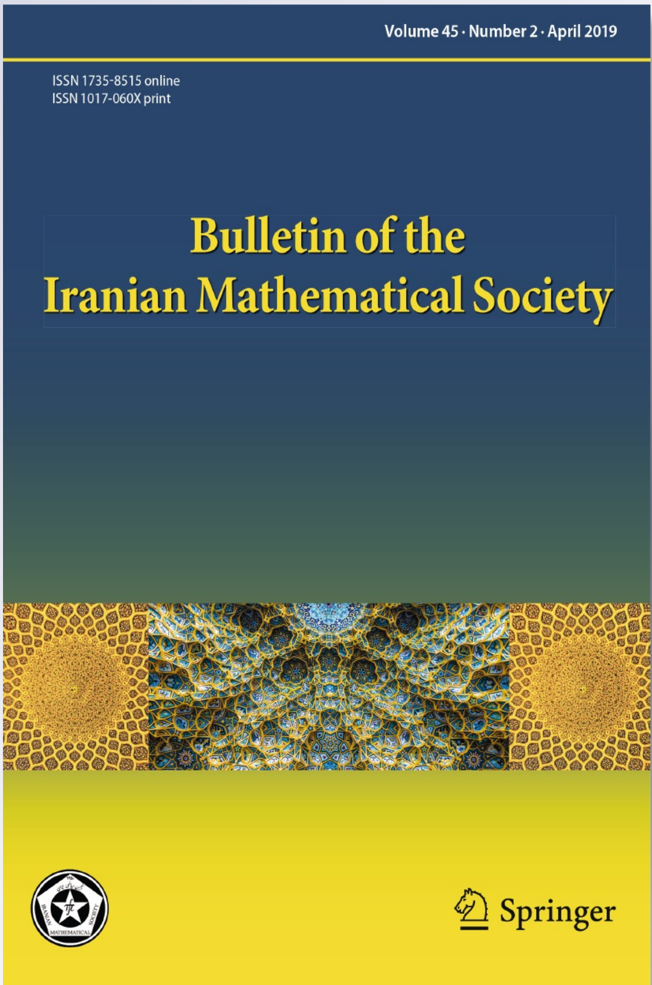
N-Relatively Invariant and N-Invariant Measure on Double Coset Spaces

F. Fahimian, R. A. Kamyabi Gol & F. Esmaeelzadeh

Bulletin of the Iranian Mathematical Society

ISSN 1017-060X
Volume 45
Number 2

Bull. Iran. Math. Soc. (2019) 45:515-525
DOI 10.1007/s41980-018-0146-2



Springer

Your article is protected by copyright and all rights are held exclusively by Iranian Mathematical Society. This e-offprint is for personal use only and shall not be self-archived in electronic repositories. If you wish to self-archive your article, please use the accepted manuscript version for posting on your own website. You may further deposit the accepted manuscript version in any repository, provided it is only made publicly available 12 months after official publication or later and provided acknowledgement is given to the original source of publication and a link is inserted to the published article on Springer's website. The link must be accompanied by the following text: "The final publication is available at link.springer.com".

Springer

Author's personal copy

Bulletin of the Iranian Mathematical Society (2019) 45:515–525
https://doi.org/10.1007/s41980-018-0146-2

ORIGINAL PAPER

N-Relatively Invariant and N-Invariant Measure on Double Coset Spaces

F. Fahimian[1] · R. A. Kamyabi Gol[1] · F. Esmaeelzadeh[2]

Received: 11 October 2017 / Revised: 3 July 2018 / Accepted: 6 July 2018 / Published online: 17 July 2018
© Iranian Mathematical Society 2018

Abstract
For a locally compact group G and two closed subgroups H, K of G let N be the normalizer group of K in G and $K\backslash G/H$ be the double coset spaces of G by H and K, respectively. The N-relatively invariant and N-invariant measures are defined for the double coset space $K\backslash G/H$ and a necessary and sufficient condition for the existence of N-relatively invariant measure is given. Among other things, conditions under which there is an N-invariant measure are investigated.

Keywords Doble coset space · Rho-function · N-invariant measure · N-relatively invariant measure

Mathematics Subject Classification Primary 47A55; Secondary 39B52

1 Introduction and Preliminaries

Let G be a locally compact group and H, K be two closed subgroups of G. The double coset space of G by H and K, respectively, is

$$K\backslash G/H = \{KxH;\ x \in G\}.$$

Communicated by Farshid Abdollahi.

