\Longleftrightarrow x \in C_H(h_j), \end{aligned}$$

where $y \in \varphi(xZ(G))$ and $h_l \in \varphi(g_l Z(G)), l = i, j$. The third part leave for the readers. □

It is clear that two isomorphic groups have isomorphic non-commuting graphs. In the following theorem we observe that two isoclinic groups are associated to two isoclinic non-commuting graphs.

Theorem 3.5 *Let G and H be two isoclinic groups. Then the non-commuting graphs associating to them are isoclinic.*

Proof If $|Z(G)| = |Z(H)|$, then $\Gamma_G \cong \Gamma_H$, by [10, Theorem 4.5]. Suppose $|Z(G)| \neq |Z(H)|$ and the pair (φ, ψ) is the isoclinism between the groups G, H. Define the map,

$$\begin{aligned} \alpha : V\left(\frac{\Gamma'_G}{T_G}\right) &\longrightarrow V\left(\frac{\Gamma'_H}{T_H}\right) \\ [t]_{T_G} &\longmapsto [s]_{T_H}, \end{aligned}$$

where $\varphi(tZ(G)) = sZ(H)$ and t, s are diameter vertices of Γ_G and Γ_H, respectively. The well-definedness of α follows by well-definedness of φ. Assume $[t_1]_{T_G} = [t_2]_{T_G}$. By the equality of these two classes we deduce that $t_1, t_2 \in xZ(G)$. Therefore $\varphi(t_1 Z(G)) = \varphi(t_2 Z(G)) = sZ(H)$ and $\alpha([t_1]_{T_G}) = \alpha([t_2]_{T_G})$. Thus this argument and the first part of Lemma 3.4 implies that the map α is well-defined. Moreover, α is one to one. Since the equality $\alpha([t_1]_{T_G}) = \alpha([t_2]_{T_G})$ implies $[s_1]_{T_H} = [s_2]_{T_H}$. Clearly $s_1, s_2 \in h_i Z(H)$. Thus $\varphi^{-1}(h_i Z(H)) = g_i Z(G)$ and so $t_1, t_2 \in g_i Z(G)$ which implies $[t_1]_{T_G} = [t_2]_{T_G}$. If $[k]_{T_H} \in V(\frac{\Gamma'_H}{T_H})$, then isomorphisms φ and ψ guarantee the existence of $[l]_{T_G} \in V(\frac{\Gamma'_G}{T_G})$ which maps to $[k]_{T_H}$ by α, which shows that α is surjective. Finally the isomorphisms φ and ψ cause that the bijection α preserves the edges.
Now, define the map

$$\beta : V\left(\frac{\frac{\Gamma_G}{T_G}}{R}\right) \longrightarrow V\left(\frac{\frac{\Gamma_H}{T_H}}{R}\right)$$

$$\left([g_i]_{T_G}\right)^* \longmapsto \left([h_i]_{T_H}\right)^*,$$

where $\varphi(g_i Z(G)) = h_i Z(H)$. If $([g_i]_{T_G})^* = ([g_j]_{T_G})^*$, then $g_j \in g_i^*$ and so $C_G(g_i) \cup \{g_j\} = C_G(g_j) \cup \{g_i\}$ (see [9]). We consider two cases

(a) If $C_G(g_i) = C_G(g_j)$, then by the second part of Lemma 3.4 we have $C_H(h_i) = C_H(h_j)$ which implies that $h_i^* = h_j^*$.

(b) If $[g_i, g_j] \neq 1$, then by [9, Remark 2.] $C_G(g_l) = \{1, g_l\}$, $l = i, j$ which implies $Z(G) = 1$. It is not possible that $|Z(G)| > |Z(H)|$. By third part of Lemma 3.4 we have $|C_H(h_i)| = |C_H(h_j)|$. Moreover, $C_H(h_l) = \langle h_l, Z(H)\rangle$ which implies that $|C_{\frac{H}{Z(H)}}(h_i Z(H))| = |C_{\frac{H}{Z(H)}}(h_j Z(H))| = 2$. Hence $([h_i]_{T_H})^* = ([h_j]_{T_H})^*$, since in the graph $\frac{\Gamma_H}{T_H}$ we consider just one element $[h_i]_{T_H}$ from $h_i Z(H)$ as a vertex.

This shows β is well-define and similar argument deduce the injectiveness of β. The map β is a graph isomorphism. □

Example 3.6 In this example we present two isoclinic groups and their isoclinic non-commuting graphs.

(i) D_8 and Q_8 are isoclinic groups, which their centers are of equal order. Their non-commuting graphs are isomorphic so they are isoclinic. Their quotient derived graphs are isomorphic to a graph with three isolated edges and their twin graphs are isomorphic to K_3.

(ii) $S_3 \times \mathbb{Z}_2$ and S_3 are isoclinic groups. Clearly, the center of these two groups does not have equal order, but their associated non-commuting graphs are isoclinic. $V\left(\frac{\Gamma'_{S_3\times\mathbb{Z}_2}}{T_{S_3\times\mathbb{Z}_2}}\right) = \{[((1\ 2\ 3), 0)]_{S_3\times\mathbb{Z}_2}, [((1\ 3\ 2), 0)]_{S_3\times\mathbb{Z}_2}, [((1\ 2), 0)]_{S_3\times\mathbb{Z}_2}, [((1\,3), 0)]_{S_3\times\mathbb{Z}_2}, [((2\,3), 0)]_{S_3\times\mathbb{Z}_2}\}$ and $V\left(\frac{\frac{S_3\times\mathbb{Z}_2}{T_{S_3\times\mathbb{Z}_2}}}{R}\right) = \{[((1\,2), 0)]^*_{S_3\times\mathbb{Z}_2}, [((1\,2\,3), 0)]^*_{S_3\times\mathbb{Z}_2}\}$. Obviously, the non-commuting graph associated to S_3 and $S_3 \times \mathbb{Z}_2$ are isoclinic.

