



α' -Corrections to DBI action via T-duality constraint

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Abstract

It is known that D_p -brane effective action at the leading order of α' in flat space–time which is given by DBI action, transforms to D_{p-1} -brane effective action under standard T-duality transformations of the open string gauge bosons and transverse scalar fields. Extending this duality to order α' , one may find corrections to the DBI action which include the second fundamental form Ω and the covariant derivative of gauge field strength DF , as well as the corrections to the T-duality transformations. Using this idea, up to two parameters, we have found all 81 covariant couplings of $DFDF$ and $\Omega\Omega$ with zero, two, four and six F 's. The four gauge field couplings that the T-duality constraint fixes are consistent with the known couplings in the literature.

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1. Introduction

One of the most exciting discoveries in string theory is T-duality [1,2]. This duality may be used to construct the effective field theory at low energy which may provide a manifestly background independent formulation of string theory [3,4]. One approach for constructing this effective action is the Double Field Theory [5–9] in which the T-duality is manifest, as the effective action is $O(D, D)$ -invariant by constructions. However, coordinate transformations in

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this approach receive α' corrections [10,11]. Another T-duality based approach for constructing the effective action, is to use the constraint that the dimensional reduction of an effective action on a circle must be invariant under the T-duality transformations [12]. In this approach, the couplings are invariant under the standard coordinate transformations, however, the T-duality transformations receive α' -corrections [13,14]. Using the T-duality constraint, the standard gravity and dilaton couplings in the effective actions at orders α' , α'^2 , α'^3 have been reproduced in [15,16]. It has been observed in [15] that the form of α' corrections to the Buscher rules depends on the scheme that one uses for the effective action.

The effective field theory of a D_p -brane in bosonic string theory includes various world-volume couplings of open string tachyon, transverse scalar fields, closed string tachyon, graviton, dilaton and B-field. Because of tachyons, the bosonic string theory and its D_p -branes are all unstable. Assuming the tachyons are frozen at the top of their corresponding tachyon potentials, the effective action at the leading order of α' in flat spacetime is then given by DBI action [17,18]:

$$S_p \supset -T_p \int d^{p+1} \sigma \sqrt{-\det(\tilde{G}_{ab} + F_{ab})} \tag{1}$$

where T_p is tension of D_p -brane, F_{ab} is gauge field strength of A_a and \tilde{G}_{ab} is metric which is pull-back of the bulk flat metric onto the world-volume¹ *i.e.*,

$$\begin{aligned} \tilde{G}_{ab} = P[\eta]_{ab} &= \frac{\partial X^\mu}{\partial \sigma^a} \frac{\partial X^\nu}{\partial \sigma^b} \eta_{\mu\nu} \\ &= \eta_{ab} + \partial_a \chi^i \partial_b \chi^j \eta_{ij} \end{aligned} \tag{2}$$

where X^μ is coordinate of space–time and $\eta_{\mu\nu}$ is flat space–time metric. In the second line the pull-back is written in the static gauge, *i.e.*, $X^a = \sigma^a$ and $X^i = \chi^i$. The DBI action (1) is invariant under the general coordinate transformations and is covariant under the standard T-duality transformation [19]. With our normalization for the gauge field, the DBI action is at the leading order of α' . It involves infinite number of F and $\partial \chi^i \partial \chi^j \eta_{ij}$. The first correction to this action is at order α' which includes $DFDF$ or $\Omega\Omega$ and infinite number of F 's. The higher derivative corrections to the Born–Infeld action in the bosonic and superstring theories, for only gauge field, have been studied in [20–25].

The world-volume couplings in the DBI action in the string frame are independent of p , the dimension of the D_p -brane. This has been used in [19] to observe that the DBI action is covariant under T-duality transformation. Assuming the higher derivative couplings on the world-volume of D_p -brane are also independent of the dimension of the brane, one expects the effective action of D_p -brane at any order of α' to be covariant under the T-duality transformation. Using this constraint, we are going to study the α' corrections to the DBI action in this paper. Since there are infinite number of F 's involved in the couplings at order α' , we consider couplings which have zero, two, four and six F 's. The couplings which have zero F are

$$S_p \supset -\alpha' T_p \int d^{p+1} \sigma \sqrt{-\det(\tilde{G}_{ab})} \left[C \tilde{\Gamma}_{\mu\nu} \tilde{G}^{ab} \tilde{G}^{cd} (\Omega_{ab}{}^\mu \Omega_{cd}{}^\nu - \Omega_{ac}{}^\mu \Omega_{bd}{}^\nu) \right] \tag{3}$$

¹ Our index convention is that the Greek letters (μ, ν, \dots) are the indices of the space–time coordinates, the Latin letters, (a, b, c, \dots) are the world-volume indices and the letters (i, j, k, \dots) are the normal bundle indices. The killing coordinate y is along the world-volume. The world-volume indices after the reduction of D_p -brane to D_{p-1} -brane are ($\tilde{a}, \tilde{b}, \tilde{c}, \dots$).

where C is a constant, \tilde{G}^{ab} is inverse of the pull-back metric and the second fundamental form Ω in the bosonic theory is defined to be [26]²

$$\Omega_{ab}{}^\alpha = \frac{\partial^2 X^\alpha}{\partial \sigma^a \partial \sigma^b} + \frac{\partial X^\mu}{\partial \sigma^a} \frac{\partial X^\nu}{\partial \sigma^b} \Gamma_{\mu\nu}{}^\alpha \tag{5}$$

The tensor $\tilde{\perp}_{\mu\nu}$ in (3) is a projection operator, *i.e.*, $\eta^{\nu\alpha} \tilde{\perp}_{\mu\nu} \tilde{\perp}_{\alpha\beta} = \tilde{\perp}_{\mu\beta}$, which projects space–time tensors to the transverse space. It is defined as $\tilde{\perp}_{\mu\nu} = \eta_{\mu\nu} - \eta_{\mu\alpha} \eta_{\nu\beta} \tilde{G}^{\alpha\beta}$ where the first fundamental form $\tilde{G}^{\mu\nu}$ is defined as

$$\tilde{G}^{\mu\nu} = \frac{\partial X^\mu}{\partial \sigma^a} \frac{\partial X^\nu}{\partial \sigma^b} \tilde{G}^{ab} \tag{6}$$

which is another projection operator, *i.e.*, $\eta_{\nu\alpha} \tilde{G}^{\mu\nu} \tilde{G}^{\alpha\beta} = \tilde{G}^{\mu\beta}$. It projects space–time tensors to the world-volume.

In flat spacetime and in static gauge, the second fundamental form (5) is zero when the space–time index α is a world volume and it is the second derivative of the transverse scalar fields when α is a transverse index, *i.e.*,

$$\Omega_{ab}{}^\alpha = \partial_a \partial_b \chi^i \delta_i{}^\alpha \tag{7}$$

The covariant action (3) includes infinite number of transverse scalar fields through the expansion of pull-back metric. We have chosen the relative coefficients of the two terms in (3) to have no corrections to the propagators of the transverse scalar fields. This action, however, is not total derivative term for terms with more than two transverse scalars. The coefficient C is a parameter which should be fixed by some calculations in string theory, *e.g.*, by studying the S-matrix element of two gravitons off the D_p -brane this parameter has been found in [26] to be $C = 1$. There are similar actions with some extra F 's, which we will find some of them in section 2. The parameters in these couplings and in (3) may be found by S-matrix calculations, however, we are interested in this paper to find them by imposing the T-duality constraint.

There are also couplings at order α' which include $DFDF$ and some extra F 's. The covariant derivative of F is

$$\begin{aligned} D_a F_{bc} &= \partial_a F_{bc} - \tilde{\Gamma}_{ab}{}^d F_{dc} - \tilde{\Gamma}_{ac}{}^d F_{bd} \\ &= \partial_a F_{bc} - \eta_{ij} \tilde{G}^{de} \partial_e \chi^i \partial_a \partial_b \chi^j F_{dc} + \eta_{ij} \tilde{G}^{de} \partial_e \chi^i \partial_a \partial_c \chi^j F_{db} \end{aligned} \tag{8}$$

where the Christoffel symbol $\tilde{\Gamma}_{ab}{}^c$ is made of the pull-back metric \tilde{G}_{ab} . As we will see in the next section, at the level of zero extra F , the couplings are total derivative terms, and at the level

² The second fundamental form in the superstring theory is defined in [27] to be

$$\Omega_{ab}{}^\alpha = \frac{\partial^2 X^\alpha}{\partial \sigma^a \partial \sigma^b} - \frac{\partial X^\alpha}{\partial \sigma^c} \tilde{\Gamma}_{ab}{}^c + \frac{\partial X^\mu}{\partial \sigma^a} \frac{\partial X^\nu}{\partial \sigma^b} \Gamma_{\mu\nu}{}^\alpha$$

where $\tilde{\Gamma}_{ab}{}^c$ is the connection made of the pull-back metric. In flat spacetime and in the static gauge it becomes

$$\Omega_{ab}{}^\alpha = \tilde{\perp}^i{}_j \partial_a \partial_b \chi^j \delta_i{}^\alpha \tag{4}$$

If one uses this expression for the couplings in (3), one would find that the resulting couplings for four transverse scalar fields are not consistent with the S-matrix element of four transverse scalar vertex operators. Moreover, we have found that this expression for the second fundamental form is not consistent with the T-duality constraint in the bosonic theory at the level of six F 's.

of two and more extra F 's, there are nontrivial couplings that their coefficients may be found by the T-duality constraint. As we will see, all parameters in the actions with zero, two, four and six extra F 's for which we have done the calculations explicitly, can be fixed up to two parameters. We choose one of them to be the coefficient C which is fixed by the S-matrix calculations to be $C = 1$.

The outline of the paper is as follows: In section 2, we find all independent couplings of $DFDF$ and $\Omega\Omega$ with two, four and six extra F 's. To this end, we first write all contractions of $DFDF$ and $\Omega\Omega$ with two, four and six F 's. The terms involving $\Omega\Omega$ are all independent, however, the terms involving $DFDF$ are not all independent as they are related by total derivative terms and the Bianchi identity. We introduce a method for imposing the Bianchi identity to find all independent couplings. In section 3, we impose the T-duality constraint on the independent couplings found in section 2 to fix their corresponding unknown coefficients in terms of two parameters. We show that the coefficients of the four gauge field couplings that the T-duality constraint fixes are consistent with the coefficients that one finds by the S-matrix method. We find also covariant couplings of six and eight gauge fields which have not been found by the S-matrix method.

2. Independent couplings

In this section we are going to find $DFDF$ and $\Omega\Omega$ couplings with two, four and six extra F 's. We begin with the couplings with two extra F 's. There are 18 contractions with structure $FFDFDF$. However, not all of them are independent.³ Some of them are related by total derivative terms and some other terms are related by the Bianchi identity $D_{[a}F_{bc]} = 0$. Note that using integration by part one can easily observe that the couplings with structure $FFFDDF$ can be written in terms of $FFDFDF$. To find the independent couplings we first construct the current I^a from 9 contractions of terms with structure $FFDFDF$. The 9 total derivative terms $D[FFDFDF]$, however, produce terms with structures $FFFDDF$ and $FFDFDF$. The two covariant derivatives in $D_a D_b F_{cd}$ can be written as symmetric and antisymmetric parts, *i.e.*,

$$D_a D_b F_{cd} = \frac{1}{2}\{D_a, D_b\}F_{cd} + \frac{1}{2}[D_a, D_b]F_{cd} \quad (9)$$

The antisymmetric part is identical to $\tilde{R}F$. On the other hand, using the Gauss–Codazzi equation

$$\tilde{R}_{abcd} = \tilde{\Gamma}_{ij}(\Omega_{ac}^i \Omega_{bd}^j - \Omega_{ad}^i \Omega_{bc}^j) \quad (10)$$

the antisymmetric part in (9) produces couplings with structure $FFFF\Omega\Omega$. They will change the unknown coefficients in the couplings with structure $FFFF\Omega\Omega$. Hence, if one uses all contractions of $FFFF\Omega\Omega$, with arbitrary coefficients, as independent couplings, one is allowed to ignore the antisymmetric part in $D_a D_b F_{cd}$, *i.e.*, the two covariant derivatives is symmetric. Using this symmetry, one finds there are 6 terms in total derivative terms which have structure $FFFDDF$. Constraining them to be zero, one finds 3 total derivative terms with structure $FFDFDF$. Adding these terms to the 18 contractions with structure $FFDFDF$, one can reduce them to 15 terms by choosing the coefficients of the total derivative terms to eliminate 3 terms. We choose to eliminate the 3 terms which do not include $D_a F^{ab}$, because as we will discuss in a moment they can be eliminated by field redefinitions.

³ We use the mathematica package 'xAct' [28] for performing the calculations in this paper.

Now one has to impose the Bianchi identity on DF -terms as well. Writing the first term on the right hand side of (8) in terms of potential $F_{ab} = \partial_a A_b - \partial_b A_a$ one can write the covariant derivative of F as $D_a F_{bc} = F'_{abc} - F'_{acb}$ where the function F'_{abc} which is not gauge invariant, is symmetric with respect to its first two indices. Writing DF in terms of F' , one can easily observe that the left hand side of the Bianchi identity, *i.e.*, $D_{[a} F_{bc]} = 0$, is zero.