✉ R. A. Kamyabi Gol
kamyabi@um.ac.ir

F. Fahimian
fatemefahimian@gmail.com

F. Esmaeelzadeh
esmaeelzadeh@bojnourdiau.ac.ir

1 Department of Mathematics, Center of Excellency in Analysis on Algebraic Structures (CEAAS), Ferdowsi University of Mashhad, Mashhad, Iran

2 Department of Mathematics, Bojnourd Branch, Islamic Azad University, Bojnourd, Iran

Author's personal copy

When K is the trivial group, the double coset space $K\backslash G/H$ is actually the homogeneous space G/H. The existence of strongly quasi-invariant measures on homogeneous spaces G/H was first proved by Mackey [12] under the assumption that G is second countable. Bruhat [3] and Loomis [11] showed how to obtain strongly quasi-invariant measures with no countability hypotheses. Also, the existence of a homomorphism rho-function causes the existence of a relatively invariant measure on G/H is in [13]. One may refer to [2,4,5,7,8,13] to find more informations about homogeneous space G/H. When $K = H$, a double coset space $K\backslash G/H$ changes to a hypergroup $G//H$ in which the homogeneous space G/H is a semi hypergroup (see more details [9]). It is worthwhile to note that the hypergroup plays important role in physics. In [6] the authors have constructed N-strongly quasi invariants measure on $K\backslash G/H$. It is worth mentioning that in [1,10] the conditions for the existence of N-relatively invariant measures and N-invariant measures are investigated, when K is a compact subgroup.

In this paper, we investigate the conditions for the existence of N-relatively invariant measure and N-invariant measure for $K\backslash G/H$, when K is an IN-group. Note that any compact group is an IN-group but not vice versa. Also, we will investigate the existence of a homomorphism rho-function ρ on G for triple (K, G, H), is necessary and sufficient for the existence of N-relatively invariant measure on $K\backslash G/H$. Some preliminaries and notations about coset space $K\backslash G/H$ and related measures on it are stated in Sect. 2. In Sects. 3 and 4, we introduce conditions of existence of N-relatively invariant measure and N-invariant measure on $K\backslash G/H$, where some relation between N-relatively invariant and N-strongly quasi invariant measures are considered.

2 Notations and Preliminary Results

Let G be a locally compact group and H, K be closed subgroups of G. Throughout this paper, we denote the left Haar measures of G, H and K, respectively, by $\mathrm{d}x$, $\mathrm{d}h$, $\mathrm{d}k$, and their modular functions by Δ_G, Δ_H and Δ_K, respectively. We recall if S is a non-empty locally compact Hausdorff space, an (left) action of G on S is a continuous map $(x, s) \mapsto xs$ from $G \times S$ to S such that $s \to xs$ is a homeomorphism of S for each $x \in G$, and $x(ys) = (xy)s$ for all $x, y \in G$ and $s \in S$. A space S equipped with an action of G is called a G-space. A G-space S is called transitive if for every $s, t \in S$ there exists $x \in G$ such that $xs = t$.

The notion of double coset space is a natural generalization of that coset space arising by two subgroups, simultaneously. Recall that if $K\backslash G/H$ is a double coset space of G by H and K, then elements of $K\backslash G/H$ are given by $\{KxH;\ x \in G\}$. The canonical mapping $q : G \to K\backslash G/H$, defined by $q(x) = KxH$, abbreviated by $\ddot{x}$, is surjective. The double coset space $K\backslash G/H$ equipped with the quotient topology, the largest topology that makes q continuous, is a locally compact Hausdorff space. In this topology, q is also an open mapping and **proper**—that is for each compact set $F \subseteq K\backslash G/H$ there is a compact set $E \subseteq G$ with $q(E) = F$. Let N be the normalizer of K in G, i.e.,

$$N = \{g \in G;\ gK = Kg\}.$$

Author's personal copy

Then, there is a naturally defined mapping

$$\varphi : N \times K\backslash G/H \rightarrow K\backslash G/H,$$

given by

$$\varphi\big(n, q(x)\big) = n.q(x) := q(nx),$$

one can verify that φ is a well-defined, continuous, transitive action of N on $K\backslash G/H$. We define the mapping Q from $C_c(G)$ onto $C_c(K\backslash G/H)$ by

