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c-Nilpotent multiplier and c-capability of the direct sum of Lie algebras

Farangis Johari*,‡, Peyman Niroomand†,§ and Mohsen Parvizi*,¶

*Department of Pure Mathematics
Ferdowsi University of Mashhad
Mashhad, Iran

†School of Mathematics and Computer Science
Damghan University, Damghan, Iran
‡farangis.johari@mail.um.ac.ir, farangisjohary@yahoo.com
§niroomand@du.ac.ir, p_niroomand@yahoo.com
¶parvizi@um.ac.ir

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In this paper, we determine the behavior of the c-nilpotent multiplier of Lie algebras with respect to the direct sum. Then we give some results on the c-capability of the direct sum of finite dimensional Lie algebras.

Keywords: c-Nilpotent multiplier; c-capable; nilpotent Lie algebra.

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1. Motivation and Introduction

Let L be a Lie algebra presented as the quotient algebra of a free Lie algebra F by an ideal R. Then the c-nilpotent multiplier of L, is defined to be

$$\mathcal{M}^{(c)}(L) = \frac{R \cap F^{c+1}}{[R,_c F]}, \quad \text{for all } c \geq 1,$$

where F^{c+1} is the $(c+1)$th term of the lower central series of F and $[R,_0 F] = R, [R,_c F] = [[R,_{c-1} F], F]$. This is analogous to the definition of the Baer-invariant of a group with respect to the variety of nilpotent groups of class at most c given by Baer in [1] (see [6–8, 11, 19] for more information on the Baer invariant of groups). The 1-nilpotent multiplier of L, is the more studied as the Schur multiplier of L, $\mathcal{M}(L) = R \cap F^2/[R, F]$, (see for instance [3–5, 13, 18]). It is proved that the Lie algebra $\mathcal{M}^{(c)}(L)$ is abelian and independent of the choice of the free Lie algebra F.

¶Corresponding author.

References [3–6, 13, 15, 16] show that the behavior of the Schur multiplier with respect to direct sum of two Lie algebras may lead us to have more results on the Schur multiplier of a Lie algebra.

From [12], the formula of the Schur multiplier for the direct product of two groups is well known. Later, the same result for the direct sum of two Lie algebras was proved in [5, 20]. Moghaddam in [11] extended this result for the c-nilpotent multiplier of the direct product of two groups and also Ellis improved the result of Moghaddam in [8]. The last two authors in [16] showed the behavior of the 2-nilpotent multipliers with respect to the direct sum of two Lie algebras. Recently, Salemkar and Aslizadeh obtained a formula for the c-nilpotent multipliers of the direct sum of Lie algebras whose abelianizations are finite dimensional (see [21, Theorem 2.5]) and also generalized it for arbitrary Lie algebras, in the case $c+1$ is a prime number or $c+1 = 4$ which is a strong restriction (see [21, Theorem 2.9]). Here, we intend to generalize the result of Salemkar *et al.* to the c-nilpotent multipliers for any arbitrary c, and then we give some results concerning the c-capability of the direct sum of Lie algebras.

2. The cth Term of the Lower Central Series of the Free Product of Two Lie Algebras

In this section, we are going to obtain the formula of the cth term of lower central series of the free product of two Lie algebras.

The definition of basic commutators plays a fundamental role in obtaining our main results.

Following Shirshov [22] for a free Lie algebra L on the set $X = \{x_1, x_2, \ldots\}$. The basic commutators on the set X defined inductively as follows.

(i) The generators $x_1, x_2, \ldots, x_n$ are basic commutators of length one and ordered by setting $x_i < x_j$ if $i < j$.

(ii) If all the basic commutators d_i of length less than t have been defined and ordered, then we may define the basic commutators of length t to be all commutators of the form $[d_i, d_j]$ such that the sum of lengths of d_i and d_j is t, $d_i > d_j$, and if $d_i = [d_s, d_u]$, then $d_j \geq d_u$. The basic commutators of length t follow those of lengths less than t. The basic commutators of the same length can be ordered in any way, but usually the lexicographical order is used.

The number of all basic commutators on a set $X = \{x_1, x_2, \ldots x_d\}$ of length n is denoted by $l_d(n)$. Thanks to [22], we have

$$l_d(n) = \frac{1}{n} \sum_{m \mid n} \mu(m) d^{\frac{n}{m}},$$

where $\mu(m)$ is the Möbius function, defined by $\mu(1) = 1, \mu(k) = 0$ if k is divisible by a square, and $\mu(p_1 \ldots p_s) = (-1)^s$ if $p_1, \ldots, p_s$ are distinct prime numbers.

