On Free Ends of Groups

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Abstract

In this talk, we introduce the ends of graphs, and then the ends of groups, which are the ends of Cayley graphs of groups. It is proven that the number of ends of group G, $e(G)$ does not depend on the Cayley graphs, and hence, it can be studied independently of generating set. We investigate free ends of graphs and group. We prove that a group G has free end if and only if it does not have more than 2 ends. As a consequence, we show that a free group has free end if and only if $G \cong \mathbb{Z}$.

Keywords: Graph-theoretical end, Cayley graph, Free end, Free group
AMS Mathematical Subject Classification [2010]: 20E06, 54d35

1 Introduction

In 1931, Freudenthal introduced the notion of ends for second countable Hausdorff spaces [5], and in particular for locally finite graphs [6]. For graphs that are not locally finite, graphs including some vertex with infinite degree, Freudenthal's definition still makes sense with the notion of topological ends for arbitrary graphs as 1-complexes [3].

One can consider the ends of a graph as the points at infinity to which its rays converge. More precisely, an end of a graph is an equivalence class of some rays. Note that in this talk all graphs are assumed to be locally finite, that is they have no vertex with infinite degree. We define a compact topological space $\overline{\Gamma}$ associated with a graph $\Gamma = (V, E, \Omega)$, where V is the set of all vertices, E is the set of all edges and Ω is the set of all of its ends.

The notion of end can be defined similarly for the Cayley graph of groups. It was proven that the number of ends of a group is constant and independent of the choice of Cayley graph. There exist theorems expressing relations on the structure of groups and the number of their ends, namely Stallings theorem stating a finitely generated group G has more than one end if and only if the group G admits a nontrivial decomposition as an amalgamated free product or an HNN extension over a finite subgroup.

In this talk, we study the end space of a finitely generated group, not depending on the Cayley graph and generating set, by Theorem 2.8. We introduce free ends of graphs and in particular free ends of groups,

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which are open in the end space of groups. We prove that group G has nontrivial free end if and only if it has exactly two distinct ends. Finally, as a consequence we show that group G has free end if and only if G has $\mathbb{Z}$ as a finite indexed subgroup.

2 Preliminaries

Ends of finitely generated groups were defined by Hopf [9], as the ends of Cayley graph of groups. First, we recall the definition of end of graphs.

2.1 Ends of Graphs

To define an end of graph Γ, first we must define rays, which make sense if Γ is necessarily infinite.

Definition 2.1. A ray is a 1-way infinite graph-theoretical path [4, p. 2645]. That is a non-empty graph $r = (V, E)$ such that

$$V = \{v_0, v_1, v_2, \ldots\}, \quad E = \{x_0x_1, x_1x_2, \ldots\},$$

where the x_i' are pairwise distinct. A ray in a graph Γ is a ray which is a subgraph of Γ (see [2, p. 6]).

Now we may define a relation on the set of all rays as follows: two rays r_1 and r_2 are equivalent if and only if no finite set F of vertices separates r_1 and r_2, that is the subrays of r_1 and r_2 in $\Gamma \setminus F$ belong to the same component. This relation is an equivalence relation on the set of all rays of Γ [3, p. 2].

Definition 2.2 ([4])**.** Every equivalence class of the latter relation is called an end of Γ. The set of all ends of Γ is denoted by $\Omega(\Gamma)$. For any subset $C \subseteq \Gamma$, we say that an end ω lives in C if every ray of ω has a subray in C.

Example 2.3. Consider the real line space $\mathbb{R}$ as a graph with the integers as vertices and open unit segments as the edges (see Figure 1). Rays in graph are of the form $n(n+1)(n+2)\ldots$ and $n(n-1)(n-2)\ldots$, where $n \in \mathbb{Z}$. Each one represents a class of rays, and hence, the graph has two distinct ends denoted by $+\infty$ and $-\infty$.

$$\cdots \longrightarrow -4 \longrightarrow -3 \longrightarrow -2 \longrightarrow -1 \longrightarrow 0 \longrightarrow 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \longrightarrow \cdots$$

Figure 1: The real line space $\mathbb{R}$ as a graph.

In 1964, some special ends were introduced and studied by Halin [8], such as free, thick and thin end.

Definition 2.4. An end ω of a graph Γ is called free if there is finite set of vertices F, such that F separates ω from the other ends of Γ.

If Γ is a graph with finitely many ends $\{\omega_1, \omega_2, \ldots, \omega_k\}$, then every end of Γ is free. To verify, note that each end is an equivalence class of rays not being separated by finite set of vertices. Thus, for two ends ω_1 and ω_i, $2 \leq i \leq k$, there exists finite set F_i of vertices, separating ω_1 from ω_i. If $F = \bigcup_{2 \leq i \leq k} F_i$, then F separates ω_1 from each other of ω_i's.

For graphs with infinitely many ends, Halin proved the following theorem.

Theorem 2.5 ([8])**.** *If Γ has infinitely many ends, then there exists an end that is not free.*

Let $\overline{\Gamma} = \Gamma \cup \Omega(\Gamma)$, the set of all points of Γ as a 1-complex together with the set of all the ends of Γ.

