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Direct product vs direct sum

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Abstract

In this talk, we intend to answer whether the direct product and direct sum of abelian groups are isomorphic or not. We show that the direct product and direct sum of an infinite family of algebraically compact torsion-free groups if have trivial reduced parts are not isomorphic.

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1 Introduction

In this talk, we intend to compare two groups of direct product and direct sum of a family of abelian groups. In the general case, the direct sum of a family of abelian groups is a subgroup of direct product. Also, if the family of groups is finite, then these two groups equal, but not for infinite families. If the family is infinite, these two groups may be non-isomorphic. In this talk, all groups are considered to be abelian.

Example 1.1. Let $\mathbb{Z}$ denotes the infinite cyclic group, $\prod_{\aleph_0}\mathbb{Z}$ denotes the direct product of countably infinite copies of infinite cyclic group, and $\sum_{\aleph_0}\mathbb{Z}$ denotes the direct sum of countably infinite copies of infinite cyclic group. It is well-known that $\sum_{\aleph_0}\mathbb{Z}$ is free abelian group and $\prod_{\aleph_0}\mathbb{Z}$ is not. Therefore $\sum_{\aleph_0}\mathbb{Z} \ncong \prod_{\aleph_0}\mathbb{Z}$.

Here, we prove that if the groups of an infinite family $\{G_i\}_{i\in I}$ are abelian, torsion-free and algebraically compact, the direct sum and direct product are not isomorphic if G_i has non-trivial reduced part for all $i \in I$. A torsion-free group is a group having no element of finite order. The definition of algebraically compact is recalled below. In [1], Chapter VII was devoted to study algebraically compact groups.

Definition 1.2 ([1]). The abelian group G is called algebraically compact if it is direct summand of every abelian group containing it as a pure subgroup. Recall that a subgroup G of group A is pure, if the equation $nx = g \in G$ is solvable in G, whenever it is solvable in the whole group A.

*Speaker

For instance, divisible groups are algebraically compact. It follows at once that a direct summand of an algebraically compact group is again algebraically compact, and a group is algebraically compact exactly if its reduced part is algebraically compact [1].

In the following theorem, we see some equivalent conditions to group G be algebraically compact [1, Theorem 38.1].

Theorem 1.3 ([1]). *The following conditions on a group G are equivalent.*

1. *G is pure injective,*
2. *G is algebraically compact,*
3. *G is a direct summand of a direct product of cocyclic groups,*
4. *G is algebraically a direct summand of a group that admits a compact topology,*
5. *if every finite subsystem of a system of equations over G has a solution in G, then the whole system is solvable in G.*

2 Main Results

In [1, Theorem 21.3], every abelian group divides to direct sum of a divisible part and a reduced part. An abeilan group is called reduced if it has no divisible subgroup other than the trivial group.

Theorem 2.1. *Let $\{G_i\}_{i\in I}$ be a family of algebraically compact abelian groups with non-trivial reduced part indexed by infinite set I. If G_i is torsion-free, for some $i \in I$, then*

$$\sum_{i\in I} G_i \ncong \prod_{i\in I} G_i.$$

Corollary 2.2. *Let G be an algebraically compact torsion-free group with non-trivial reduced part, and c be an infinite cardinal number. Then $\prod_c G \ncong \sum_c G$.*

Theorem 2.1 and Corollary 2.2 does not hold if the condition of having non-trivial reduced part is omitted as stated below.

Example 2.3. Let $G = \mathbb{R}$ be the additive group of real numbers set. Now, we see that $\sum_{\aleph_0} \mathbb{R} \cong \prod_{\aleph_0} \mathbb{R}$. By [1, Page 105], since $\mathbb{R}$ is divisible torsion-free group with continuum cardinality, $\mathbb{R} \cong \sum_c \mathbb{Q}$. Again, by the same argument $\sum_{\aleph_0} \mathbb{R} \cong \sum_c \mathbb{Q}$, and also $\prod_{\aleph_0} \mathbb{R} \cong \sum_c \mathbb{Q}$. Hence, $\sum_{\aleph_0} \mathbb{R} \cong \prod_{\aleph_0} \mathbb{R}$.

References

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