Using the topside statement and looking [18, Lemma 1.1] and [22], we have the next theorem.

Theorem 2.1. *Let F be a free Lie algebra on a set X, then F^c/F^{c+i} is an abelian Lie algebra with the basis of all basic commutators on X of lengths $c, c+1, \ldots, c+i-1$ for all i, $1 \leq i \leq c$. In particular, F^c/F^{c+1} is an abelian Lie algebra of dimension $l_d(c)$.*

The following definition is vital and will be used in the rest.

Definition 2.2. Let $c \geq 1$. Consider the free product $A * B$ of two Lie algebras A and B. Let us impose the ordering $A < B$. The set of basic commutators of length c on two letters A and B is denoted by M. We define

$$\sum(A * B)_c = \langle [P_1, \ldots, P_c]_\lambda \,|\, \lambda \in M \rangle,$$

where $P_i = A$ or $P_i = B$. In fact, $\sum(A * B)_c$ is the subalgebra generated by all the basic commutator subalgebras $[P_1, \ldots, P_c]_\lambda$ such that $\lambda \in M$.

For example, we have $\sum(A*B)_1 = A*B$, $\sum(A*B)_2 = [A, B]$ and $\sum(A*B)_3 = \langle [B, A, A], [B, A, B] \rangle$.

Lemma 2.3. *Let A and B be two Lie algebras. Then $\sum(A * B)_c = \langle [\sum(A * B)_{c-1}, A], [\sum(A * B)_{c-1}, B] \rangle = [\sum(A * B)_{c-1}, A * B]$ and $\sum(A * B)_c$ is an ideal of $A * B$ where $c \geq 2$.*

Proof. Clearly, $\sum(A * B)_{c+1} = \langle [\sum(A * B)_c, A], [\sum(A * B)_c, B] \rangle$. We proceed by induction on c. If $c = 2$, then $\sum(A * B)_2 = [A, B]$. Let $c \geq 3$. By the induction hypothesis, we have $\sum(A * B)_c$ is an ideal of $A * B$ and

$$\begin{aligned}\sum(A * B)_c &= \left\langle \left[\sum(A * B)_{c-1}, A\right], \left[\sum(A * B)_{c-1}, B\right] \right\rangle \\ &= \left[\sum(A * B)_{c-1}, A * B\right].\end{aligned}$$

We claim that

$$\left\langle \left[\sum(A * B)_c, A\right], \left[\sum(A * B)_c, B\right] \right\rangle = \left[\sum(A * B)_c, A * B\right].$$

Clearly, $\langle [\sum(A*B)_c, A], [\sum(A*B)_c, B] \rangle \subseteq [\sum(A*B)_c, A*B]$. It is enough to show that

$$\left[\sum(A * B)_c, A * B\right] \subseteq \left\langle \left[\sum(A * B)_c, A\right], \left[\sum(A * B)_c, B\right] \right\rangle.$$

Let $l = a + b + w \in A * B$, $w = \sum_{i=1}^n [a_i, b_i]$ and $x \in \sum(A * B)_c$ such that $a, a_i \in A$, $b, b_i \in B$ and $w \in [A, B]$. We show that $[x, l] \in \langle [\sum(A * B)_c, A], [\sum(A * B)_c, B] \rangle$. For this, we know $[x, l] = [x, a + b + w] = [x, a] + [x, b] + [x, w]$. Clearly,

$$[x, a] + [x, b] \in \left\langle \left[\sum(A * B)_c, A\right], \left[\sum(A * B)_c, B\right] \right\rangle.$$

Since $[x, w] = [x, \sum_{i=1}^n [a_i, b_i]] = \sum_{i=1}^n [x, [a_i, b_i]]$, it is enough to see that $[x, [a_i, b_i]] \in \langle [\sum(A * B)_c, A], [\sum(A * B)_c, B] \rangle$. The induction hypothesis implies

$[b_i, x], [x, a_i] \in \sum(A*B)_c$. By the Jacobian identity, $[x, [a_i, b_i]] = [b_i, x, a_i] + [x, a_i, b_i]$ and so

$$[x, [a_i, b_i]] = [b_i, x, a_i] + [x, a_i, b_i] \in \left\langle \left[\sum(A * B)_c, A\right], \left[\sum(A * B)_c, B\right] \right\rangle.$$