We define a topology on $\overline{\Gamma}$, and from now on by $\overline{\Gamma}$ we mean the topological space equipped with the topology introduced in Theorem 2.6.

Theorem 2.6 ([4])**.** *Let Γ be a locally finite graph. The topology generated by the sets as follows, makes $\overline{\Gamma}$ a compact space, and $\Omega(\Gamma)$ a totally disconnected space with the subspace topology.*

1. *Open sets of G as a 1-complex,*
2. *and the sets $\widehat{C}(F,\omega)$ defined for every end ω and every finite set F of vertices which consists of: 1) $C(F,\omega) =: C$, the unique component of $\Gamma \setminus F$ in which ω lives, 2) the set of all the ends of Γ that live in C, 3) and the finitely many open edges between F and C.*

The set of all ends of Γ with the subspace topology of $\overline{\Gamma}$, is called end space and denoted by $\Omega(\Gamma)$.

2.2 Ends of groups

The ends are also defined for groups corresponded to the ends of Cayley graphs of groups. First, we recall the definition of Cayley graph of given group G with respect to generating set S, and we denote it by $\Gamma(G, S)$.

Definition 2.7 ([7])**.** Suppose that G is a group and S is a generating set for G. Then the Cayley graph of group G with respect to generating set S, is graph $\Gamma(G, S)$ with the set of elements of G as vertices, and its edges is defined as follows: for any two vertices g and h, gh is an edge if and only if there is $s \in S \cup S^{-1}$, such that $h = gs$ [7, Page 11].

Since S is a generating set for the group G, each element G is generated by a finite sequences of the elements of S, and therefore from each vertex g, there is a finite path in $\Gamma(G, S)$ from g to the identity element of G. Therefore, for any group G and generating set S, $\Gamma(G, S)$ is connected.

By Definition 2.7, the Cayley graph of any group G depends on the choice of the generating set S. The ends of the group are defined by using its Cayley graph, but in Theorem 2.8, we see that the the end space of the Cayley graph of finitely generated groups is independent of the generating set ([7, Corollary 4.5] and [7, p. 11]).

Theorem 2.8 ([7])**.** *Suppose that G is a group with a finite generating set S and $\Gamma(G, S)$ be the Cayley graph of G with respect to S. If S' is another finite generating set for G and $\Gamma(G, S')$ is the Cayley graph of the group G with respect to S', then $\Omega(\Gamma(G, S))$ is homeomorphic to $\Omega(\Gamma(G, S'))$.*

According to Theorem 2.8, the end space can be discussed for finitely generated groups independently of the Cayley graphs and generating sets. Thus, we denote the end space of finitely generated group G by $\Omega(G)$.

Theorem 2.9. *Let G be a finitely generated group. Then $\Omega(G)$ is either a discrete space with 1 or 2 elements, or Cantor set.*

Also, the ends of Cayley graphs of G are called the ends of group G, and their numbers $|\Omega(\Gamma(G, S))|$, which may equal 0, 1, 2 or continuum cardinal, is denoted by $e(G)$.

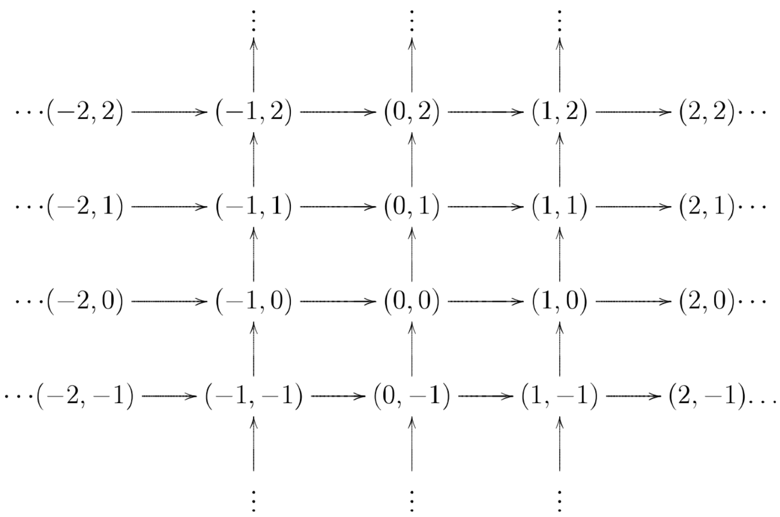

Figure 2: The Cayley graph of $\mathbb{Z} \oplus \mathbb{Z}$ with respect to $\{(1,0),(0,1)\}$.

Example 2.10. 1. Let $\mathbb{Z} \oplus \mathbb{Z}$ denotes the free abelian group of rank 2. This group has only one end. Cayley graph $\Gamma(\mathbb{Z} \oplus \mathbb{Z}, (1,0), (0,1))$ is as drawn in Figure 2.

2. Let $\mathbb{Z}$ be the infinite cyclic group with generating set $\{2, 3\}$. This group has two ends as shown in Figure 3.