When one rewrites the 15 couplings in terms of F' , one would find 7 independent couplings. Therefore, the Bianchi identity reduces the 15 couplings to 7 independent couplings when they are written in terms F'_{abc} . There are different ways to write the 7 independent couplings in terms of field strength F_{ab} . One particular choice for the couplings is

$$\begin{aligned}
 &F_{de} F^{de} D_a F_{bc} D^a F^{bc}, \quad F_c^e F_{de} D^a F^{bc} D_a F_b^d \\
 &F_a^e F_{de} D^a F^{bc} D^d F_{bc}, \quad F_c^e F_{de} D^a F_a^c D^b F_b^d \\
 &F_{cd} F_{be} D^a F_a^c D^b F^{de}, \quad F_{de} F^{de} D^a F_a^c D^b F_{cb} \\
 &F_b^e F_{de} D^a F_a^c D^b F_c^d
 \end{aligned} \tag{11}$$

where the indices are raised by the inverse metric \tilde{G}^{ab} . Our notation for F_a^b is that the earlier alphabet index appears first. All other choices for the couplings are identical to the above couplings after using the Bianchi identity, *i.e.*, they all are identical when they are written in terms of potential F'_{abc} . Similar calculations for $DFDF$ with zero extra F produces no independent coupling.

The last four terms in (11) include $D_a F^{ab}$. Under field redefinition $A_a \rightarrow A_a + \delta A_a$, $\chi^i \rightarrow \chi^i + \delta \chi^i$ the DBI action produces the couplings

$$\sqrt{-\det(\tilde{G})} \left[\frac{1}{2} D_a F^{ab} \delta A_b + \tilde{G}^{ab} \Omega_{ab}{}^j \delta \chi^i \eta_{ij} + \dots \right] \tag{12}$$

where dots represent terms which have some powers of F . Hence, the coefficients of the couplings which include $D_a F^{ab}$ or $\Omega_a{}^i$ can be changed under field redefinitions. On the other hand, it has been observed in [15] that the corrections to the T-duality transformations depend on the scheme that one uses for the field variables. For simplicity we use the scheme in which there are minimum number of couplings, *i.e.*, we use the field redefinitions to eliminate all terms which include $D_a F^{ab}$. So up to field redefinitions, there are 3 independent couplings in (11).

There are 5 independent couplings with structure $FF\Omega\Omega$, *i.e.*,

$$\begin{aligned}
 &F_{ab} F_{cd} \Omega^{aci} \Omega^{bd}{}_i, \quad F_a^c F_{bc} \Omega^{adi} \Omega^b{}_{di}, \quad F_{bc} F^{bc} \Omega_{adi} \Omega^{adi}, \\
 &F_{bc} F^{bc} \Omega_a{}^i \Omega^d{}_{di}, \quad F_c^d F_{bd} \Omega_a{}^i \Omega^{cb}{}_i
 \end{aligned} \tag{13}$$

where the world-volume indices are raised by \tilde{G}^{ab} and the transverse indices are lowered by $\tilde{\perp}_{ij}$. Using the variation (12), one can use a scheme in which the last two terms are eliminated by appropriate field redefinitions.⁴ All together, up to field redefinitions there are 6 independent terms at two extra F level, *i.e.*,

$$S_p \supset -\alpha' T_p \int d^{p+1} \sigma \sqrt{-\det(\tilde{G}_{ab})} \left[C_1 F_{ab} F_{cd} \Omega^{aci} \Omega^{bd}{}_i + C_2 F_a^c F_{bc} \Omega^{adi} \Omega^b{}_{di} \right]$$

⁴ One could also use field redefinition to remove the first term in (3), however, the absence of this term changes the propagator of the scalar fields. In that case, the α' corrections to the T-duality transformations would have linear term as well as nonlinear terms. We work in this paper with the couplings (3).

$$\begin{aligned}
 &+C_3 F_{bc} F^{bc} \Omega_{adi} \Omega^{adi} + N_1 F_{de} F^{de} D_a F_{bc} D^a F^{bc} \\
 &+N_2 F_c^e F_{de} D^a F^{bc} D_a F_b^d + N_3 F_a^e F_{de} D^a F^{bc} D^d F_{bc} \Big] \tag{14}
 \end{aligned}$$

The coefficients $C_1, C_2, C_3, N_1, N_2, N_3$ and C in (3) are 7 parameters that can be found by the S-matrix elements of four open string vertex operators [22,29]. They are

$$C = 1 ; C_1 = C_2 = 1 , C_3 = -\frac{1}{4} ; N_1 = -\frac{1}{24} , N_2 = -\frac{1}{3} , N_3 = \frac{1}{6} \tag{15}$$

However, we are going to find them in the next section by imposing the T-duality constraint.

At the level of four extra F 's, there are 56 contractions with structure $FFFFDFDF$. To find the total derivative terms, we note that there are 21 total derivative terms with structure $D[FFFFDFDF]$. Using their coefficients to eliminate the terms with structure $FFFFFDDF$, one finds 7 total derivative terms with structure $FFFFDFDF$. Using them one can eliminate 7 terms in the contractions $FFFFDFDF$. Using the Bianchi identity as we have done in the previous case, one finds 23 independent terms. 10 of them have $D_a F^{ab}$ which can be eliminated by appropriate field redefinitions. So up to field redefinitions there are the following 13 independent structures:

$$\begin{aligned}
 S_p \supset -\alpha' T_p \int d^{p+1} \sigma \sqrt{-\det(\tilde{G}_{ab})} \Big[&T_1 F_{ae} F_{bf} F_c^g F_{dg} D^a F^{bc} D^d F^{ef} \\
 &+T_2 F_{ab} F_c^g F_{dg} F_{ef} D^a F^{bc} D^d F^{ef} + T_3 F_a^f F_c^g F_{dg} F_{ef} D^a F^{bc} D^d F_b^e \\
 &+T_4 F_a^f F_c^g F_{df} F_{eg} D^a F^{bc} D_b F^{de} + T_5 F_a^f F_{cf} F_d^g F_{eg} D^a F^{bc} D^d F_b^e \\
 &+T_6 F_{ae} F_{bd} F_c^g F_{fg} D^a F^{bc} D^d F^{ef} + T_7 F_a^f F_c^g F_{de} F_{fg} D^a F^{bc} D_b F^{de} \\
 &+T_8 F_{ab} F_{ce} F_d^g F_{fg} D^a F^{bc} D^d F^{ef} + T_9 F_a^f F_{ce} F_d^g F_{fg} D^a F^{bc} D^d F_b^e \\
 &+T_{10} F_{ad} F_c^f F_e^g F_{fg} D^a F^{bc} D_b F^{de} + T_{11} F_{ae} F_{cd} F_{fg} F^{fg} D^a F^{bc} D^d F_b^e \\
 &+T_{12} F_{ad} F_{ce} F_{fg} F^{fg} D^a F^{bc} D_b F^{de} + T_{13} F_{ac} F_{de} F_{fg} F^{fg} D^a F^{bc} D_b F^{de} \Big] \tag{16}
 \end{aligned}$$

The coefficients T_1, \dots, T_{13} are 13 parameters that we are going to find them by the T-duality constraint.

There are 12 independent terms with structure $FFFF\Omega\Omega$. The terms that have trace of the second fundamental form may be eliminated by appropriate field redefinitions. The remaining terms are

$$\begin{aligned}
 S_p \supset -\alpha' T_p \int d^{p+1} \sigma \sqrt{-\det(\tilde{G}_{ab})} \Big[&W_1 F_a^b F^{af} F_c^e F^{cd} \Omega_{fd}^i \Omega_{bei} \\
 &+W_2 F_a^b F^{af} F_f^c F^{de} \Omega_{bd}^i \Omega_{cei} + W_3 F_{af} F^{af} F^{bc} F^{de} \Omega_{bd}^i \Omega_{cei} \\
 &+W_4 F_a^b F^{af} F_c^e F^{cd} \Omega_{fb}^i \Omega_{dei} + W_5 F_a^b F^{af} F_f^c F_b^d \Omega_c^e \Omega_{dei} \\
 &+W_6 F_{af} F^{af} F_b^d F^{bc} \Omega_c^e \Omega_{dei} + W_7 F_a^b F^{af} F_f^c F_{bc} \Omega_{dei} \Omega^{dei} \\
 &+W_8 F_{af} F^{af} F_{bc} F^{bc} \Omega_{dei} \Omega^{dei} \Big] \tag{17}
 \end{aligned}$$

The coefficients W_1, \dots, W_8 are 8 parameters that we are going to find them by the T-duality constraint. The parameters T_1, \dots, T_{13} and W_1, \dots, W_8 may also be found from studying the S-matrix element of six open string vertex operators. However, as far as we know, because of the very lengthy calculations involved in the S-matrix elements, these coefficients have not been found in the literature.

At the level of six extra F 's, one finds the following 37 independent couplings for $DFDF$:

$$\begin{aligned}
 S_p \supset -\alpha' T_p \int d^{p+1} \sigma \sqrt{-\det(\tilde{G}_{ab})} [& Z_1 F_{de} F^{de} F_{fg} F^{fg} F_{hu} F^{hu} D^a F^{bc} D_b F_{ac} \\
 & + Z_2 F_a^g F_{bg} F_c^h F_d^u F_{eh} F_{fu} D^a F^{bc} D^d F^{ef} + Z_3 F_{ae} F_b^g F_c^h F_d^u F_{fh} F_{gh} D^a F^{bc} D^d F^{ef} \\
 & + Z_4 F_a^g F_{be} F_c^h F_d^u F_{fh} F_{gu} D^a F^{bc} D^d F^{ef} + Z_5 F_a^f F_c^g F_d^h F_e^u F_{fh} F_{gu} D^a F^{bc} D^d F_b^e \\
 & + Z_6 F_b^f F_c^g F_d^h F_e^u F_{fh} F_{gu} D^a F^{bc} D_a F^{de} + Z_7 F_{ad} F_b^g F_c^h F_{eg} F_f^u F_{hu} D^a F^{bc} D^d F^{ef} \\
 & + Z_8 F_a^f F_c^g F_{df} F_e^h F_g^u F_{hu} D^a F^{bc} D_b F^{de} + Z_9 F_a^f F_c^g F_{df} F_e^h F_g^u F_{hu} D^a F^{bc} D^d F_b^e \\
 & + Z_{10} F_a^f F_{cf} F_d^g F_e^h F_g^u F_{hu} D^a F^{bc} D^d F_b^e + Z_{11} F_{ae} F_{bd} F_c^g F_f^h F_g^u F_{hu} D^a F^{bc} D^d F^{ef} \\
 & + Z_{12} F_{ad} F_{be} F_c^g F_f^h F_g^u F_{hu} D^a F^{bc} D^d F^{ef} + Z_{13} F_{ab} F_c^g F_{de} F_f^h F_g^u F_{hu} D^a F^{bc} D^d F^{ef} \\
 & + Z_{14} F_{ab} F_{ce} F_d^g F_f^h F_g^u F_{hu} D^a F^{bc} D^d F^{ef} + Z_{15} F_{ad} F_c^f F_e^g F_f^h F_g^u F_{hu} D^a F^{bc} D_b F^{de} \\
 & + Z_{16} F_{ad} F_c^f F_e^g F_f^h F_g^u F_{hu} D^a F^{bc} D^d F_b^e + Z_{17} F_{bd} F_c^f F_e^g F_f^h F_g^u F_{hu} D^a F^{bc} D_a F^{de} \\
 & + Z_{18} F_c^e F_d^f F_e^g F_f^h F_g^u F_{hu} D^a F^{bc} D_a F_b^d + Z_{19} F_c^e F_d^f F_e^g F_f^h F_g^u F_{hu} D^a F^{bc} D_b F_a^d \\
 & + Z_{20} F_d^f F^{de} F_e^g F_f^h F_g^u F_{hu} D^a F^{bc} D_b F_{ac} + Z_{21} F_{ad} F_{ce} F_f^h F^{fg} F_g^u F_{hu} D^a F^{bc} D^d F_b^e \\
 & + Z_{22} F_{bd} F_{ce} F_f^h F^{fg} F_g^u F_{hu} D^a F^{bc} D_a F^{de} + Z_{23} F_c^e F_{de} F_f^h F^{fg} F_g^u F_{hu} D^a F^{bc} D_a F_b^d \\
 & + Z_{24} F_c^e F_{de} F_f^h F^{fg} F_g^u F_{hu} D^a F^{bc} D_b F_a^d + Z_{25} F_{de} F^{de} F_f^h F^{fg} F_g^u F_{hu} D^a F^{bc} D_b F_{ac} \\
 & + Z_{26} F_a^f F_c^g F_{df} F_{eg} F_{hu} F^{hu} D^a F^{bc} D^d F_b^e + Z_{27} F_b^f F_c^g F_{df} F_{eg} F_{hu} F^{hu} D^a F^{bc} D_a F^{de} \\
 & + Z_{28} F_a^f F_{cf} F_d^g F_{eg} F_{hu} F^{hu} D^a F^{bc} D^d F_b^e + Z_{29} F_{ae} F_{bd} F_c^g F_{fg} F_{hu} F^{hu} D^a F^{bc} D^d F^{ef} \\
 & + Z_{30} F_{ad} F_{be} F_c^g F_{fg} F_{hu} F^{hu} D^a F^{bc} D^d F^{ef} + Z_{31} F_{ab} F_c^g F_{de} F_{fg} F_{hu} F^{hu} D^a F^{bc} D^d F^{ef} \\
 & + Z_{32} F_a^f F_c^g F_{de} F_{fg} F_{hu} F^{hu} D^a F^{bc} D_b F^{de} + Z_{33} F_c^e F_d^f F_e^g F_{fg} F_{hu} F^{hu} D^a F^{bc} D_a F_b^d \\
 & + Z_{34} F_c^e F_d^f F_e^g F_{fg} F_{hu} F^{hu} D^a F^{bc} D_b F_a^d + Z_{35} F_{ad} F_{ce} F_{fg} F^{fg} F_{hu} F^{hu} D^a F^{bc} D^d F_b^e \\
 & + Z_{36} F_c^e F_{de} F_{fg} F^{fg} F_{hu} F^{hu} D^a F^{bc} D_a F_b^d + Z_{37} F_c^e F_{de} F_{fg} F^{fg} F_{hu} F^{hu} D^a F^{bc} D_b F_a^d]
 \end{aligned}$$