Hence $[x, w] \in \langle [\sum(A * B)_c, A], [\sum(A * B)_c, B]\rangle$. Therefore

$$\sum(A * B)_{c+1} = \left[\sum(A * B)_c, A * B\right].$$

Now, we show that $\sum(A * B)_{c+1}$ is an ideal of $A * B$. Let $y \in A * B$ and $x = \sum_{i=1}^{n}[x_i, l_i] \in \sum(A * B)_{c+1}$ such that $x_i \in \sum(A * B)_c$ and $l_i \in A * B$. Since $[x, y] = \sum_{i=1}^{n}[x_i, l_i, y]$ and $[x_i, l_i] \in \sum(A * B)_c$ for all $1 \leq i \leq n$, we conclude $[x, y] \in \sum(A * B)_{c+1}$. Hence $\sum(A * B)_c$ is an ideal and result follows. □

We give the structures of the cth term of the the lower central series of the free product of two Lie algebras.

Proposition 2.4. *Let A and B be two Lie algebras. Then*

$$(A * B)^c = A^c + B^c + \sum(A * B)_c$$

for $c \geq 1$.

Proof. We proceed by induction on c. If $c = 1$, then $A * B = A + B + [A, B]$ and the result holds. Let $c \geq 2$. By the induction hypothesis, $(A*B)^c = A^c + B^c + \sum(A*B)_c$. It is easy to see that

$$\begin{aligned}(A * B)^{c+1} &= [(A * B)^c, A * B] = \left[A^c + B^c + \sum(A * B)_c, A * B\right] \\ &= [A^c, A * B] + [B^c, A * B] + \left[\sum(A * B)_c, A * B\right].\end{aligned}$$

By Lemma 2.3, $[\sum(A * B)_c, A * B] = \sum(A * B)_{c+1}$. We conclude that

$$\begin{aligned}&A^{c+1} + B^{c+1} + \sum(A * B)_{c+1} \\ &\quad \subseteq (A * B)^{c+1} = [A^c, A * B] + [B^c, A * B] + \sum(A * B)_{c+1}.\end{aligned}$$

Now, we show that

$$[A^c, A * B] + [B^c, A * B] + \sum(A * B)_{c+1} \subseteq A^{c+1} + B^{c+1} + \sum(A * B)_{c+1}.$$

Let $x \in A^c, y \in B^c$ and $l = a + b + w \in A * B$, $w = \sum_{i=1}^{n}[a_i, b_i]$ such that $a, a_i \in A$, $b, b_i \in B$ and $w \in [A, B]$. Since

$$[x, l] = [x, a] + [x, b] + [x, w] \quad \text{and} \quad [y, l] = [y, a] + [y, b] + [y, w],$$

we can see that $[x, b] + [x, w], [y, a] + [y, w] \in \sum(A * B)_{c+1}$. Hence $[A^c, A * B] + [B^c, A * B] + \sum(A * B)_{c+1} = A^{c+1} + B^{c+1} + \sum(A * B)_{c+1}$. It completes the proof. □

The next step gives a generating set for $\sum(F_1 * F_2)_{c+1}$ in terms of the free generators of F_1 and F_2.

Proposition 2.5. *Let F_1 and F_2 be two free Lie algebras generated by the set X and Y, respectively, and $F = F_1 * F_2$ for all $c \geq 1$. Then $(\sum(F_1 * F_2)_{c+1} + F^{c+2})/F^{c+2}$ is an abelian Lie algebra with the basis of all basic commutators λ of length $c+1$ in the set $X \cup Y$ which have at least one x_i and at least one y_j.*

Proof. By Proposition 2.4, we have

$$\begin{aligned} F^{c+1}/F^{c+2} &= (F_1^{c+1} + F_2^{c+1} + \sum(F_1 * F_2)_{c+1})/F^{c+2} \\ &\cong (F_1^{c+1}/F_1^{c+2}) \oplus (F_2^{c+1}/F_2^{c+2}) \oplus \left(\left(\sum(F_1 * F_2)_{c+1} + F^{c+2}\right)/F^{c+2}\right). \end{aligned}$$

By Theorem 2.1, F^{c+1}/F^{c+2} , F_1^{c+1}/F_1^{c+2} and F_2^{c+1}/F_2^{c+2} are the abelian Lie algebras with the basis of all basic commutators of length $c+1$ on $X \cup Y, X$ and Y, respectively. We conclude that $(\sum(F_1 * F_2)_{c+1} + F^{c+2})/F^{c+2}$ is an abelian Lie algebra with the basis of all basic commutators λ of length c in $X \cup Y$ which involve at least one x_i and at least one y_i, as required. □