$$Q(f)(KxH) = \int_K \int_H f(k^{-1}xh)\mathrm{d}h\mathrm{d}k,$$

then Q is a well-defined continuous linear map, as well as $supp\big(Q(f)\big) \subseteq q(supp\ f)$. We recall that a locally compact group G is called an IN-group if there is a compact unit neighbourhood U in G which is invariant under inner automorphisms, that is, $xUx^{-1} = U$ for all $x \in G$. It is known that IN-groups are unimodular. In [6], it is shown that when K is an IN-group then $Q(L_n f) = L_n Q(f)$ for all $n \in N$ and $f \in C_c(G)$.

Suppose that μ is a positive Radon measure on $K\backslash G/H$. The measure μ is called N-relatively invariant measure if there is a positive real character χ on N such that

$$\int_{K\backslash G/H} Q(f)(n\ddot{x})\mathrm{d}\mu(\ddot{x}) = \chi(n) \int_{K\backslash G/H} Q(f)(\ddot{x})\mathrm{d}\mu(\ddot{x}),$$

for all $n \in N$ and $f \in C_c(G)$. A measure μ is said to be an N- invariant measure if χ is identically 1. For a positive Radon measure μ let μ_n denote its translate by $n \in N$, that is $\mu_n(E) = \mu(n.E)$ for any Borel set E in $K\backslash G/H$. A measure μ is called N-strongly quasi invariant measure if there exists a positive continuous function λ associate to μ from $N \times K\backslash G/H$ such that $\mathrm{d}\mu_n(\ddot{y}) = \lambda(n, \ddot{y})\mathrm{d}\mu(\ddot{y})$. A rho-function for triple (K, G, H) is a positive locally integrable function ρ on G such that

$$\rho(kxh) = \frac{\Delta_H(h)\Delta_K(k)}{\Delta_G(h)}\rho(x),$$

for all $x \in G, h \in H, k \in K$.
In [6] it is shown that for each triple (K, G, H) there exists a rho-function ρ which construct N-strongly quasi invariant measure μ that satisfies:

$$\int_{K\backslash G/H} Q(f)(\ddot{x})\mathrm{d}\mu(\ddot{x}) = \int_G f(x)\rho(x)\mathrm{d}x,$$

for each $f \in C_c(G)$. In this case, we have

$$\lambda(n, \ddot{y}) = \frac{\rho(ny)}{\rho(y)}, \quad (n \in N, \quad \ddot{y} \in K\backslash G/H.) \tag{2.1}$$

Author's personal copy

Furthermore, λ satisfying:

(i) $\lambda(k, \ddot{x}) \cdot \lambda(k^{-1}, \ddot{x}) = 1$ for all $k \in K$

(ii) $\lambda(n, Kn^{-1}H) \cdot \lambda(n^{-1}, KH) = 1$ for all $n \in N$.

(iii) If for each $n \in N$, $\lambda_n : K\backslash G/H \to (0, +\infty)$ is constant function then $\lambda_{mn} = \lambda_m \cdot \lambda_n$, for all $m \in N$.

At this point, we recall the following theorem which has been proved in [6] and for completeness and readers' convenience, we include a proof.

Theorem 2.1 *With the assumptions as above, if μ is a positive Radon measure on $K\backslash G/H$, then the positive Radon measure $\tilde{\mu}$ on G defined by*

$$\int_G f(x)d\tilde{\mu}(x) = \int_{K\backslash G/H} Q(f)(\ddot{x})\mathrm{d}\mu(\ddot{x}), \tag{2.2}$$

satisfying

$$\int_G f(kxh^{-1})\mathrm{d}\tilde{\mu}(x) = \Delta_K(k)\Delta_H(h)\int_G f(x)\mathrm{d}\tilde{\mu}(x). \tag{2.3}$$

Conversely, if a positive Radon measure $\tilde{\mu}$ on G satisfying (2.3), then Eq. (2.2) defines a positive Radon measure μ on $K\backslash G/H$.