(18)

And the following 16 couplings for $\Omega\Omega$:

$$\begin{aligned}
 S_p \supset -\alpha' T_p \int d^{p+1} \sigma \sqrt{-\det(\tilde{G}_{ab})} [& Y_1 F_a^b F^{ac} F_c^d F_e^f F^{eg} F_g^h \Omega_{bf}^i \Omega_{dhi} \\
 & + Y_2 F_a^b F^{ac} F_c^d F_b^e F_f^g F^{fh} \Omega_{dh}^i \Omega_{egi} + Y_3 F_{ab} F^{ab} F_c^d F^{ce} F_f^g F^{fh} \Omega_{eh}^i \Omega_{dgi} \\
 & + Y_4 F_a^b F^{ac} F_c^d F_b^e F_d^f F^{gh} \Omega_{eg}^i \Omega_{fhi} + Y_5 F_{ab} F^{ab} F_c^d F^{ce} F_e^f F^{gh} \Omega_{dg}^i \Omega_{fhi} \\
 & + Y_6 F_a^b F^{ac} F_c^d F_{bd} F^{ef} F^{gh} \Omega_{eg}^i \Omega_{fhi} + Y_7 F_{ab} F^{ab} F_{cd} F^{cd} F^{ef} F^{gh} \Omega_{eg}^i \Omega_{fhi} \\
 & + Y_8 F_a^b F^{ac} F_c^d F_b^e F_f^g F^{fh} \Omega_{de}^i \Omega_{hgi} + Y_9 F_{ab} F^{ab} F_c^d F^{ce} F_f^g F^{fh} \Omega_{ed}^i \Omega_{hgi} \\
 & + Y_{10} F_a^b F^{ac} F_c^d F_b^e F_d^f F_e^g \Omega_f^h \Omega_{ghi} + Y_{11} F_{ab} F^{ab} F_c^d F^{ce} F_e^f F_d^g \Omega_f^h \Omega_{ghi} \\
 & + Y_{12} F_a^b F^{ac} F_c^d F_{bd} F_e^f F^{eg} \Omega_g^h \Omega_{fhi} + Y_{13} F_{ab} F^{ab} F_{cd} F^{cd} F_e^f F^{eg} \Omega_g^h \Omega_{fhi} \\
 & + Y_{14} F_a^b F^{ac} F_c^d F_b^e F_d^f F_{ef} \Omega_{ghi} \Omega^{ghi} + Y_{15} F_{ab} F^{ab} F_c^d F^{ce} F_e^f F_{df} \Omega_{ghi} \Omega^{ghi} \\
 & + Y_{16} F_{ab} F^{ab} F_{cd} F^{cd} F_{ef} F^{ef} \Omega_{ghi} \Omega^{ghi}]
 \end{aligned}$$

(19)

The coefficients Z_1, \dots, Z_{37} and Y_1, \dots, Y_{16} are 53 parameters that we are going to find them by the T-duality constraint. This construction of independent terms can be used to find higher order couplings in which we are not interested in this paper. We will show in the next section that almost all parameters in the above couplings can be fixed by the T-duality constraint except two of them.

3. T-duality constraint

The T-duality relates the bosonic string theory compactified on a circle with radius ρ to the same theory compactified on another circle with radius $\frac{\alpha'}{\rho}$. It relates the tension of D_p -brane to the tension of D_{p-1} -brane or D_{p+1} -brane, depending on whether the original D_p -brane is along or orthogonal to the circle, respectively. Assuming the world-volume couplings of the D_p -brane in the string frame are independent of p , we expect the T-duality also relates the world-volume effective action of D_p -brane to the effective action of D_{p-1} -brane or D_{p+1} -brane, *i.e.*,

$$S_{D_p} \xrightarrow{T} S_{D_{p\pm 1}} \tag{20}$$

This action can be expanded at low energy, *i.e.*,

$$S_{D_p} = \sum_{n=0}^{\infty} (\alpha')^n S_{D_p}^{(n)} \tag{21}$$

At order α'^0 the action is given by the DBI action (1). At order α' , there are infinite number of couplings depending on the number of extra F 's in $DFDF$ and $\Omega\Omega$ couplings. At zero extra F , it is given by (3), at two extra F 's it is given by (14), at four extra F 's it is given by (16) and (17), and so on. We are not interested in this paper in the couplings at order α' with eight and higher extra F 's, and on the couplings at higher orders of α' .

When the T-duality transformations act along the killing coordinate y , and the y -direction is a world-volume, then the transformations at the leading order of α' are:

$$\begin{aligned} A^y &\xrightarrow{T^{(0)}} \chi^y \\ A^{\tilde{a}} &\xrightarrow{T^{(0)}} A^{\tilde{a}}, \quad \chi^i \xrightarrow{T^{(0)}} \chi^i \end{aligned} \tag{22}$$

where \tilde{a} is the world-volume index which does not include the y -direction. These transformations are expected to receive α' corrections. That is, the T-duality operator has an α' expansion:

$$T = \sum_{n=0}^{\infty} (\alpha')^n T^{(n)} \tag{23}$$

where $T^{(0)}$ is the transformation (22).

The invariance of the effective actions at order $(\alpha')^0$ then means that

$$S_{D_p}^{(0)} \xrightarrow{T^{(0)}} S_{D_{p-1}}^{(0)} \tag{24}$$

where $S_{D_p}^{(0)}$ is the reduction of D_p -brane action at order α'^0 on the circle. At order α' , the action has two terms, *i.e.*, $S_{D_p} = S_{D_p}^{(0)} + \alpha' S_{D_p}^{(1)}$. The invariance then means

$$\begin{aligned}
 S_{D_p}^{(1)} &\xrightarrow{T^{(0)}} S_{D_{p-1}}^{(1)} + \delta S^{(1)} \\
 S_{D_p}^{(0)} &\xrightarrow{T^{(0)+T^{(1)}}} S_{D_{p-1}}^{(0)} + \delta S'^{(1)} + \dots
 \end{aligned}
 \tag{25}$$

where dots represent terms at higher orders of α' . The above relation indicates that the extra term $\delta S^{(1)}$ which is produced by applying the T-duality transformation (22) on the reduction of action $S_{D_p}^{(1)}$ on the circle, should be canceled by applying the T-duality transformations at order α' on the reduction of the action $S_{D_p}^{(0)}$. Since the transformations are on the actions, one may add total derivative terms $J^{(1)}$ to make the cancellation happens. That is why we call the α' order term in the second line of (25) to be $\delta S'^{(1)}$, i.e., $\delta S'^{(1)} + \delta S^{(1)} + J^{(1)} = 0$. Note that $\delta S^{(1)}$ contain only terms which involve χ^y , so the corrections to the T-duality transformations and the total derivative terms in $J^{(1)}$ should include only terms which contain χ^y . Similar T-duality transformations exist for the effective actions at the higher orders of α' .

Since the T-duality transformations affect A_a and χ^i , it is convenient to expand the effective action, the T-duality transformations and total derivative terms at order α'^n in terms of powers of F and $\partial\chi$ as well,⁵ i.e.,

$$\begin{aligned}
 S_{D_p}^{(n)} &= \sum_{m=0}^{\infty} S_{D_p}^{(m,n)} \\
 T^{(n)} &= \sum_{m=0}^{\infty} T^{(m,n)} \\
 J^{(n)} &= \sum_{m=0}^{\infty} J^{(m,n)}
 \end{aligned}
 \tag{26}$$

where m is the power of F , ∂F , $\partial\chi$, $\partial\partial\chi$ in $S_{D_p}^{(m,n)}$ and $J^{(m,n)}$, and it is the extra power of F and $\partial\chi$ on the right hand side of the T-duality transformation $T^{(m,n)}$. For example, for $m = 2$ the action at order α'^0 is

$$S_{D_p}^{(2,0)} = -T_p \int d^{p+1}\sigma \left[\frac{1}{4} F_{ab} F_{cd} \eta^{ac} \eta^{bd} + \frac{1}{2} \partial_a \chi^i \partial_b \chi^j \eta_{ij} \eta^{ab} \right]
 \tag{27}$$

and the T-duality transformation is $T^{(2,0)} = 0$. In fact, $T^{(0,0)}$ is given by (22) and $T^{(m,0)} = 0$ for $m \neq 0$. The transformation $T^{(m,1)}$ is

$$\begin{aligned}
 A^y &\xrightarrow{T^{(m,1)}} \alpha' (\delta\chi^y)^{(m,1)} \\
 A^{\tilde{a}} &\xrightarrow{T^{(m,1)}} \alpha' (\delta A^{\tilde{a}})^{(m,1)}, \quad \chi^i \xrightarrow{T^{(m,1)}} \alpha' (\delta\chi^i)^{(m,1)}
 \end{aligned}
 \tag{28}$$

where $(\delta\chi^y)^{(m,1)}$, $(\delta\chi^i)^{(m,1)}$, $(\delta A^{\tilde{a}})^{(m,1)}$ are all contractions of one $\partial\partial\chi^y$, $\partial\partial\chi^i$ or ∂F and m number of F , $\partial\chi^y$ or $\partial\chi^i$ with arbitrary parameters. Each term should have at least one χ^y . We expect these parameters to be found by the T-duality constraint.

⁵ Using the transformations (22), one may find the T-duality transformations of the covariant objects F , DF , \tilde{G} and Ω . Then one may find the α' corrections to these objects by using the T-duality constraint. In this paper, however, we use perturbation to rewrite the covariant action in terms of F and $\partial\chi$ and then use the T-duality transformations (22) and their corresponding α' -corrections.

The invariance of the effective actions at order $(\alpha')^0$ then means

$$S_{D_p}^{(m,0)} \xrightarrow{T^{(0,0)}} S_{D_{p-1}}^{(m,0)} \tag{29}$$

for any number of m . Using the T-duality transformation (22), one finds that the transformation (29) is satisfied for any number of m . That means the DBI action is covariant under the T-duality transformation (22), as expected.

The invariance at order α' means

$$\begin{aligned} S_{D_p}^{(m,1)} &\xrightarrow{T^{(0,0)}} S_{D_{p-1}}^{(m,1)} + \delta S^{(m,1)} \\ S_{D_p}^{(n,0)} &\xrightarrow{T^{(0,0)} + T^{(m-n,1)}} S_{D_{p-1}}^{(n,0)} + \delta S_n^{(m,1)} + \dots \end{aligned}$$

where $2 \leq n \leq m - 2$, and dots represent terms at higher orders of α' . Adding total derivative terms at order $J^{(m,1)}$, one finds the T-duality constraint

$$\sum_{n=2}^{m-2} \delta S_n^{(m,1)} + \delta S^{(m,1)} + J^{(m,1)} = 0 \tag{30}$$

There are similar constraints for the couplings at higher orders of α' .

The above constraint may be used at each level of m to fix the parameters of independent couplings that we have found in the previous section. The simplest case is the action at the level of $m = 2$. Since we have chosen the coefficient in (3) to make no correction to the propagator, $S_{D_p}^{(2,1)}$ is a total derivative term. Hence, the T-duality constraint does not fix the parameter C in this action. However, one expects it should be related to all other parameters at orders $m > 2$, because this coefficient appears in all couplings with $m \geq 2$.

3.1. Two extra F 's

At order α' , and at the level of $m = 4$, there are two contributions to the action $S_{D_p}^{(4,1)}$. One contribution is coming from (3) and the other one from (14). The parameters $C, C_1, C_2, C_3, N_1, N_2, N_3$ appear in $S_{D_p}^{(4,1)}$. Then one should reduce it on the circle along the y -direction and use the T-duality transformation (22). Then one should compare the result with $S_{D_{p-1}}^{(4,1)}$. One finds

$$S_{D_p}^{(4,1)} \xrightarrow{T^{(0,0)}} S_{D_{p-1}}^{(4,1)} + \delta S^{(4,1)} \tag{31}$$

where $\delta S^{(4,1)}$ contains some non-zero terms at the level of $m = 4$ which includes all parameters $C, C_1, C_2, C_3, N_1, N_2, N_3$. They can not be canceled even by total derivative terms. This indicates that the T-duality transformations (22) at order α'^0 must receive α' corrections if the parameters $C, C_1, C_2, C_3, N_1, N_2, N_3$ are non-zero.

Since we have chosen the couplings (3) to have no corrections to the propagators, we expect the α' -corrections to the T-duality transformations (22) have no linear term. This stems from the fact that the S-matrix elements in string theory which have standard propagators, satisfy the Ward identity corresponding to the T-duality [12]. In other words, the field theory with standard propagators, should have no α' -correction to the T-duality transformations at the linear order, *i.e.*,

$$T^{(0,n)} = 0 ; n > 0 \tag{32}$$

Hence, the corrections to the T-duality transformations (22) are at orders $T^{(2,1)}$, $T^{(4,1)}$, $T^{(6,1)}$, \dots , $T^{(2,2)}$, $T^{(4,2)}$, $T^{(6,2)}$, \dots , and so on.