Corollary 2.6. *Let F_1 and F_2 be two free Lie algebras generated by X and Y with d_1 and d_2 elements, respectively, and $F = F_1 * F_2$. Let S be the set of all basic commutators λ of length c in $X \cup Y$ which have at least one x and at least one y. Suppose that the order is defined as $x_i < x_j < y_t < y_d$ for $i < j$ and $t < d$, where x_i, $x_j \in X$ and $y_t, y_d \in Y$. Then $\dim\langle S\rangle = l_{d_1+d_2}(c) - l_{d_1}(c) - l_{d_2}(c)$.*

Proof. S is the set of all basic commutators λ of length c in $X \cup Y$ such that λ is not a basic commutator of length c on the set X or Y. By Theorem 2.1, $\dim\langle S\rangle = l_{d_1+d_2}(c) - l_{d_1}(c) - l_{d_2}(c)$. The result follows. □

Corollary 2.7. *Let F_1 and F_2 be two free Lie algebras generated by the set X and Y, respectively, and $F = F_1 * F_2$. Let S be the set of all basic commutators λ of length $c+1$ on $X \cup Y$ which have at least one x_i and at least one y_i. Let we have the order $x_i < x_j < y_t < y_d$ for $i < j$ and $t < d$, where x_i, $x_j \in X$ and $y_t, y_d \in Y$. Then $\sum(F_1 * F_2)_{c+1} = \sum(F_1 * F_2)_{c+2} + \langle S\rangle$.*

Proof. By Proposition 2.5, we have $(\sum(F_1 * F_2)_{c+1} + F^{c+2})/F^{c+2} = (\langle S\rangle + F^{c+2})/F^{c+2}$ and so $\sum(F_1 * F_2)_{c+1} + F^{c+2} = \langle S\rangle + F^{c+2}$. Thus,

$$\begin{aligned} \sum(F_1 * F_2)_{c+1} &= (\langle S\rangle + F^{c+2}) \cap \sum(F_1 * F_2)_{c+1} \\ &= \left(\sum(F_1 * F_2)_{c+1} \cap F^{c+2}\right) + \langle S\rangle. \end{aligned}$$

By Propositions 2.4, $\sum(F_1 * F_2)_{c+1} = ((F_1^{c+2} + F_2^{c+2} + \sum(F_1 * F_2)_{c+2}) \cap \sum(F_1 * F_2)_{c+1}) + \langle S\rangle = ((F_1^{c+2} + F_2^{c+2}) \cap \sum(F_1 * F_2)_{c+1}) + \sum(F_1 * F_2)_{c+2} + \langle S\rangle$, as required. □

3. The c-Nilpotent Multiplier of a Direct Sum of Lie Algebras

In this section, we study the c-nilpotent multiplier with respect to the direct sum of two Lie algebras, and then we give some results on the c-capability of the direct sum of two Lie algebras. The techniques are used here are based on the notion of free products of Lie algebras and the expansion of an element of a free Lie algebra in terms of basic commutators. Every element of a free Lie algebra can be expressed as a sum of some basic commutators. For the elements of a free product of two free Lie algebras also we have a similar expression except that the basic commutators are on the union of the bases of the two free Lie algebras taken to form the free product. It is easy to see that the free product of two free Lie algebras is in fact a free Lie algebra on the disjoint union of the bases of the two chosen free Lie algebras (see [22] for more details).

Let L_1 and L_2 be two Lie algebras with the following free presentations

$$0 \to R_1 \to F_1 \to L_1 \to 0 \quad \text{and} \quad 0 \to R_2 \to F_2 \to L_2 \to 0,$$

respectively. Then the free presentation of the direct sum $L_1 \oplus L_2$ is given in the following.

Lemma 3.1 ([16, Lemma 2.1]). *Let $F = F_1 * F_2$ be the free product of two free Lie algebras F_1 and F_2. Then $0 \to R \to F \to L_1 \oplus L_2 \to 0$ is the free presentation for $L_1 \oplus L_2$ in which $R = R_1 + R_2 + [F_2, F_1]$.*

The following lemma plays a key role in the main result, also it is different from [21, Proposition 2.1], since we use the concept of free products in order to obtain the free presentation of a direct sum of Lie algebras.