By the second part of Example 3.6 we deduce that the isomorphism $\frac{\frac{\Gamma_G}{T_G}}{R} \cong \frac{\frac{\Gamma_H}{T_H}}{R}$ does not imply $\frac{\Gamma_G}{R} \cong \frac{\Gamma_H}{R}$, generally.

If $H \leq G$ and $N \trianglelefteq G$, then $H \sim HZ(G)$ and $G/N \sim G/N \cap G'$ by [4, Lemma 3.5]. Thus by Theorem 3.5 the non-commuting graph associated to them are isoclinic.

By Theorem 3.5 and Example 3.6 we observe that the class of all non-commuting graphs can be classify to isoclinic graphs such that each class has a stem as representative. Furthermore, we guess that the converse of the Theorem 3.5 does not hold generally. Studying of the structure of the groups whose non-commuting graphs are isoclinic is not easy. As it was mentioned, graph isoclinism is weaker than graph isomorphism and checking the structure of the groups whose non-commuting graphs are isomorphic is the start of difficult research.

In [8], the non-centralizer graph of the group G was introduced which helps to recognize big class of the non-commuting graphs. Let us recall its definition, since it is important in the sequel. Note that in the main definition the vertex set of the graph is whole elements of G, here we reduced it to $G\backslash Z(G)$ as required.

Definition 3.7 We construct a graph whose vertices are the non-central elements of the group G and whose edges are obtained by joining any two vertices x and y whenever $C_G(x) \neq C_G(y)$. We call this graph the non-centralizer graph of G, and we denote it by Υ_G.

Let us recall that a group G all of whose centralizers are abelian is called an AC-group. The non-commuting graph and the non-centralizer graph of a given group are not isomorphic in general. In [8, Theorem 2.11] we observed that being AC-group cause that the non-centralizer graph of G coincides to its non-commuting graph.

Lemma 3.8 Γ_G *is a non-commuting graph isomorphic to the complete* k*-partite graph if and only if* G *is an AC-group.*

Proof Assume t_1, t_2 are vertices of the graph such that $t_1, t_2 \in C_G(x)$, where $x \in V(\Gamma_G)$. Since they commute with x and the graph is complete k-partite, we deduce that they are in the same part of the graph. Hence $[t_1, t_2] = 1$ and the result is clear. □

Theorem 3.9 *Every two complete 3-partite non-commuting graphs of finite groups are isoclinic graphs.*

Proof Let us consider two complete 3-partite non-commuting graph K_{n_1,n_2,n_3} and K_{m_1,m_2,m_3}, where n_i, m_i are positive integer and $i = 1, 2, 3$. By Lemma 3.8 $\Gamma_G \cong K_{n_1,n_2,n_3}$ and $\Gamma_H \cong K_{m_1,m_2,m_3}$, where G and H are AC-groups. By [8, Proposition 2.18] $G/Z(G) \cong H/Z(H) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Therefore, $G \sim H$ by [4, Proposition 3.7]. Hence Theorem 3.5 complete the proof. □

By similar argument, we conclude the following result.

Corollary 3.10 (i) *There are 2 class of isoclinic graphs for the complete 4-partite non-commuting graphs of finite groups.*
(ii) *There are 4 class of isoclinic graphs for the complete 5-partite non-commuting graphs of finite groups.*
(iii) *There are 3 class of isoclinic graphs for the complete 6-partite non-commuting graphs of finite groups.*
(iv) *There are 3 class of isoclinic graphs for the complete 7-partite non-commuting graphs of finite groups.*
(v) *There are 2 class of isoclinic graphs for the complete 8-partite non-commuting graphs of finite groups.*

Proof Let us explain the first part, the rest will prove similarly. Suppose K_{n_1,n_2,n_3,n_4} and K_{m_1,m_2,m_3,m_4} are two complete 4-partite non-commuting graph, where n_i, m_i are positive integer and $i = 1, 2, 3, 4$. Therefore they are associated with AC-groups G and H. By [8, Proposition 2.18] $X/Z(X) \cong S_3$ or $\mathbb{Z}_3 \times \mathbb{Z}_3$, where $X = G$ or H. If $G/Z(G) \cong H/Z(H)$, then $G \sim H$ by [4, Proposition 3.7] and so K_{n_1,n_2,n_3,n_4} and K_{m_1,m_2,m_3,m_4} are isoclinic by Theorem 3.5. Moreover, if $G/Z(G) \cong S_3$ and $H/Z(H) \cong \mathbb{Z}_3 \times \mathbb{Z}_3$, then $|V(\frac{\Gamma'_G}{T_G})| \neq |V(\frac{\Gamma'_H}{T_H})|$ and so K_{n_1,n_2,n_3,n_4} and K_{m_1,m_2,m_3,m_4} are not isoclinic. □

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