Therefore, the extra terms in $\delta S^{(4,1)}$ should be canceled by the T-duality transformation $T^{(2,1)}$ on the reduction of the action $S_{D_p}^{(2,0)}$ in (27), i.e.,

$$S_{D_p}^{(2,0)} \xrightarrow{T^{(0,0)}+T^{(2,1)}} S_{D_{p-1}}^{(2,0)} + \delta S_2^{(4,1)} + \delta S_2^{(6,2)} \tag{33}$$

where $\delta S_2^{(6,2)}$ contains some non-zero terms at order α'^2 and at the level of $m = 6$ in which we are not interested. The reduction of (27) is

$$S_{D_p}^{(2,0)} = -T_p(2\pi\rho) \int d^p\sigma \left[\frac{1}{2}\partial_{\tilde{a}}A_y\partial^{\tilde{a}}A^y + \frac{1}{2}\partial_{\tilde{a}}\chi_i\partial^{\tilde{a}}\chi^i - \frac{1}{2}\partial_{\tilde{a}}A_{\tilde{b}}\partial^{\tilde{b}}A^{\tilde{a}} + \frac{1}{2}\partial_{\tilde{b}}A_{\tilde{a}}\partial^{\tilde{b}}A^{\tilde{a}} \right] \tag{34}$$

The T-duality transformation $T^{(2,1)}$ for $(\delta\chi^y)^{(2,1)}$, $(\delta A^{\tilde{a}})^{(2,1)}$, $(\delta\chi^i)^{(2,1)}$ are all contractions of the following expressions by the flat metric $\eta^{\tilde{a}\tilde{b}}$ and with arbitrary coefficients:

$$\begin{aligned} (\delta\chi^y)^{(2,1)} &\sim \partial_{\tilde{a}}\partial_{\tilde{b}}\chi^y\partial_{\tilde{c}}\chi^y\partial_{\tilde{d}}\chi^y + \partial_{\tilde{a}}\partial_{\tilde{b}}\chi^y\partial_{\tilde{c}}\chi^i\partial_{\tilde{d}}\chi^j\eta_{ij} + \partial_{\tilde{a}}\partial_{\tilde{b}}\chi^i\partial_{\tilde{c}}\chi^j\partial_{\tilde{d}}\chi^y\eta_{ij} \\ &\quad + \partial_{\tilde{a}}\partial_{\tilde{b}}\chi^yF_{\tilde{c}\tilde{d}}F_{\tilde{e}\tilde{f}} + \partial_{\tilde{a}}\chi^yF_{\tilde{b}\tilde{c}}\partial_{\tilde{d}}F_{\tilde{e}\tilde{f}}, \\ (\delta A^{\tilde{a}})^{(2,1)} &\sim \partial_{\tilde{a}}\chi^y\partial_{\tilde{b}}\chi^y\partial_{\tilde{c}}F_{\tilde{d}\tilde{e}} + \partial_{\tilde{a}}\partial_{\tilde{b}}\chi^y\partial_{\tilde{c}}\chi^yF_{\tilde{d}\tilde{e}}, \\ (\delta\chi^i)^{(2,1)} &\sim \partial_{\tilde{a}}\partial_{\tilde{b}}\chi^y\partial_{\tilde{c}}\chi^y\partial_{\tilde{d}}\chi^i + \partial_{\tilde{a}}\partial_{\tilde{b}}\chi^i\partial_{\tilde{c}}\chi^y\partial_{\tilde{d}}\chi^y. \end{aligned} \tag{35}$$

Since the contractions involve derivatives of the field strength, one should impose the Bianchi identity $\partial_{[\tilde{a}}F_{\tilde{b}\tilde{c}]} = 0$ to find independent terms. We impose this identity at the end after finding the parameters by the T-duality constraint. Applying the above T-duality transformations on (33), one can find $\delta S^{(4,1)}$ which contains the arbitrary parameters in (35). To compare it with $\delta S^{(4,1)}$ in (31), one should also take into account the total derivative terms.

The total derivative terms can be written as

$$J^{(4,1)} = -T_{p-1} \int d^p\sigma \eta^{\tilde{a}\tilde{b}}\partial_{\tilde{a}}I_{\tilde{b}}^{(4,1)} \tag{36}$$

where $I_{\tilde{b}}^{(4,1)}$ is all contractions with arbitrary parameters of the following expression with $\eta^{\tilde{a}\tilde{b}}$:

$$\begin{aligned} &\partial_{\tilde{a}}\partial_{\tilde{b}}\chi^y\partial_{\tilde{c}}\chi^y\partial_{\tilde{d}}\chi^y\partial_{\tilde{e}}\chi^y + \partial_{\tilde{a}}\partial_{\tilde{b}}\chi^i\partial_{\tilde{c}}\chi^j\partial_{\tilde{d}}\chi^y\partial_{\tilde{e}}\chi^y\eta_{ij} + \partial_{\tilde{a}}\chi^i\partial_{\tilde{b}}\chi^j\partial_{\tilde{c}}\partial_{\tilde{d}}\chi^y\partial_{\tilde{e}}\chi^y\eta_{ij} \\ &\quad + \partial_{\tilde{a}}\partial_{\tilde{b}}\chi^y\partial_{\tilde{c}}\chi^yF_{\tilde{d}\tilde{e}}F_{\tilde{f}\tilde{g}} + \partial_{\tilde{a}}\chi^y\partial_{\tilde{b}}\chi^yF_{\tilde{c}\tilde{d}}\partial_{\tilde{e}}F_{\tilde{f}\tilde{g}} \end{aligned} \tag{37}$$

Note that all terms above and the terms in (35) involve χ^y .

The T-duality constraint

$$\delta S^{(4,1)} + \delta S^{(4,1)} + J^{(4,1)} = 0 \tag{38}$$

Then gives some algebraic equations between the effective action parameters, the parameters in (35) and the parameters in the total derivative terms. On general ground, we do not expect the T-duality constraint fixes the overall coefficients of the T-dual multiplets. We choose $C = 1$ which is fixed by the S-matrix calculation. Then if there is only one T-dual multiplet, its overall coefficient then should be fixed. The solution to the above equation produces the following relations between the effective action parameters $C_1, C_2, C_3, N_1, N_2, N_3$:

$$\begin{aligned}
 C_2 &\rightarrow 1, & C_1 &\rightarrow 2 + 24N_1, & C_3 &\rightarrow -\frac{1}{4}, \\
 N_3 &\rightarrow -4N_1, & N_2 &\rightarrow -1 - 16N_1
 \end{aligned}
 \tag{39}$$

where the parameter N_1 remain arbitrary. This indicates that there are two T-dual multiplets, one multiplet with the overall coefficient $C = 1$ and the second one with the overall coefficient N_1 . As we will see, even though the parameter N_1 appears in the T-duality constraint at the levels $m > 4$, the T-duality constraint at the levels of $m = 6, 8$ that we have done the calculations, can not fix this parameter. The above parameters are consistent with the S-matrix calculation results (15), *i.e.*, if we choose the overall coefficient of the second multiplet to be $N_1 = -1/24$, then the above parameters become exactly the S-matrix results in (15).

The algebraic equations at the level of $m = 4$, gives the following α' -corrections to the T-duality transformations:

$$\begin{aligned}
 A^y \xrightarrow{T^{(2,1)}} &\alpha' [E_1 F^{\tilde{b}\tilde{c}} \partial_{\tilde{a}} F_{\tilde{b}\tilde{c}} \partial^{\tilde{a}} \chi^y - (1 + 12N_1) \partial_{\tilde{a}} \chi^y \partial^{\tilde{a}} \chi_y \partial_{\tilde{b}} \partial^{\tilde{b}} \chi^y \\
 &+ E_3 \partial_{\tilde{a}} \chi^i \partial^{\tilde{a}} \chi^y \partial_{\tilde{b}} \partial^{\tilde{b}} \chi_i - (1 + 24N_1) \partial^{\tilde{a}} \chi^y \partial_{\tilde{b}} \partial_{\tilde{a}} \chi_y \partial^{\tilde{b}} \chi^y \\
 &+ 2E_1 \partial^{\tilde{a}} \chi^y F^{\tilde{b}\tilde{c}} \partial_{\tilde{c}} F_{\tilde{a}\tilde{b}} + E_2 \partial^{\tilde{a}} \chi^y F_{\tilde{a}}^{\tilde{b}} \partial_{\tilde{c}} F_{\tilde{b}}^{\tilde{c}} \\
 &- (2 + 24N_1) F_{\tilde{a}}^{\tilde{c}} F^{\tilde{a}\tilde{b}} \partial_{\tilde{c}} \partial_{\tilde{b}} \chi^y + (\frac{1}{4} + 2N_1) \partial_{\tilde{c}} \partial^{\tilde{c}} \chi^y F_{\tilde{a}\tilde{b}} F^{\tilde{a}\tilde{b}}] \\
 \tilde{A}^{\tilde{a}} \xrightarrow{T^{(2,1)}} &\alpha' [-4N_1 \partial_{\tilde{b}} \chi^y \partial^{\tilde{b}} \chi_y \partial_{\tilde{c}} F^{\tilde{a}\tilde{c}} + (1 + 16N_1) \partial^{\tilde{a}} \chi^y \partial^{\tilde{b}} \chi_y \partial_{\tilde{c}} F_{\tilde{b}}^{\tilde{c}} \\
 &+ (3 + 40N_1) \partial_{\tilde{c}} \partial^{\tilde{a}} \chi_y \partial^{\tilde{b}} \chi^y F_{\tilde{b}}^{\tilde{c}} + (2 + 32N_1 - E_2) \partial_{\tilde{c}} \partial^{\tilde{c}} \chi_y \partial^{\tilde{b}} \chi^y F_{\tilde{b}}^{\tilde{a}} \\
 &+ (1 + 24N_1) \partial^{\tilde{b}} \chi^y \partial^{\tilde{c}} \chi_y \partial_{\tilde{c}} F_{\tilde{b}}^{\tilde{a}}] \\
 \chi^i \xrightarrow{T^{(2,1)}} &\alpha' [-E_3 \partial_{\tilde{a}} \chi^i \partial^{\tilde{a}} \chi^y \partial_{\tilde{b}} \partial^{\tilde{b}} \chi_y + \partial^{\tilde{a}} \chi^y \partial_{\tilde{b}} \partial_{\tilde{a}} \chi^i \partial^{\tilde{b}} \chi_y \\
 &- \frac{1}{2} \partial_{\tilde{a}} \chi^y \partial^{\tilde{a}} \chi_y \partial_{\tilde{b}} \partial^{\tilde{b}} \chi^i]
 \end{aligned}
 \tag{40}$$

where E_1, E_2 and E_3 are three other arbitrary parameters. However, the terms with coefficient E_1 cancels by using the Bianchi identity $\partial_{[\tilde{a}} F_{\tilde{b}\tilde{c}]} = 0$. So one can set $E_1 = 0$. The other two parameters may be fixed by studying the T-duality constraint at order $S^{(6,2)}$. Note that the above transformations are non-zero for any values for the parameters E_2, E_3, N_1 . Hence, the T-duality constraint forces the leading order T-duality transformations (22) to receive higher derivative corrections.

If we have used the field redefinition freedom to remove the first term in (3), the constraint (39) would not change, however, there would be a linear term $\partial \partial \chi^y$ in the T-duality transformation of A^y and the coefficients of all terms in (40) would also change. The reason is that the T-duality transformations (40) are in fact the field redefinitions in the reduced space. The field redefinitions depends on whether or not we keep the first term in (3).

3.2. Four extra F 's

At the order α' and at the level of $m = 6$, there are three contributions to the action $S_{D_p}^{(6,1)}$. One contribution is coming from (3), another one is coming from (14) and the last one is coming from the couplings in (16) and (17). The parameter N_1 which has not been fixed in (39) and the

parameters T_1, \dots, T_{13} and W_1, \dots, W_8 appear in $S_{D_p}^{(6,1)}$. One should reduce $S_{D_p}^{(6,1)}$ on the circle along the y -direction and use the T-duality transformation (22). Then one should compare the result with $S_{D_{p-1}}^{(6,1)}$. One finds

$$S_{D_p}^{(6,1)} \xrightarrow{T^{(0,0)}} S_{D_{p-1}}^{(6,1)} + \delta S^{(6,1)} \tag{41}$$

where $\delta S^{(6,1)}$ contains some non-zero terms at the level of $m = 6$ which includes all above parameters. Each term in $\delta S^{(6,1)}$ has the scalar field χ^y .