By applying Lemma 3.1 and the above notation, we can compute the c-nilpotent multiplier of $L_1 \oplus L_2$ in terms of F_i's and R_i's as follows

$$\mathcal{M}^{(c)}(L_1 \oplus L_2) = \frac{R \cap F^{c+1}}{[R,_c F]} = \frac{(R_1 + R_2 + [F_2, F_1]) \cap (F_1 * F_2)^{c+1}}{[R_1 + R_2 + [F_2, F_1],_c F_1 * F_2]}.$$

Define

$$\begin{aligned} \eta : \mathcal{M}^{(c)}(L_1 \oplus L_2) = \frac{R \cap F^{c+1}}{[R,_c F]} &\to \frac{R_1 \cap F_1^{c+1}}{[R_1,_c F_1]} \oplus \frac{R_2 \cap F_2^{c+1}}{[R_2,_c F_2]} \\ &= \mathcal{M}^{(c)}(L_1) \oplus \mathcal{M}^{(c)}(L_2). \end{aligned} \tag{3.1}$$

which is induced by the canonical homomorphism from $F = F_1 * F_2 \to F_1 \times F_2$. Then we have

Lemma 3.2. *Let L_1 and L_2 be two Lie algebras. Then*

$$\mathcal{M}^{(c)}(L_1 \oplus L_2) \cong \mathcal{M}^{(c)}(L_1) \oplus \mathcal{M}^{(c)}(L_2) \oplus K,$$

for all $c, c \geq 1$ and $K = \ker\eta$.

Proof. Let $F = F_1 * F_2$. Then the epimorphism $F \to F_1 \times F_2$ induces the above epimorphism η. Consider the map

$$\beta : \frac{R_1 \cap F_1^{c+1}}{[R_1,_c F_1]} \oplus \frac{R_2 \cap F_2^{c+1}}{[R_2,_c F_2]} \to \frac{R \cap F^{c+1}}{[R,_c F]}$$

defined by $(x_1 + [R_1,_c F_1], x_2 + [R_2,_c F_2]) \mapsto x_1 + x_2 + [R,_c F]$. Clearly, β is a well-defined homomorphism. It is easy to see that β is a left inverse to η in Eq. (3.1). Therefore the sequence

$$0 \to K \to \mathcal{M}^{(c)}(L_1 \oplus L_2) \to \mathcal{M}^{(c)}(L_1) \oplus \mathcal{M}^{(c)}(L_2) \to 0$$

splits and the result holds. □

Now we compute the kernel of the epimorphism η in Eq. 3.1.

Theorem 3.3. *Let*

$$\eta : \mathcal{M}^{(c)}(L_1 \oplus L_2) \to \mathcal{M}^{(c)}(L_1) \oplus \mathcal{M}^{(c)}(L_2)$$

be the epimorphism defined in Eq. (3.1). *Then*

$$\ker \eta = \sum (F_1 * F_2)_{c+1} + [R,_c F]/[R,_c F].$$

Proof. Clearly $(\sum (F_1 * F_2)_{c+1} + [R,_c F])/[R,_c F] \subseteq \ker \eta$. Let $w + [R,_c F] \in \ker \eta$. Using Proposition 2.4, we have $w + [R,_c F] = a + b + z + [R,_c F] \in \ker \eta$ such that $a \in F_1^{c+1}$, $b \in F_2^{c+1}$ and $z \in \sum (F_1 * F_2)_{c+1}$. The definition of η implies $a \in [R_1,_c F_1]$ and $b \in [R_2,_c F_2]$ so that $w + [R,_c F] = z + [R,_c F]$, as required. □

The following corollary is an immediate consequence of Lemma 3.2 and Theorem 3.3.

Corollary 3.4. *Let L_1 and L_2 be two Lie algebras. Then*

$$\mathcal{M}^{(c)}(L_1 \oplus L_2) \cong \mathcal{M}^{(c)}(L_1) \oplus \mathcal{M}^{(c)}(L_2) \oplus \left(\sum (F_1 * F_2)_{c+1} + [R,_c F] \right) / [R,_c F]$$

for all $c, c \geq 1$.

Lemma 3.5. *Let F_1 and F_2 be two free Lie algebras generated by the sets X and Y, respectively, and $F = F_1 * F_2$. Let S be the set of all basic commutators λ of length $c+1$ on $X \cup Y$ which involve at least one x_i and at least one y_i. Let we have the order $x_i < x_j < y_t < y_d$ for $i < j$ and $t < d$, where x_i, $x_j \in X$ and $y_t, y_d \in Y$. Then $(\sum (F_1 * F_2)_{c+1} + [R,_c F])/[R,_c F] = \langle S \rangle + [R,_c F]/[R,_c F]$.*

Proof. By Corollary 2.7, we have $\sum (F_1 * F_2)_{c+1} = \sum (F_1 * F_2)_{c+2} + \langle S \rangle$. Also by Lemma 2.3, we have $\sum (F_1 * F_2)_{c+2} = [\sum (F_1 * F_2)_{c+1}, F]$. Since $[R,_c F] = [R_1 + R_2 + [F_2, F_1],_c F]$, so $\sum (F_1 * F_2)_{c+2} = [\sum (F_1 * F_2)_{c+1}, F] \subseteq [[F_2, F_1],_c F] \subseteq [R,_c F] = [R_1 + R_2 + [F_2, F_1],_c F]$. The result follows. □

A similar definition to the following can be found in [8].