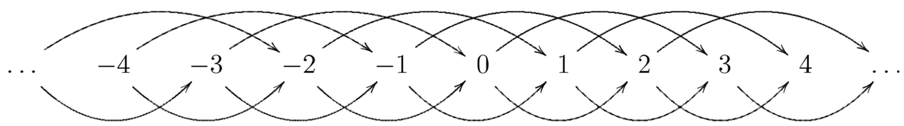

Figure 3: The Cayley graph of $\mathbb{Z}$ with respect to $\{2, 3\}$.

3. Let $\mathbb{Z} * \mathbb{Z}$ be the free group of rank 2. This group has cantor set as the end space, and then has infinitely many ends as shown in Figure 4.

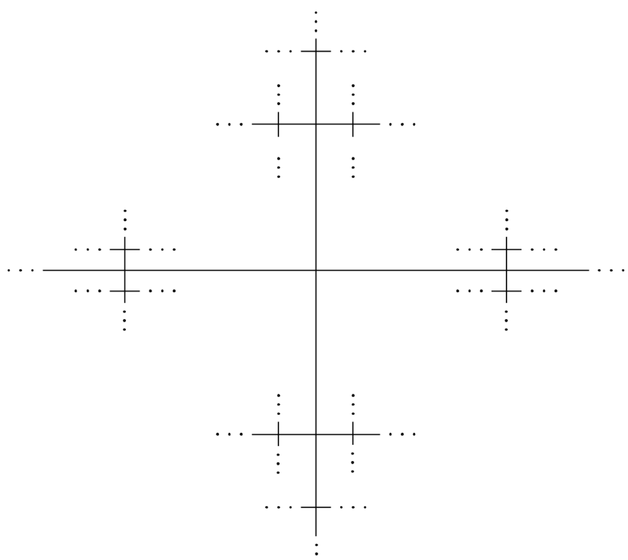

Figure 4: The Cayley graph of $\mathbb{Z} * \mathbb{Z}$ with respect to generators $\{a, b\}$.

Classifying groups by the number of its ends begun by Hopf [9], and more specifically for finitely generated groups by Stallings [10], and completed for all groups by Dicks and Dunwoody [1]. These classifications are briefly stated in Theorem 2.12. For this, we need recall the definitions of amalgamated free product and HNN extension.

Definition 2.11 ([7]). 1. Suppose that G_1 and G_2 are groups with subgroups H_1 and H_2 respectively. Further suppose that $\varphi : H_1 \to H_2$ is an isomorphism. Then the amalgamated free product of G_1 and G_2 with respect to φ is the group

$$G_1 *_\varphi G_2 := \langle G_1, G_2 | \varphi(h)h^1, \ , h \in H_1 \rangle.$$

A more common notation for this amalgamated free product is $G_1 *_H G_2$, where H is a group isomorphic to both H_1 and H_2.

2. Suppose that G is a group, H_1 and H_2 are subgroups of G, and $\varphi : H_1 \to H_2$ is an isomorphism. Then the HNN extension of G via φ is the group

$$G*_\varphi := \langle G, t | tht^1 \varphi(h)^1, h \in H_1 \rangle,$$

where t is an element not in G, called the stable letter of $G*_\varphi$. A more common notation for the HNN extension of G via φ is $G*_H$, where H is a group isomorphic to both H_1 and H_2.

Theorem 2.12. *Let G be any group. Then $e(G) = 0, 1, 2$, or c, where c is the continuum cardinal.*

i) $e(G) = 0$ if and only if G is finite.

ii) $e(G) = 2$ if and only if G has $\mathbb{Z}$ as a subgroup with finite index.

iii) $e(G) = c$ if and only if one of the following holds

a) $G \cong B *_C D$, where C is finite and $B \neq C \neq D$, or

b) $G \cong B*_C$, where C is finite, or

c) G is countably infinite and locally finite.

iv $e(G) = 1$ if and only if G is not of type (i), (ii), or (iii).

3 Main results

In this section, we investigate free ends of groups and eliminate trivial cases. Free end was defined on graphs. An end ω of finitely generated group G is free if it is free as an end of a Cayley graph $\Gamma(G, S)$, for some finite generating set S.

The topology defined in Theorem 2.5, implies the following lemma.

Lemma 3.1. *Let Γ be a locally finite graph. Then an end ω is free if and only if $\{\omega\}$ is open in $\Omega(\Gamma)$.*

By lemma 3.1, the definition of free end of group G depends only on the structure of $\Omega(G)$ not depending on the structure of Cayley graph.

Theorem 3.2. *Let G be a finitely generated group. Then G has free end if and only if $e(G) \leq 2$.*

It is well-known that free groups has more than 2 ends. Thus, the following corollary is concluded from Theorem 3.2.

Corollary 3.3. *Let G be a free group. Then G has a free end if and only if $G \cong \mathbb{Z}$.*

Obviously, for the cases $e(G) = 0$ and $e(G) = 1$, the ends are free, because there is no other end to be separated from ω. By eliminating these obvious cases, we can study free ends only for groups with more than one end, which led to the following corollary.

Corollary 3.4. *Let G be finitely generated group. Then G has free end if and only if G has $\mathbb{Z}$ as a subgroup with finite index.*

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