The extra terms in $\delta S^{(6,1)}$ should be canceled by the T-duality transformation $T^{(4,1)}$ on the reduction of the action $S_{D_p}^{(2,0)}$ in (34), and by the T-duality transformation $T^{(2,1)}$ in (40) on the reduction of the action $S_{D_p}^{(4,0)}$, i.e.,

$$\begin{aligned} S_{D_p}^{(2,0)} &\xrightarrow{T^{(0,0)}+T^{(4,1)}} S_{D_{p-1}}^{(2,0)} + \delta S_2^{(6,1)} + \delta S_2^{(10,2)} \\ S_{D_p}^{(4,0)} &\xrightarrow{T^{(0,0)}+T^{(2,1)}} S_{D_{p-1}}^{(4,0)} + \delta S_4^{(6,1)} + \delta S_4^{(8,2)} + \delta S_4^{(10,3)} + \delta S_4^{(12,4)} \end{aligned} \tag{42}$$

where $\delta S_2^{(10,2)}$, $\delta S_4^{(8,2)}$, $\delta S_4^{(10,3)}$ and $\delta S_4^{(12,4)}$ contains some non-zero terms at higher orders of α' in which we are not interested. It is straightforward to extract the action $S_{D_p}^{(4,0)}$ from the DBI action (1) and then reduce it on the circle along the y -direction. The T-duality transformation $T^{(2,1)}$ is given in (40), and the T-duality transformation $T^{(4,1)}$ for $(\delta\chi^y)^{(4,1)}$, $(\delta A^{\tilde{a}})^{(4,1)}$, $(\delta\chi^i)^{(4,1)}$ are all contractions of the following expressions by the flat metric $\eta^{\tilde{a}\tilde{b}}$ and with arbitrary coefficients:

$$\begin{aligned} (\delta\chi^y)^{(4,1)} &\sim \partial\partial\chi^y\partial\chi^y\partial\chi^y\partial\chi^y + \partial\partial\chi^y\partial\chi^y\partial\chi^y\partial\chi + \partial\partial\chi\partial\chi\partial\chi^y\partial\chi^y\partial\chi^y \\ &\quad + \partial\partial\chi^y\partial\chi\partial\chi FF + \partial\partial\chi^y\partial\chi\partial\chi\partial\chi\partial\chi + \partial\partial\chi\partial\chi\partial\chi\partial\chi\partial\chi^y + \partial\partial\chi^y FFFF \\ &\quad + \partial\chi^y FFF\partial F + \partial\partial\chi\partial\chi\partial\chi^y FF + \partial\chi^y\partial\chi\partial\chi F\partial F + \partial\partial\chi^y\partial\chi^y\partial\chi^y FF \\ &\quad + \partial\chi^y\partial\chi^y\partial\chi^y F\partial F, \\ (\delta A^{\tilde{a}})^{(4,1)} &\sim \partial\chi^y\partial\chi^y FF\partial F + \partial\partial\chi^y\partial\chi^y FFF + \partial\chi^y\partial\chi^y\partial\chi\partial\chi\partial F + \partial\partial\chi^y\partial\chi^y\partial\chi\partial\chi F \\ &\quad + \partial\partial\chi\partial\chi\partial\chi^y\partial\chi^y F + \partial\chi^y\partial\chi^y\partial\chi^y\partial\chi^y\partial F + \partial\partial\chi^y\partial\chi^y\partial\chi^y\partial\chi^y F, \\ (\delta\chi^i)^{(4,1)} &\sim \partial\partial\chi\partial\chi^y\partial\chi^y\partial\chi^y\partial\chi^y + \partial\partial\chi^y\partial\chi^y\partial\chi^y\partial\chi^y\partial\chi + \partial\partial\chi\partial\chi\partial\chi\partial\chi^y\partial\chi^y \\ &\quad + \partial\partial\chi^y\partial\chi^y\partial\chi\partial\chi\partial\chi + \partial\partial\chi\partial\chi^y\partial\chi^y FF + \partial\partial\chi^y\partial\chi^y\partial\chi FF \\ &\quad + \partial\chi^y\partial\chi^y\partial\chi F\partial F \end{aligned} \tag{43}$$

where ∂ and F have $(\tilde{a}, \tilde{b}, \tilde{c}, \dots)$ indices and χ has (i, j, k, \dots) indices.

We have to also consider total derivative terms, i.e.,

$$J^{(6,1)} = -T_{p-1} \int d^p \sigma \eta^{\tilde{a}\tilde{b}} \partial_{\tilde{a}} I_{\tilde{b}}^{(6,1)} \tag{44}$$

where $I_{\tilde{b}}^{(6,1)}$ is all contractions with arbitrary parameters of the following expression with $\eta^{\tilde{a}\tilde{b}}$:

$$\begin{aligned} &\partial\partial\chi^y\partial\chi^y\partial\chi^y\partial\chi^y\partial\chi^y + \partial\partial\chi^y\partial\chi^y\partial\chi^y\partial\chi^y\partial\chi\partial\chi + \partial\partial\chi\partial\chi\partial\chi^y\partial\chi^y\partial\chi^y\partial\chi^y \\ &\quad + \partial\partial\chi\partial\chi\partial\chi\partial\chi\partial\chi^y\partial\chi^y + \partial\partial\chi^y\partial\chi^y\partial\chi\partial\chi\partial\chi\partial\chi + \partial\partial\chi^y\partial\chi^y\partial\chi\partial\chi FF \end{aligned}$$

$$\begin{aligned}
 & +\partial\partial\chi\partial\chi\partial\chi^y\partial\chi^yFF + \partial\chi^y\partial\chi^y\partial\chi\partial\chi F\partial F + \partial\partial\chi^y\partial\chi^y\partial\chi^y\partial\chi^yFF \\
 & +\partial\chi^y\partial\chi^y\partial\chi^y\partial\chi^yF\partial F + \partial\partial\chi^y\partial\chi^yFFFF + \partial\chi^y\partial\chi^yFFF\partial F
 \end{aligned} \tag{45}$$

Note that all terms above and the terms in (43) involve χ^y .

Then the T-duality constraint

$$\delta S^{(6,1)} + \delta S_2^{(6,1)} + \delta S_4^{(6,1)} + J^{(6,1)} = 0 \tag{46}$$

generates some algebraic equations between all parameters. The solution to these equations produce the following numbers for the effective action parameters in (17) and (16):

$$\begin{aligned}
 W_1 &\rightarrow 8N_1, & W_2 &\rightarrow -2 - 16N_1, & W_3 &\rightarrow \frac{1}{2} + 4N_1, & W_4 &\rightarrow -1 - 8N_1, \\
 W_5 &\rightarrow 16N_1, & W_6 &\rightarrow \frac{1}{4} - 2N_1, & W_7 &\rightarrow \frac{1}{8}, & W_8 &\rightarrow -\frac{1}{32}, \\
 T_1 &\rightarrow 1, & T_2 &\rightarrow 0, & T_3 &\rightarrow -\frac{2}{5} + \frac{24}{5}N_1, & T_4 &\rightarrow -\frac{2}{5} + \frac{24}{5}N_1, \\
 T_5 &\rightarrow \frac{7}{5} + \frac{96}{5}N_1, & T_6 &\rightarrow \frac{2}{5} - \frac{24}{5}N_1, & T_7 &\rightarrow -\frac{1}{5} + \frac{12}{5}N_1, & T_8 &\rightarrow \frac{7}{5} - \frac{24}{5}N_1, \\
 T_9 &\rightarrow -\frac{3}{5} - \frac{64}{5}N_1, & T_{10} &\rightarrow -\frac{6}{5} - \frac{48}{5}N_1, & T_{11} &\rightarrow 2N_1, & T_{12} &\rightarrow \frac{1}{4} + 2N_1, & T_{13} &\rightarrow N_1
 \end{aligned} \tag{47}$$

The parameters in the first two lines fix the action (17). The other parameters fix the action (16). The parameter N_1 could not be fixed by the calculation at the level $m = 6$. So at this level there are two T-dual multiplets. However, from the S-matrix calculations in $m = 4$ we know that $N_1 = -1/24$. It would be interesting to fix the parameters in (16), (17) by the S-matrix calculations in $m = 6$ and compare the result with the above numbers.

The parameters E_2, E_3 in the T-duality transformations $T^{(2,1)}$ appear in above calculations, however, the above T-duality constrain at the level $m = 6$ could not fix them. There are also many parameters in the T-duality transformations $T^{(4,1)}$ which are not fix by the above calculations. The T-duality transformations $T^{(4,1)}$ that our calculation fixes appear in the appendix.

3.3. Six extra F 's

At the order α' and at the level of $m = 8$, there are four contributions to the action $S_{D_p}^{(8,1)}$. One contribution is coming from expanding (3) and keeping $m = 8$ terms, the second contribution is coming from expanding (14) with the coefficients (39), the third contribution is coming from expanding the couplings in (16) and (17) with the parameters (47), and the last one is coming from the couplings in (18) and (19). The parameter N_1 and the parameters Z_1, \dots, Z_{37} and Y_1, \dots, Y_{16} appear in $S_{D_p}^{(8,1)}$. One should reduce $S_{D_p}^{(8,1)}$ on the circle along the y -direction and use the T-duality transformation (22). Then one should compare the result with $S_{D_{p-1}}^{(8,1)}$. One finds

$$S_{D_p}^{(8,1)} \xrightarrow{T^{(0,0)}} S_{D_{p-1}}^{(8,1)} + \delta S^{(8,1)} \tag{48}$$

where $\delta S^{(8,1)}$ contains some non-zero terms at the level of $m = 8$ which includes all above parameters. Each term in $\delta S^{(8,1)}$ has the scalar field χ^y .

The extra terms in $\delta S^{(8,1)}$ should be canceled by the T-duality transformation $T^{(6,1)}$ on the reduction of the action $S_{D_p}^{(2,0)}$, by the T-duality transformation $T^{(4,1)}$ on the reduction of the action $S_{D_p}^{(4,0)}$, and by the T-duality transformation $T^{(2,1)}$ on the reduction of the action $S_{D_p}^{(6,0)}$, *i.e.*,

$$\begin{aligned}
 S_{D_p}^{(2,0)} &\xrightarrow{T^{(0,0)}+T^{(6,1)}} S_{D_{p-1}}^{(2,0)} + \delta S_2^{(8,1)} + \delta S_2^{(16,2)} \\
 S_{D_p}^{(4,0)} &\xrightarrow{T^{(0,0)}+T^{(4,1)}} S_{D_{p-1}}^{(4,0)} + \delta S_4^{(8,1)} + \delta S_4^{(12,2)} + \delta S_4^{(16,3)} + \delta S_4^{(20,4)} \\
 S_{D_p}^{(6,0)} &\xrightarrow{T^{(0,0)}+T^{(2,1)}} S_{D_{p-1}}^{(6,0)} + \delta S_6^{(8,1)} + \delta S_6^{(10,2)} + \delta S_6^{(12,3)} + \delta S_6^{(14,4)} + \delta S_6^{(16,5)} + \delta S_6^{(18,6)}
 \end{aligned} \tag{49}$$

where $\delta S_2^{(16,2)}, \dots, \delta S_6^{(18,6)}$ contains some non-zero terms at higher orders of α' in which we are not interested. The T-duality transformation $T^{(2,1)}$ is given in (40), the T-duality transformation $T^{(4,1)}$ is given in the appendix and $T^{(6,1)}$ can easily be constructed with some arbitrary parameters similar to (43). Similar to (45), one can construct the total derivative terms $J^{(8,1)}$. Then the T-duality constraint

$$\delta S^{(8,1)} + \delta S_2^{(8,1)} + \delta S_4^{(8,1)} + \delta S_6^{(8,1)} + J^{(8,1)} = 0 \tag{50}$$

generates some algebraic equations between all unknown parameters in the T-duality transformation, the total derivative terms and the parameters in (18) and (19).

The solution to equation (50) produces the following numbers for the effective action parameters in (19):

$$\begin{aligned}
 Y_1 &\rightarrow \frac{7}{5} + \frac{56}{5}N_1, & Y_2 &\rightarrow \frac{2}{5} - \frac{24}{5}N_1, & Y_3 &\rightarrow 0, & Y_4 &\rightarrow \frac{14}{5} + \frac{192}{5}N_1, \\
 Y_5 &\rightarrow -\frac{3}{4} - 10N_1, & Y_6 &\rightarrow -\frac{1}{4} - 3N_1, & Y_7 &\rightarrow \frac{1}{16} + \frac{3}{4}N_1, & Y_8 &\rightarrow \frac{3}{5} - \frac{56}{5}N_1, \\
 Y_9 &\rightarrow 2N_1, & Y_{10} &\rightarrow 1, & Y_{11} &\rightarrow -\frac{1}{4}, & Y_{12} &\rightarrow -\frac{1}{8}, \\
 Y_{13} &\rightarrow \frac{1}{32}, & Y_{14} &\rightarrow -\frac{1}{12}, & Y_{15} &\rightarrow \frac{1}{32}, & Y_{16} &\rightarrow -\frac{1}{384}.
 \end{aligned} \tag{51}$$

And the following numbers for the effective action in (18):

$$\begin{aligned}
 Z_1 &\rightarrow \frac{1+8N_1}{1920}, & Z_2 &\rightarrow \frac{-7-56N_1}{5}, & Z_3 &\rightarrow \frac{-4-12N_1}{5}, & Z_4 &\rightarrow \frac{-1+72N_1}{5}, \\
 Z_5 &\rightarrow 1, & Z_6 &\rightarrow -\frac{7}{10} + \frac{72}{5}N_1, & Z_7 &\rightarrow \frac{23-96N_1}{35}, & Z_8 &\rightarrow \frac{-8+96N_1}{5}, \\
 Z_9 &\rightarrow \frac{4-48N_1}{35}, & Z_{10} &\rightarrow \frac{-9-72N_1}{5}, & Z_{11} &\rightarrow \frac{-7+144N_1}{5}, & Z_{12} &\rightarrow \frac{36-552N_1}{35}, \\
 Z_{13} &\rightarrow \frac{1+88N_1}{5}, & Z_{14} &\rightarrow \frac{-8+56N_1}{5}, & Z_{15} &\rightarrow \frac{-2-176N_1}{5}, & Z_{16} &\rightarrow \frac{17+976N_1}{35}, \\
 Z_{17} &\rightarrow \frac{1+88N_1}{5}, & Z_{18} &\rightarrow \frac{-3+96N_1}{5}, & Z_{19} &\rightarrow \frac{-1-48N_1}{5}, & Z_{20} &\rightarrow -\frac{7}{60} - \frac{34}{15}N_1, \\
 Z_{21} &\rightarrow -\frac{3}{20} - \frac{11}{5}N_1, & Z_{22} &\rightarrow \frac{3}{80} + \frac{9}{5}N_1, & Z_{23} &\rightarrow -\frac{1}{4}, & Z_{24} &\rightarrow \frac{3}{20} + \frac{6}{5}N_1, \\
 Z_{25} &\rightarrow \frac{3}{80} + \frac{3}{10}N_1, & Z_{26} &\rightarrow \frac{1}{20} + \frac{15}{5}N_1, & Z_{27} &\rightarrow \frac{3}{20} + \frac{11}{5}N_1, & Z_{28} &\rightarrow -\frac{1}{20} - \frac{22}{5}N_1,
 \end{aligned}$$

$$\begin{aligned}
Z_{29} &\rightarrow \frac{1+8N_1}{5}, & Z_{30} &\rightarrow -\frac{1}{10} - \frac{4}{5}N_1, & Z_{31} &\rightarrow -\frac{1}{20} + \frac{8}{5}N_1, & Z_{32} &\rightarrow 3N_1, \\
Z_{33} &\rightarrow \frac{3}{10} + \frac{32}{5}N_1, & Z_{34} &\rightarrow -2N_1, & Z_{35} &\rightarrow -\frac{1}{160} + \frac{N_1}{5}, & Z_{36} &\rightarrow \frac{-1-3N_1}{20}, \\
Z_{37} &\rightarrow \frac{1}{160} + \frac{N_1}{20}
\end{aligned} \tag{52}$$

The solution to the equation (50) produces also the T-duality transformation $T^{(6,1)}$ which is very lengthy expression and has many unfixed parameters. It is not illuminating, so we do not write it. It is interesting to note that the T-duality constraint could fix all parameters in the actions (18) and (19). The parameter N_1 could not be fixed by the T-duality constraint even at the level of $m = 8$. So the two T-dual multiplets remain independent at the level of $m = 8$. It seems if one extends the above calculation to $m > 8$, one would find only higher F -corrections to the two T-dual multiplets.