Definition 3.6. Let $c \geq 1, K$ and H be two abelian Lie algebras with bases $\{a_i\}_{i\in I}$ and $\{b_j\}_{j\in j}$, respectively. Let we have the ordering $K < H$ and the set of basic commutators of length c on the letters K and H which contains at least one H and one K, is denoted by S_1. We define

$$\tau(K, H)_c = \oplus_{\lambda\in S_1}(P_1 \otimes \cdots \otimes P_c), \tag{3.2}$$

where $P_i = K$ or $P_i = H$.

Note that descriptions of $\tau(K, H)_c$ follow from basic properties of the tensor products of abelian Lie algebras and the definition of basic commutators.

$$\begin{aligned}
&\tau(K, H)_1 = 0,\\
&\tau(K, H)_2 = (H \otimes K),\\
&\tau(K, H)_3 = (H \otimes K \otimes K) \oplus (H \otimes K \otimes H),\\
&\tau(K, H)_4 = (H \otimes K \otimes K \otimes K) \oplus (H \otimes K \otimes K \otimes H) \oplus (H \otimes K \otimes H \otimes H).
\end{aligned}$$

Multi linearity of the generating elements of $\sum(F_1 * F_2)_{c+1}(\mathrm{mod}[R,_c F])$ imposes a connection between $\sum(F_1 * F_2)_{c+1}(\mathrm{mod}[R,_c F])$ and $\tau(F_1^{ab}, F_2^{ab})_{c+1}$.

Lemma 3.7. *With the notations and assumptions of Proposition 2.5, we have* $\tau(L_1^{ab}, L_2^{ab})_{c+1} \cong (\sum(F_1 * F_2)_{c+1} + [R,_c F])/[R,_c F]$ *for all* $c \geq 1$.

Proof. By Lemma 3.5, the map $\alpha_1 : (\sum(F_1 * F_2)_{c+1} + [R,_c F])/[R,_c F] \to \tau(F_1^{ab}, F_2^{ab})_{c+1}$ given by $[f_1, \ldots, f_{c+1}] + [R,_c F] \mapsto \tilde{f_1} + H^2 \otimes \cdots \otimes \tilde{f_{c+1}} + H^2$, where $H = L_1^2$ or $H = L_2^2$ is a Lie homomorphism. Conversely, we may check that $\alpha_2 : \tau(L_1^{ab}, L_2^{ab})_{c+1} \to (\sum(F_1 * F_2)_{c+1} + [R,_c F])/[R,_c F]$ given by $\tilde{f_1} + H^2 \otimes \cdots \otimes \tilde{f_{c+1}} + H^2 \mapsto [f_1, \ldots, f_{c+1}] + [R,_c F]$, where $H = L_1^2$ or $H = L_2^2$ is a Lie homomorphism too. Now $\alpha_1\alpha_2$ and $\alpha_2\alpha_1$ are identity homomorphisms, so the result follows. $\square$

The following theorem generalizes [21, Theorems 2.5 and 2.9] and states a formula for the c-nilpotent multiplier of a direct sum of two arbitrary Lie algebras without any restriction on c.

Theorem 3.8. *Let* L_1 *and* L_2 *be arbitrary Lie algebras. Then*

$$\mathcal{M}^{(c)}(L_1 \oplus L_2) \cong \mathcal{M}^{(c)}(L_1) \oplus \mathcal{M}^{(c)}(L_2) \oplus \tau(L_1^{ab}, L_2^{ab})_{c+1}$$

for all $c, c \geq 1$.

Proof. The result follows immediately from Corollary 3.4 and Lemma 3.7. $\square$

4. The c-Capability of a Direct Sum of Finite Dimensional Lie Algebras

In this section, we are going to determine the c-capability of a direct sum of finite dimensional non-abelian Lie algebras.