4. Discussion

In this paper, we have found that the constraint that the covariant effective actions must be invariant under the T-duality transformation (22) plus their appropriate higher derivative corrections, fixes the independent couplings in the effective actions at order α' up to two parameters, *i.e.*, (39), (47), (51) and (52). Hence, the T-duality constraint dictates that there are two T-dual multiplets. One with overall factor C and the other one with the overall factor N_1 . We have chosen the overall factor of the first multiplet to be $C = 1$ which is dictated by the S-matrix calculations. The S-matrix also fixes the overall coefficient of the other T-dual multiplet to be $N_1 = -1/24$.

Another approach for imposing the T-duality constraint is that one considers non-covariant action and constrain it to be invariant under the standard T-duality (22) without α' -corrections. Then one should use non-covariant field redefinitions and total derivative terms to convert the non-covariant action to the covariant form [31]. This method has been used in [31] to reproduce the known bulk effective action of the bosonic string theory at order α' . We have used this method and found exactly the relations (39) at four-field level and (47) at six-field level. That is, we have written all contractions of $F, \partial F, \partial\chi, \partial\partial\chi$ at order α' and at the level of $m = 4$. Then we constrain it to be invariant under the T-duality transformation (22). The resulting action converted to (14) by appropriate non-covariant field redefinitions and total derivative terms provided that the relations (39) are satisfied. Similar calculation at the level of $m = 6$ produces the coefficients in (47).

A specific non-covariant D-brane action at order α' in the bosonic string theory has been written in [29] which is invariant under T-duality transformations (22) and includes all powers of F . It includes $\partial F, \partial\partial\chi$ and some matrices that contains all powers of F and $\partial\chi$. We have expanded that action at the level of $m = 4$ and use non-covariant field redefinitions and total derivative terms to convert it to the covariant action (14). We have succeeded at the level of $m = 4$, however, we could not found covariant action at the level of $m = 6$. That means the action proposed in [29] does not produce the result of the S-matrix calculations at the level of $m > 4$. In fact the F and $\partial\chi$ in the matrices used in [29] must be constant. The same matrices have been used in [30] to construct the effective action of two massless closed strings and infinite number of constant F . It has been shown in [30] that the result is consistent with the S-matrix element of two closed string vertex operators in the presence of constant F .

We have found the couplings at order α' with zero, two, four and six extra F . In general there are non-zero couplings with more than six extra F as well. One may try to find a closed expression for all couplings at order α' . One suggestion may be to extend the pull-back metric \tilde{G}^{ab} in (3) to include F 's as well. An extension is the following symmetric matrix:

$$G^{ab} = \left(\frac{1}{\tilde{G} + F} \tilde{G} \frac{1}{\tilde{G} - F} \right)^{ab} \quad (53)$$

In the absence of the transverse scalar fields χ^i , it is the open string metric which appears in the effective action when it is written in terms of non-commutative variables [32]. In terms of the commutative variables which we are working with, the above matrix may be used to rewrite the couplings we have found by the T-duality constraint in a closed expression. For example, all the couplings which have $\Omega_{ab}^i \Omega^{ab}_i$ can be written as

$$S_p \supset \alpha' T_p \int d^{p+1} \sigma \sqrt{-\det(G_{ab})} \left[\Omega_{ab}^i \Omega^{ab}_i \right] \quad (54)$$

where $\det(G_{ab}) = \det(\tilde{G}_{ab} + F_{ab})$. Expanding the DBI part, it produces all couplings we have found in (39), (47), (51) and (52) which includes the structure $\Omega_{ab}^i \Omega^{ab}_i$. To be able to rewrite all other couplings in a closed expression, one may also need the following antisymmetric matrix as well:

$$\Theta^{ab} = \left(\frac{1}{\tilde{G} + F} F \frac{1}{\tilde{G} - F} \right)^{ab} \quad (55)$$

It would be interesting to find a closed expression for the couplings that the T-duality constraint fixes. That expression would produce correct couplings with arbitrary number of F 's.

We have found the world-volume couplings at order α' . One may be interested in extending these couplings to the order α'^2 . In this case, one should first find the independent couplings at order α'^2 as we have done in section 2 for the couplings at order α' . Then one should transform them under the T-duality transformation (22) at order α'^0 to find $\delta S^{(m,2)}$. It should be canceled by total derivative terms $J^{(m,2)}$ and by $\delta S_n^{(m,2)}$ terms which are resulted from transforming the DBI action under the T-duality transformations at order α'^2 and from transforming the couplings at order α' under the T-duality transformations at order α' that we have found in this paper. This later term makes the calculation in the bosonic theory to be very lengthy. However, in the superstring theory there are no couplings at order α' . Hence, the calculation would be much easier to perform. It would be interesting to find the α'^2 corrections to the DBI and WZ actions in the superstring theory by the T-duality constraint and compare them with the couplings found in [23,25] by the boundary state formalism in superstring theory.

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Appendix A

In this appendix, we write the T-duality transformation $T^{(4,1)}$ that the T-duality constraint (46) fixes. The transformation for A^y is

$$\begin{aligned}
A^y T^{(4,1)} \longrightarrow & \alpha' [-16N_1 F_a^{\bar{c}} F^{\bar{a}\bar{b}} F_b^{\bar{d}} F_c^{\bar{e}} \partial_{\bar{d}} \partial_{\bar{e}} \chi^y + (\frac{1}{2} + 10N_1) F_{\bar{a}\bar{b}} F^{\bar{a}\bar{b}} F_c^{\bar{e}} F^{\bar{c}\bar{d}} \partial_{\bar{d}} \partial_{\bar{e}} \chi^y \\
& - (\frac{19}{10} + \frac{19}{5} N_1) F_b^{\bar{d}} F^{\bar{b}\bar{c}} \partial_{\bar{c}} \partial_{\bar{d}} \chi^y \partial_{\bar{a}} \chi^y \partial^{\bar{a}} \chi^y + (\frac{1}{4} + 4N_1) F_{\bar{b}\bar{c}} F^{\bar{b}\bar{c}} \partial^{\bar{d}} \partial_{\bar{d}} \chi^y \partial_{\bar{a}} \chi^y \partial^{\bar{a}} \chi^y \\
& + (2 + 24N_1 - E_3) F_b^{\bar{d}} F^{\bar{b}\bar{c}} \partial_{\bar{c}} \partial_{\bar{d}} \chi^i \partial_{\bar{a}} \chi^i \partial^{\bar{a}} \chi^y + E_4 F_{\bar{b}\bar{c}} F^{\bar{b}\bar{c}} \partial^{\bar{d}} \partial_{\bar{d}} \chi^i \partial_{\bar{a}} \chi^i \partial^{\bar{a}} \chi^y \\
& + (\frac{1}{4} + 8N_1) F_{\bar{c}\bar{d}} F^{\bar{c}\bar{d}} \partial_{\bar{a}} \partial_{\bar{b}} \chi^y \partial^{\bar{a}} \chi^y \partial^{\bar{b}} \chi^y + \frac{2}{5} (-1 + 12N_1) F_a^{\bar{c}} F_c^{\bar{d}} \partial_{\bar{b}} \partial_{\bar{d}} \chi^y \partial^{\bar{a}} \chi^y \partial^{\bar{b}} \chi^y \\
& + \frac{1}{5} (22 + 256N_1 - 5E_2) F_a^{\bar{c}} F_b^{\bar{d}} \partial_{\bar{c}} \partial_{\bar{d}} \chi^y \partial^{\bar{a}} \chi^y \partial^{\bar{b}} \chi^y \\
& - (\frac{1}{2} + 6N_1) \partial^{\bar{c}} \partial_{\bar{c}} \chi^y \partial_{\bar{a}} \chi^y \partial^{\bar{a}} \chi^y \partial_{\bar{b}} \chi^y \partial^{\bar{b}} \chi^y \\
& + (1 + 12N_1 - E_3) \partial^{\bar{c}} \partial_{\bar{c}} \chi^y \partial_{\bar{a}} \chi^i \partial^{\bar{a}} \chi^y \partial_{\bar{b}} \chi^i \partial^{\bar{b}} \chi^y \\
& + E_5 \partial^{\bar{c}} \partial_{\bar{c}} \chi^i \partial_{\bar{a}} \chi^y \partial^{\bar{a}} \chi^y \partial_{\bar{b}} \chi^i \partial^{\bar{b}} \chi^y - (\frac{1}{4} + 2N_1) F_{\bar{c}\bar{d}} F^{\bar{c}\bar{d}} \partial_{\bar{a}} \partial_{\bar{b}} \chi^y \partial^{\bar{a}} \chi^i \partial^{\bar{b}} \chi^i \\
& - (\frac{1}{2} + 4N_1) F_a^{\bar{c}} F_{\bar{b}\bar{c}} \partial^{\bar{d}} \partial_{\bar{d}} \chi^y \partial^{\bar{a}} \chi^i \partial^{\bar{b}} \chi^i + E_6 \partial^{\bar{c}} \partial_{\bar{c}} \chi^j \partial_{\bar{a}} \chi^i \partial^{\bar{a}} \chi^y \partial_{\bar{b}} \chi^j \partial^{\bar{b}} \chi^i \\
& - E_2 F_a^{\bar{c}} F_b^{\bar{d}} \partial_{\bar{c}} \partial_{\bar{d}} \chi^i \partial^{\bar{a}} \chi^y \partial^{\bar{b}} \chi^i + E_7 F_a^{\bar{c}} F_{\bar{b}\bar{c}} \partial^{\bar{d}} \partial_{\bar{d}} \chi^i \partial^{\bar{a}} \chi^y \partial^{\bar{b}} \chi^i \\
& + E_8 \partial^{\bar{c}} \partial_{\bar{c}} \chi^i \partial_{\bar{a}} \chi^i \partial^{\bar{a}} \chi^y \partial_{\bar{b}} \chi^j \partial^{\bar{b}} \chi^j - 3(1 + 8N_1) \partial_{\bar{b}} \partial_{\bar{c}} \chi^y \partial_{\bar{a}} \chi^y \partial^{\bar{a}} \chi^y \partial^{\bar{b}} \chi^y \partial^{\bar{c}} \chi^y \\
& + (2 + 24N_1 - E_3) \partial_{\bar{b}} \partial_{\bar{c}} \chi^i \partial_{\bar{a}} \chi^i \partial^{\bar{a}} \chi^y \partial^{\bar{b}} \chi^y \partial^{\bar{c}} \chi^y \\
& - (4 + 56N_1 - E_2) F_a^{\bar{d}} \partial^{\bar{a}} \chi^y \partial^{\bar{b}} \chi^y \partial_{\bar{c}} F_{\bar{b}\bar{d}} \partial^{\bar{c}} \chi^y \\
& + (1 + 12N_1) \partial_{\bar{b}} \partial_{\bar{c}} \chi^y \partial_{\bar{a}} \chi^y \partial^{\bar{a}} \chi^y \partial^{\bar{b}} \chi^i \partial^{\bar{c}} \chi^i + E_2 F_a^{\bar{d}} \partial^{\bar{a}} \chi^y \partial^{\bar{b}} \chi^i \partial_{\bar{c}} F_{\bar{b}\bar{d}} \partial^{\bar{c}} \chi^i \\
& - E_3 \partial_{\bar{b}} \partial_{\bar{c}} \chi^i \partial_{\bar{a}} \chi^i \partial^{\bar{a}} \chi^y \partial^{\bar{b}} \chi^j \partial^{\bar{c}} \chi^j + \frac{1}{2} F^{\bar{c}\bar{d}} \partial_{\bar{a}} \chi^y \partial^{\bar{a}} \chi^y \partial^{\bar{b}} \chi^y \partial_{\bar{d}} F_{\bar{b}\bar{c}} \\
& + E_9 F_b^{\bar{c}} \partial_{\bar{a}} \chi^y \partial^{\bar{a}} \chi^y \partial^{\bar{b}} \chi^y \partial_{\bar{d}} F_c^{\bar{d}} + E_{10} F_b^{\bar{c}} \partial_{\bar{a}} \chi^i \partial^{\bar{a}} \chi^y \partial^{\bar{b}} \chi^i \partial_{\bar{d}} F_c^{\bar{d}} \\
& + E_{11} F_a^{\bar{c}} \partial^{\bar{a}} \chi^y \partial_{\bar{b}} \chi^i \partial^{\bar{b}} \chi^i \partial_{\bar{d}} F_c^{\bar{d}} + E_{12} F_{\bar{a}\bar{b}} \partial^{\bar{a}} \chi^y \partial^{\bar{b}} \chi^i \partial^{\bar{c}} \chi^i \partial_{\bar{d}} F_c^{\bar{d}} \\
& + \frac{3}{5} (1 + 8N_1) F_b^{\bar{d}} F^{\bar{b}\bar{c}} F_c^{\bar{e}} \partial^{\bar{a}} \chi^y \partial_{\bar{e}} F_{\bar{a}\bar{d}} - \frac{1}{4} F_{\bar{b}\bar{c}} F^{\bar{b}\bar{c}} F^{\bar{d}\bar{e}} \partial^{\bar{a}} \chi^y \partial_{\bar{e}} F_{\bar{a}\bar{d}} \\
& + \frac{1}{5} (17 + 256N_1 - 5E_2) F_a^{\bar{b}} F_c^{\bar{e}} F^{\bar{c}\bar{d}} \partial^{\bar{a}} \chi^y \partial_{\bar{e}} F_{\bar{a}\bar{d}} + E_{13} F_a^{\bar{b}} F_{\bar{c}\bar{d}} F^{\bar{c}\bar{d}} \partial^{\bar{a}} \chi^y \partial_{\bar{e}} F_b^{\bar{e}} \\
& - F_a^{\bar{b}} F_b^{\bar{c}} F^{\bar{d}\bar{e}} \partial^{\bar{a}} \chi^y \partial_{\bar{e}} F_{\bar{c}\bar{d}} + E_{14} F_a^{\bar{b}} F_b^{\bar{c}} F_c^{\bar{d}} \partial^{\bar{a}} \chi^y \partial_{\bar{e}} F_d^{\bar{e}} \\
& + (2 + 48N_1) \partial_{\bar{b}} \partial_{\bar{c}} \chi^y \partial_{\bar{a}} \chi^i \partial^{\bar{a}} \chi^y \partial^{\bar{b}} \chi^y \partial^{\bar{c}} \chi^i + (2 + 24N_1) F_a^{\bar{c}} F_b^{\bar{d}} \partial_{\bar{c}} \partial_{\bar{d}} \chi^y \partial^{\bar{a}} \chi^i \partial^{\bar{b}} \chi^i \\
& - (4 + 48N_1) F_a^{\bar{c}} F_c^{\bar{d}} \partial_{\bar{b}} \partial_{\bar{d}} \chi^y \partial^{\bar{a}} \chi^i \partial^{\bar{b}} \chi^i - \frac{1}{8} F_a^{\bar{c}} F^{\bar{a}\bar{b}} F_b^{\bar{d}} F_{\bar{c}\bar{d}} \partial^{\bar{e}} \partial_{\bar{e}} \chi^y \\
& - (\frac{1}{32} + \frac{N_1}{2}) F_{\bar{a}\bar{b}} F^{\bar{a}\bar{b}} F_{\bar{c}\bar{d}} F^{\bar{c}\bar{d}} \partial^{\bar{e}} \partial_{\bar{e}} \chi^y \\
& - (\frac{5}{2} + 32N_1 - E_2) F_a^{\bar{c}} F_{\bar{b}\bar{c}} \partial^{\bar{d}} \partial_{\bar{d}} \chi^y \partial^{\bar{a}} \chi^y \partial^{\bar{b}} \chi^y] \tag{56}
\end{aligned}$$

where E_2, E_3 are the parameters that appear also in (40) which could not be fixed by the constraint (46). The other parameters E_4, \dots, E_{20} in above and the following transformations are

also the free parameters that the constraint (46) could not fix. They may be fixed by considering the higher order constraints. The transformation for $A^{\tilde{a}}$ is

$$\begin{aligned}
 A^{\tilde{a}} \xrightarrow{T^{(4,1)}} & \alpha' \left[-\left(\frac{1}{4} + 10N_1\right) F_b^{\tilde{c}} F_{\tilde{d}\tilde{e}}^{\tilde{c}} F^{\tilde{d}\tilde{e}} \partial^{\tilde{a}} \partial_{\tilde{c}} \chi^y \partial^{\tilde{b}} \chi_y - \frac{1}{5} (1 + 88N_1) F_b^{\tilde{c}} F_c^{\tilde{d}} F_{\tilde{d}}^{\tilde{e}} \partial^{\tilde{a}} \partial_{\tilde{e}} \chi^y \partial^{\tilde{b}} \chi_y \right. \\
 & - \frac{1}{4} F^{\tilde{a}\tilde{c}} F_{\tilde{d}\tilde{e}}^{\tilde{c}} F^{\tilde{d}\tilde{e}} \partial_{\tilde{b}} \partial_{\tilde{c}} \chi^y \partial^{\tilde{b}} \chi_y + (4 + 64N_1 + E_{14} - E_2) F^{\tilde{a}\tilde{c}} F_b^{\tilde{d}} F_{\tilde{c}\tilde{d}}^{\tilde{e}} \partial^{\tilde{e}} \partial_{\tilde{e}} \chi^y \partial^{\tilde{b}} \chi_y \\
 & - \frac{3}{5} (1 + 8N_1) F^{\tilde{a}\tilde{c}} F_b^{\tilde{d}} F_{\tilde{d}}^{\tilde{e}} \partial_{\tilde{c}} \partial_{\tilde{e}} \chi^y \partial^{\tilde{b}} \chi_y - \frac{1}{5} (17 + 256N_1) F^{\tilde{a}\tilde{c}} F_b^{\tilde{d}} F_c^{\tilde{e}} \partial_{\tilde{d}}^{\tilde{e}} \partial_{\tilde{e}} \chi^y \partial^{\tilde{b}} \chi_y \\
 & + \left(\frac{17}{5} + \frac{216}{5} N_1 + E_2\right) F_b^{\tilde{a}} F_c^{\tilde{e}} F^{\tilde{c}\tilde{d}} \partial_{\tilde{d}} \partial_{\tilde{e}} \chi^y \partial^{\tilde{b}} \chi_y - F^{\tilde{a}\tilde{c}} F_c^{\tilde{d}} F_{\tilde{d}}^{\tilde{e}} \partial_{\tilde{b}} \partial_{\tilde{e}} \chi^y \partial^{\tilde{b}} \chi_y \\
 & - \left(\frac{3}{4} + 14N_1 + E_{13}\right) F_b^{\tilde{a}} F_{\tilde{c}\tilde{d}}^{\tilde{c}} F^{\tilde{c}\tilde{d}} \partial^{\tilde{e}} \partial_{\tilde{e}} \chi^y \partial^{\tilde{b}} \chi_y \\
 & + \frac{1}{5} (17 + 176N_1) F_b^{\tilde{d}} \partial_{\tilde{c}} \partial_{\tilde{d}} \chi^y \partial^{\tilde{a}} \chi_y \partial^{\tilde{b}} \chi^y \partial^{\tilde{c}} \chi_y \\
 & - \frac{1}{10} (29 + 352N_1) F_c^{\tilde{d}} \partial^{\tilde{a}} \partial_{\tilde{d}} \chi^y \partial_{\tilde{b}} \chi_y \partial^{\tilde{b}} \chi^y \partial^{\tilde{c}} \chi_y \\
 & + \frac{1}{10} (1 + 48N_1) F^{\tilde{a}\tilde{d}} \partial_{\tilde{c}} \partial_{\tilde{d}} \chi^y \partial_{\tilde{b}} \chi_y \partial^{\tilde{b}} \chi^y \partial^{\tilde{c}} \chi_y \\
 & - (8N_1 + E_9 - E_2) F_c^{\tilde{a}} \partial^{\tilde{d}} \partial_{\tilde{d}} \chi^y \partial_{\tilde{b}} \chi_y \partial^{\tilde{b}} \chi^y \partial^{\tilde{c}} \chi_y \\
 & - (6 + 80N_1 + E_{15}) F_c^{\tilde{d}} \partial^{\tilde{a}} \partial_{\tilde{d}} \chi_i \partial_{\tilde{b}} \chi^i \partial^{\tilde{b}} \chi^y \partial^{\tilde{c}} \chi_y \\
 & - (2 + 32N_1 + E_{16} - E_2 + E_3) F_c^{\tilde{a}} \partial^{\tilde{d}} \partial_{\tilde{d}} \chi_i \partial_{\tilde{b}} \chi^i \partial^{\tilde{b}} \chi^y \partial^{\tilde{c}} \chi_y \\
 & + E_{15} F_c^{\tilde{d}} \partial^{\tilde{a}} \partial_{\tilde{d}} \chi^y \partial_{\tilde{b}} \chi^i \partial^{\tilde{b}} \chi_y \partial^{\tilde{c}} \chi_i \\
 & - \frac{1}{5} (7 + 136N_1) F_b^{\tilde{d}} F_{\tilde{d}}^{\tilde{e}} \partial^{\tilde{b}} \chi^y \partial_{\tilde{c}} F_{\tilde{e}}^{\tilde{a}} \partial^{\tilde{c}} \chi_y + \frac{1}{5} (17 + 256N_1) F^{\tilde{a}\tilde{d}} F_{\tilde{d}}^{\tilde{e}} \partial^{\tilde{b}} \chi^y \partial_{\tilde{c}} F_{\tilde{b}\tilde{e}}^{\tilde{c}} \partial^{\tilde{c}} \chi_y \\
 & + (E_{12} + E_2) F_{\tilde{b}\tilde{c}}^{\tilde{d}} \partial^{\tilde{d}} \partial_{\tilde{d}} \chi^y \partial^{\tilde{a}} \chi^i \partial^{\tilde{b}} \chi_y \partial^{\tilde{c}} \chi_i - \left(\frac{1}{4} + 10N_1\right) F_{\tilde{d}\tilde{e}}^{\tilde{c}} F^{\tilde{d}\tilde{e}} \partial^{\tilde{b}} \chi^y \partial_{\tilde{c}} F_b^{\tilde{a}} \partial^{\tilde{c}} \chi_y \\
 & - (2 + 32N_1 + E_{10} - E_3) F_c^{\tilde{a}} \partial^{\tilde{d}} \partial_{\tilde{d}} \chi^y \partial_{\tilde{b}} \chi^i \partial^{\tilde{b}} \chi_y \partial^{\tilde{c}} \chi_i \\
 & - (1 + 16N_1) F_c^{\tilde{d}} \partial_{\tilde{b}} \partial_{\tilde{d}} \chi_i \partial^{\tilde{a}} \chi^y \partial^{\tilde{b}} \chi_y \partial^{\tilde{c}} \chi^i \\
 & - (1 + 16N_1 - E_{17}) F_{\tilde{b}\tilde{c}}^{\tilde{d}} \partial^{\tilde{d}} \partial_{\tilde{d}} \chi_i \partial^{\tilde{a}} \chi^y \partial^{\tilde{b}} \chi_y \partial^{\tilde{c}} \chi^i + 4N_1 F_c^{\tilde{d}} \partial^{\tilde{a}} \partial_{\tilde{d}} \chi_i \partial_{\tilde{b}} \chi^y \partial^{\tilde{b}} \chi_y \partial^{\tilde{c}} \chi^i \\
 & + \left(\frac{1}{2} + 4N_1 - E_{18}\right) F_c^{\tilde{a}} \partial^{\tilde{d}} \partial_{\tilde{d}} \chi_i \partial_{\tilde{b}} \chi^y \partial^{\tilde{b}} \chi_y \partial^{\tilde{c}} \chi^i - E_{11} F_b^{\tilde{a}} \partial^{\tilde{d}} \partial_{\tilde{d}} \chi^y \partial^{\tilde{b}} \chi_y \partial_{\tilde{c}} \chi_i \partial^{\tilde{c}} \chi^i \\
 & + 4N_1 \partial_{\tilde{b}} \chi^i \partial^{\tilde{b}} \chi^y \partial_{\tilde{c}} \chi_i \partial^{\tilde{c}} \chi_y \partial_{\tilde{d}} F^{\tilde{a}\tilde{d}} + \left(\frac{1}{2} + 4N_1\right) \partial^{\tilde{a}} \chi^y \partial_{\tilde{b}} \chi_y \partial^{\tilde{b}} \chi^y \partial^{\tilde{c}} \chi_y \partial_{\tilde{d}} F_c^{\tilde{d}} \\
 & + E_{19} \partial^{\tilde{a}} \chi^i \partial_{\tilde{b}} \chi_i \partial^{\tilde{b}} \chi^y \partial^{\tilde{c}} \chi_y \partial_{\tilde{d}} F_c^{\tilde{d}} - (1 + 16N_1 - E_{19}) \partial^{\tilde{a}} \chi^y \partial_{\tilde{b}} \chi^i \partial^{\tilde{b}} \chi_y \partial_{\tilde{c}} \chi_i \partial_{\tilde{d}} F_c^{\tilde{d}} \\
 & + \frac{3}{5} (1 + N_1) F^{\tilde{a}\tilde{d}} F_b^{\tilde{e}} \partial^{\tilde{b}} \chi^y \partial^{\tilde{c}} \chi_y \partial_{\tilde{d}} F_{\tilde{c}\tilde{e}} \\
 & + \left(\frac{12}{5} + \frac{96}{5} N_1 + E_2\right) F_b^{\tilde{a}} \partial_{\tilde{c}} \partial_{\tilde{d}} \chi^y \partial^{\tilde{b}} \chi_y \partial^{\tilde{c}} \chi^y \partial^{\tilde{d}} \chi_y \\
 & \left. - \left(\frac{1}{2} + 16N_1\right) \partial_{\tilde{b}} \chi^y \partial^{\tilde{b}} \chi_y \partial^{\tilde{c}} \chi^y \partial_{\tilde{d}} F_c^{\tilde{a}} \partial^{\tilde{d}} \chi_y \right]
 \end{aligned}$$