Recall that from [19] a Lie algebra L is c-capable if there exists some Lie algebra H such that $L \cong H/Z_c(H)$, where $Z_c(H)$ is the cth center of H. Evidently, L is 1-capable if and only if it is an inner derivation Lie algebra, and L is c-capable ($c \geq 2$) if and only if it is an inner derivation Lie algebra of a $(c-1)$-capable Lie algebra.

In [19], the c-epicenter of a Lie algebra L, $Z_c^*(L)$, is defined to be the smallest ideal M of L such that L/M is c-capable. For $c = 1$, the 1-epicenter of L is equal to $Z^*(L)$ for a Lie algebra L which was defined in [17]. It is obvious that $Z_c^*(L)$ is a characteristic ideal of L contained in $Z_c(L)$, and $Z_c^*(L/Z_c^*(L)) = 0$. So L is c-capable if and only if $Z_c^*(L) = 0$.

The proof of the following lemma is similar to the proof of [14, Theorem 2.7].

Lemma 4.1. *Let A and B be two Lie algebras. Then $Z_c^*(A \oplus B) \subseteq Z_c^*(A) \oplus Z_c^*(B)$ for all $c, c \geq 1$.*

Proof. Since $(A \oplus B)/(Z_c^*(A) \oplus Z_c^*(B)) \cong (A/Z_c^*(A)) \oplus (B/Z_c^*(B))$, we have $Z_c^*(A \oplus B) \subseteq Z_c^*(A) \oplus Z_c^*(B)$, as required. □

The following results show that the c-capability of the direct product of a non-abelian Lie algebra and an abelian Lie algebra depends only on the c-capability of its non-abelian factor.

Proposition 4.2. *Let L be a finite dimensional Lie algebra. Then $L \cong T \oplus A$ in which A is an abelian Lie algebra and $Z(L) \cap L^2 = Z(T)$. Moreover, $Z_c^*(L) = Z_c^*(T) \subseteq T^2$ for all $c \geq 1$.*

Proof. By applying [9, Proposition 3.1], we have $L \cong T \oplus A$ such that $Z(L) \cap L^2 = Z(T)$ and A is an abelian Lie algebra. By using [21, Corollary 3.2], we have L/L^2 and T/T^2 are c-capable and so $Z_c^*(L) \subseteq Z_c^*(T) \subseteq T^2$ for all $c, c \geq 1$. If $Z_c^*(T) = 0$, then $Z_c^*(L) = Z_c^*(T) = 0$. Now let $Z_c^*(T) \neq 0$. We claim that $Z_c^*(T) \subseteq Z_c^*(L)$. By invoking Theorem 3.8, we have

$$\mathcal{M}^{(c)}(L) \cong \mathcal{M}^{(c)}(T) \oplus \mathcal{M}^{(c)}(A) \oplus \tau(T/T^2, A)_{c+1}$$

and

$$\mathcal{M}^{(c)}(L/Z_c^*(T)) \cong \mathcal{M}^{(c)}(T/Z_c^*(T)) \oplus \mathcal{M}^{(c)}(A) \oplus \tau(T/T^2, A)_{c+1}.$$

Now [19, Corollary 2.4] implies $\dim \mathcal{M}^{(c)}(T) = \dim \mathcal{M}^{(c)}(T/Z_c^*(T)) - \dim(Z_c^*(T) \cap L^{c+1})$. Thus,

$$\begin{aligned}\dim \mathcal{M}^{(c)}(L) &= \dim \mathcal{M}^{(c)}(T/Z_c^*(T)) + \dim \mathcal{M}^{(c)}(A) \\ &\quad + \dim \tau(T/T^2, A)_{c+1} - \dim(Z_c^*(T) \cap L^{c+1}) \\ &= \dim \mathcal{M}^{(c)}(L/Z_c^*(T)) - \dim(Z_c^*(T) \cap L^{c+1}).\end{aligned}$$

Again by [19, Corollary 2.4], $Z_c^*(T) \subseteq Z_c^*(L)$, as required. □

The following corollary is an immediate consequence of Proposition 4.2.