$$\begin{aligned}
& -(4 + 64N_1 + E_{15})\partial_{\bar{b}}\chi^i\partial^{\bar{b}}\chi^y\partial_{\bar{c}}F_{\bar{d}}^{\bar{a}}\partial^{\bar{c}}\chi^y\partial^{\bar{d}}\chi_i \\
& -(6 + 80N_1 + E_{15})F_{\bar{b}\bar{c}}\partial^{\bar{a}}\partial_{\bar{d}}\chi^y\partial^{\bar{b}}\chi_y\partial^{\bar{c}}\chi^i\partial^{\bar{d}}\chi_i \\
& -(2 + 32N_1 - E_2)F_{\bar{b}}^{\bar{a}}\partial_{\bar{c}}\partial_{\bar{d}}\chi^y\partial^{\bar{b}}\chi_y\partial^{\bar{c}}\chi^i\partial^{\bar{d}}\chi_i \\
& +(2 + 16N_1 + E_{15})\partial_{\bar{b}}\chi^i\partial^{\bar{b}}\chi^y\partial^{\bar{c}}\chi_y\partial_{\bar{d}}F_{\bar{c}}^{\bar{a}}\partial^{\bar{d}}\chi_i + 4N_1\partial_{\bar{b}}\chi^y\partial^{\bar{b}}\chi_y\partial^{\bar{c}}\chi^i\partial_{\bar{d}}F_{\bar{c}}^{\bar{a}}\partial^{\bar{d}}\chi_i \\
& -(1 + 16N_1)\partial^{\bar{a}}\chi^y\partial^{\bar{b}}\chi_y\partial^{\bar{c}}\chi^i\partial_{\bar{d}}F_{\bar{b}\bar{c}}\partial^{\bar{d}}\chi_i \\
& -(2 + 16N_1 + E_{15})F_{\bar{b}\bar{c}}\partial^{\bar{a}}\partial_{\bar{c}}\chi_i\partial^{\bar{b}}\chi^y\partial^{\bar{c}}\chi_y\partial^{\bar{d}}\chi^i \\
& -\frac{1}{5}(3 + 104N_1)F_{\bar{b}}^{\bar{d}}F_{\bar{d}}^{\bar{c}}\partial^{\bar{b}}\chi^y\partial^{\bar{c}}\chi_y\partial_{\bar{e}}F_{\bar{c}}^{\bar{a}} - (\frac{1}{2} + 16N_1)F_{\bar{c}}^{\bar{e}}F^{\bar{c}\bar{d}}\partial_{\bar{b}}\chi^y\partial^{\bar{b}}\chi_y\partial_{\bar{e}}F_{\bar{d}}^{\bar{a}} \\
& -4N_1F_{\bar{b}}^{\bar{d}}F_{\bar{c}\bar{d}}\partial^{\bar{b}}\chi^y\partial^{\bar{c}}\chi_y\partial_{\bar{e}}F^{\bar{a}\bar{e}} + \frac{4}{5}(3 + 44N_1)F_{\bar{c}}^{\bar{e}}F^{\bar{c}\bar{d}}\partial^{\bar{a}}\chi^y\partial^{\bar{b}}\chi_y\partial_{\bar{e}}F_{\bar{b}\bar{d}} \\
& -(\frac{1}{4} + 6N_1)F_{\bar{c}\bar{d}}F^{\bar{c}\bar{d}}\partial^{\bar{a}}\chi^y\partial^{\bar{b}}\chi_y\partial_{\bar{e}}F_{\bar{b}}^{\bar{e}} - F_{\bar{b}}^{\bar{c}}F^{\bar{d}\bar{e}}\partial^{\bar{a}}\chi^y\partial^{\bar{b}}\chi_y\partial_{\bar{e}}F_{\bar{c}\bar{d}} \\
& +\frac{1}{2}F^{\bar{a}\bar{c}}F^{\bar{d}\bar{e}}\partial_{\bar{b}}\chi^y\partial^{\bar{b}}\chi_y\partial_{\bar{e}}F_{\bar{c}\bar{d}} - F_{\bar{b}}^{\bar{a}}F^{\bar{d}\bar{e}}\partial^{\bar{b}}\chi^y\partial^{\bar{c}}\chi_y\partial_{\bar{e}}F_{\bar{c}\bar{d}} \\
& +E_{20}F^{\bar{a}\bar{d}}F_{\bar{b}\bar{d}}\partial^{\bar{b}}\chi^y\partial^{\bar{c}}\chi_y\partial_{\bar{e}}F_{\bar{c}}^{\bar{e}} - (3 + 48N_1 - E_{20})F_{\bar{b}}^{\bar{c}}F_{\bar{c}}^{\bar{d}}\partial^{\bar{a}}\chi^y\partial^{\bar{b}}\chi_y\partial_{\bar{e}}F_{\bar{d}}^{\bar{e}} \\
& +(\frac{1}{2} + 12N_1)F^{\bar{a}\bar{c}}F_{\bar{c}}^{\bar{d}}\partial_{\bar{b}}\chi^y\partial^{\bar{b}}\chi_y\partial_{\bar{e}}F_{\bar{d}}^{\bar{e}} - (1 + 16N_1 + E_2)F_{\bar{b}}^{\bar{a}}F_{\bar{c}}^{\bar{d}}\partial^{\bar{b}}\chi^y\partial^{\bar{c}}\chi_y\partial_{\bar{e}}F_{\bar{d}}^{\bar{e}} \\
& -(2 + 24N_1)F_{\bar{d}}^{\bar{a}}\partial_{\bar{b}}\partial_{\bar{c}}\chi_i\partial^{\bar{b}}\chi^y\partial^{\bar{c}}\chi_y\partial^{\bar{d}}\chi^i + \frac{4}{5}(3 + 44N_1)F_{\bar{b}}^{\bar{d}}F_{\bar{c}}^{\bar{e}}\partial^{\bar{b}}\chi^y\partial^{\bar{c}}\chi_y\partial_{\bar{e}}F_{\bar{d}}^{\bar{a}} \\
& +2N_1\partial_{\bar{b}}\chi^y\partial^{\bar{b}}\chi_y\partial_{\bar{c}}\chi^y\partial^{\bar{c}}\chi_y\partial_{\bar{d}}F^{\bar{a}\bar{d}} + 2N_1F_{\bar{c}\bar{d}}F^{\bar{c}\bar{d}}\partial_{\bar{b}}\chi^y\partial^{\bar{b}}\chi_y\partial_{\bar{e}}F^{\bar{a}\bar{e}}] \tag{57}
\end{aligned}$$

The transformation for χ^i is

$$\begin{aligned}
\chi^i & \xrightarrow{T^{(4,1)}} \alpha'[-(\frac{1}{2} + 4N_1)F_{\bar{b}}^{\bar{d}}F^{\bar{b}\bar{c}}\partial_{\bar{c}}\partial_{\bar{d}}\chi^i\partial_{\bar{a}}\chi^y\partial^{\bar{a}}\chi_y + E_3F_{\bar{b}}^{\bar{d}}F^{\bar{b}\bar{c}}\partial_{\bar{c}}\partial_{\bar{d}}\chi^y\partial_{\bar{a}}\chi^i\partial^{\bar{a}}\chi_y \\
& -(\frac{1}{4} + 2N_1 + E_4)F_{\bar{b}\bar{c}}F^{\bar{b}\bar{c}}\partial^{\bar{d}}\partial_{\bar{d}}\chi^y\partial_{\bar{a}}\chi^i\partial^{\bar{a}}\chi_y \\
& -(\frac{1}{2} + 2N_1)F_{\bar{c}\bar{d}}F^{\bar{c}\bar{d}}\partial_{\bar{a}}\partial_{\bar{b}}\chi^i\partial^{\bar{a}}\chi^y\partial^{\bar{b}}\chi_y \\
& -(2 + 32N_1)F_{\bar{a}}^{\bar{c}}F_{\bar{c}}^{\bar{d}}\partial_{\bar{b}}\partial_{\bar{d}}\chi^i\partial^{\bar{a}}\chi^y\partial^{\bar{b}}\chi_y + (1 + 16N_1)F_{\bar{a}}^{\bar{c}}F_{\bar{b}}^{\bar{d}}\partial_{\bar{c}}\partial_{\bar{d}}\chi^i\partial^{\bar{a}}\chi^y\partial^{\bar{b}}\chi_y \\
& +(\frac{1}{2} + 12N_1 - E_5 + E_3)\partial^{\bar{c}}\partial_{\bar{c}}\chi^y\partial_{\bar{a}}\chi_y\partial^{\bar{a}}\chi^y\partial_{\bar{b}}\chi^i\partial^{\bar{b}}\chi_y \\
& +E_3\partial^{\bar{c}}\partial_{\bar{c}}\chi_j\partial_{\bar{a}}\chi^i\partial^{\bar{a}}\chi^y\partial_{\bar{b}}\chi^j\partial^{\bar{b}}\chi_y \\
& -(E_7 - E_2)F_{\bar{a}}^{\bar{c}}F_{\bar{b}\bar{c}}\partial^{\bar{d}}\partial_{\bar{d}}\chi^y\partial^{\bar{a}}\chi_y\partial^{\bar{b}}\chi^i - (E_6 + E_3)\partial^{\bar{c}}\partial_{\bar{c}}\chi^y\partial_{\bar{a}}\chi^j\partial^{\bar{a}}\chi_y\partial_{\bar{b}}\chi_j\partial^{\bar{b}}\chi^i \\
& -E_8\partial^{\bar{c}}\partial_{\bar{c}}\chi^y\partial_{\bar{a}}\chi^i\partial^{\bar{a}}\chi_y\partial_{\bar{b}}\chi_j\partial^{\bar{b}}\chi^j + E_3\partial_{\bar{b}}\partial_{\bar{c}}\chi^y\partial_{\bar{a}}\chi^i\partial^{\bar{a}}\chi_y\partial^{\bar{b}}\chi^y\partial^{\bar{c}}\chi_y \\
& +E_3\partial_{\bar{b}}\partial_{\bar{c}}\chi^y\partial_{\bar{a}}\chi^i\partial^{\bar{a}}\chi_y\partial^{\bar{b}}\chi^j\partial^{\bar{c}}\chi_j + E_{17}F_{\bar{a}\bar{c}}\partial^{\bar{a}}\chi^y\partial^{\bar{b}}\chi_y\partial^{\bar{c}}\chi^i\partial_{\bar{d}}F_{\bar{b}}^{\bar{d}} \\
& +E_{16}F_{\bar{b}}^{\bar{c}}\partial_{\bar{a}}\chi^i\partial^{\bar{a}}\chi^y\partial^{\bar{b}}\chi_y\partial_{\bar{d}}F_{\bar{c}}^{\bar{d}} - (\frac{1}{2} - 12N_1)\partial_{\bar{b}}\partial_{\bar{c}}\chi^i\partial_{\bar{a}}\chi^y\partial^{\bar{a}}\chi_y\partial^{\bar{b}}\chi^y\partial^{\bar{c}}\chi_y \\
& +E_{18}F_{\bar{b}}^{\bar{c}}\partial_{\bar{a}}\chi^y\partial^{\bar{a}}\chi_y\partial^{\bar{b}}\chi^i\partial_{\bar{d}}F_{\bar{c}}^{\bar{d}} + \frac{1}{8}F_{\bar{b}\bar{c}}F^{\bar{b}\bar{c}}\partial^{\bar{d}}\partial_{\bar{d}}\chi^i\partial_{\bar{a}}\chi^y\partial^{\bar{a}}\chi_y]
\end{aligned}$$

$$\begin{aligned}
& +\frac{3}{8}\partial_{\bar{c}}\tilde{\chi}^i\partial_{\bar{a}}\chi^y\partial^{\bar{a}}\chi_y\partial_{\bar{b}}\chi^y\partial^{\bar{b}}\chi_y+\frac{1}{2}\partial_{\bar{c}}\tilde{\chi}^i\partial_{\bar{a}}\chi^j\partial^{\bar{a}}\chi^y\partial_{\bar{b}}\chi_j\partial^{\bar{b}}\chi_y \\
& -2\partial_{\bar{b}}\partial_{\bar{c}}\tilde{\chi}^i\partial_{\bar{a}}\chi^j\partial^{\bar{a}}\chi^y\partial^{\bar{b}}\chi_y\partial^{\bar{c}}\chi_j+\frac{1}{2}\partial_{\bar{b}}\partial_{\bar{c}}\tilde{\chi}^i\partial_{\bar{a}}\chi^y\partial^{\bar{a}}\chi_y\partial^{\bar{b}}\chi^j\partial^{\bar{c}}\chi_j]
\end{aligned} \tag{58}$$

In above transformations we have removed the terms that are canceled by the Bianchi identity $\partial_{[\bar{a}}F_{\bar{b}\bar{c}]}=0$. Note that the above transformations are non-zero for any specific values for the parameters E_2, \dots, E_{20} . Hence, it is impossible to find solution for the T-duality constraint (46) without adding corrections to the standard T-duality transformations (22).

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