Corollary 4.3. *Let $L = T \oplus A(n)$ be a finite dimensional Lie algebra such that T is a non-abelian Lie algebra. Then L is c-capable if and only if T is c-capable.*

Theorem 4.4. *Let $L = L_1 \oplus L_2$ such that L_1 and L_2 are finite dimensional non-abelian Lie algebras. Then $Z_c^*(L_1 \oplus L_2) = Z_c^*(L_1) \oplus Z_c^*(L_2)$.*

Proof. We have $L_i = T_i \oplus A_i$ and $Z_c^*(T_i) = Z_c^*(L_i)$ for $1 \leq i \leq 2$, by Proposition 4.2. Therefore $L = T_1 \oplus T_2 \oplus A$ and $Z_c^*(L) = Z_c^*(T_1 \oplus T_2)$, where $A = A_1 \oplus A_2$, by Proposition 4.2. We claim that $Z_c^*(T_1 \oplus T_2) = Z_c^*(T_1) \oplus Z_c^*(T_2)$. Lemma 4.1 implies $Z_c^*(T_1 \oplus T_2) \subseteq Z_c^*(T_1) \oplus Z_c^*(T_2)$. Now we show that $Z_c^*(T_i) \subseteq Z_c^*(T_1 \oplus T_2)$ for $i = 1, 2$. If $Z_c^*(T_i) = 0$ for $i = 1, 2$, then $Z_c^*(T) = Z_c^*(T_1) \oplus Z_c^*(T_2) = 0$. Now we have $Z_c^*(T_i) \neq 0$ for $i = 1, 2$, or $Z_c^*(T_1) \neq 0$ and $Z_c^*(T_2) = 0$, or $Z_c^*(T_2) \neq 0$ and$Z_c^*(T_1) = 0$. First consider $Z_c^*(T_i) \neq 0$ for $i = 1, 2$. By invoking Theorem 3.8, we have

$$\mathcal{M}^{(c)}(T_1 \oplus T_2) \cong \mathcal{M}^{(c)}(T) \oplus \mathcal{M}^{(c)}(T_2) \oplus \tau(T_1/T_1^2, T_2/T_2^2)_{c+1},$$

$$\mathcal{M}^{(c)}(T_1 \oplus T_2/Z_c^*(T_1)) \cong \mathcal{M}^{(c)}(T_1/Z_c^*(T_1)) \oplus \mathcal{M}(T_2) \oplus \tau(T_1/T_1^2, T_2/T_2^2)_{c+1}$$

and

$$\mathcal{M}^{(c)}(T_1 \oplus T_2/Z_c^*(T_2)) \cong \mathcal{M}^{(c)}(T_1) \oplus \mathcal{M}(T_2/Z_c^*(T_2)) \oplus \tau(T_1/T_1^2, T_2/T_2^2)_{c+1}.$$

Now [19, Corollary 2.4] implies $\dim \mathcal{M}^{(c)}(T_1) = \dim \mathcal{M}^{(c)}(T_1/Z_c^*(T_1)) - \dim Z_c^*(T_1) \cap (T_1)^{c+1}$ and $\dim \mathcal{M}^{(c)}(T_2) = \dim \mathcal{M}^{(c)}(T_2/Z_c^*(T_2)) - \dim Z_c^*(T_2) \cap (T_2)^{c+1}$. Thus $\dim \mathcal{M}^{(c)}(T_1 \oplus T_2) = \dim \mathcal{M}^{(c)}(T_1/Z_c^*(T_1)) - \dim Z_c^*(T_1) \cap (T_1)^{c+1} + \dim \mathcal{M}^{(c)}(T_2) + \tau(T_1/T_1^2, T_2/T_2^2)_{c+1} < \mathcal{M}^{(c)}(T_1 \oplus T_2/Z_c^*(T_1))$ and

$$\begin{aligned}\dim \mathcal{M}^{(c)}(T_1 \oplus T_2) &= \dim \mathcal{M}^{(c)}(T_2/Z_c^*(T_2)) - \dim Z_c^*(T_2) \cap (T_2)^{c+1} \\ &\quad + \dim \mathcal{M}^{(c)}(T_1) + \tau(T_1/T_1^2, T_2/T_2^2)_{c+1} \\ &< \mathcal{M}^{(c)}(T_1 \oplus T_2/Z_c^*(T_2)).\end{aligned}$$

By using [19, Corollary 2.4], we have $Z_c^*(T_i) \subseteq Z_c^*(T)$, for $i = 1, 2$. Thus $Z_c^*(L) = Z_c^*(T_1 \oplus T_2) = Z_c^*(T_1) \oplus Z_c^*(T_2) = Z_c^*(L_1) \oplus Z_c^*(L_2)$. The proof is complete. □

Now we can state the following corollary which is interesting enough to be considered when studying the c-capability of finite dimensional non-abelian Lie algebras.

Corollary 4.5. *Let L_1 and L_2 be two finite dimensional non-abelian Lie algebras and $c \geq 1$. Then $L_1 \oplus L_2$ is c-capable if and only if L_1 and L_2 are c-capable.*

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