Proof Suppose that μ is a positive Radon measure on $K\backslash G/H$, then $\tilde{\mu}$ defined by (2.2) is clearly a positive Radon measure on G. Also, for each $h_0 \in H$, $k_0 \in K$ and $f \in C_c(G)$, we have

$$\begin{aligned}
\int_G f(k_0xh_0^{-1})\mathrm{d}\tilde{\mu}(x) &= \int_{K\backslash G/H}\int_K\int_H L_{k_0^{-1}} \circ R_{h_0^{-1}} f(k^{-1}xh)\mathrm{d}h\mathrm{d}k\mathrm{d}\tilde{\mu}(x)\\
&= \int_{K\backslash G/H}\int_K\int_H f(k_0k^{-1}xhh_0^{-1})\mathrm{d}h\mathrm{d}k\mathrm{d}\mu(\ddot{x})\\
&= \Delta_H(h_0)\Delta_K(k_0)\int_{K\backslash G/H} Q(f)(\ddot{x})\mathrm{d}\mu(\ddot{x})\\
&= \Delta_H(h_0)\Delta_K(k_0)\int_G f(x)\mathrm{d}\tilde{\mu}(x).
\end{aligned}$$

Conversely, suppose that the positive Radon measure $\tilde{\mu}$ on G satisfying (2.3). Then for $\varphi \in C_c(K\backslash G/H)$, define

$$\mu : C_c(K\backslash G/H) \to (0, +\infty),$$

by

$$\mu(\varphi) = \int_G f(x)\mathrm{d}\tilde{\mu},$$

in which $f \in C_c(G)$ such that $\varphi = Q(f)$. Now, μ is well-defined. Indeed, if $f \in C_c(G)$ such that $Q(f) = 0$. It can be seen, there is g in $C_c(G)$ such that $Q(g) \equiv 1$ on $Q(supp\ f)$. Using the Fubini's Theorem, we have

Author's personal copy

$$\begin{aligned}
\int_G f(x)\mathrm{d}\tilde{\mu} &= \int_G f(x)Q\big(g\big)\big(q(x)\big)\mathrm{d}\tilde{\mu}(x)\\
&= \int_G f(x)\int_K\int_H g(k^{-1}xh)\mathrm{d}h\mathrm{d}k\mathrm{d}\tilde{\mu}(x)\\
&= \int_K\int_H\int_G f(kxh^{-1})g(x)\Delta_K(k)\Delta_H(h)\mathrm{d}\tilde{\mu}(x)\mathrm{d}h\mathrm{d}k\\
&= \int_G g(x)\int_H\int_K f(k^{-1}xh)\mathrm{d}k\mathrm{d}h\mathrm{d}\tilde{\mu}(x)\\
&= \int_G g(x)Q(f)\big(q(x)\big)\mathrm{d}\tilde{\mu}(x) = 0.
\end{aligned}$$

It is easy to check that μ is a positive linear functional, therefore, it induces a positive Radon measure μ on $K\backslash G/H$ such that,

$$\int_G f(x)d\tilde{\mu}(x) = \int_{K\backslash G/H} Q(f)(\ddot{x})\mathrm{d}\mu(\ddot{x}).$$

□

3 *N*-Relatively Invariant Measure on *K**G*/*H*

In this section, we give some results concerning N-relatively invariant measure for the double coset space $K\backslash G/H$, when K is an IN-group. From now on we consider K as an IN-group. Recall that a positive Radon measure μ on $K\backslash G/H$ is called N-relatively invariant measure if μ is not identically zero and there is a positive real character χ on N such that

$$\int_{K\backslash G/H} Q(f)(n\ddot{x})\mathrm{d}\mu(\ddot{x}) = \chi(n)\int_{K\backslash G/H} Q(f)(\ddot{x})\mathrm{d}\mu(\ddot{x}),$$

for all $n \in N$ and $f \in C_c(G)$. Such a χ is called the modular function of μ. An N-relatively invariant measure μ is said to be N-invariant if its modular function is identically 1.

In the next proposition it is shown that there exists an N-relatively invariant measure on double coset space under some reasonable assumptions.

Proposition 3.1 *If ξ is a real character on G such that $\Delta_G(h) = \Delta_H(h)\xi(h)$ for all $h \in H$ and $\xi\big|_K = \Delta_K$, then there is an N-relatively invariant measure μ on $K\backslash G/H$ such that supp $\mu = K\backslash G/H$ and $\xi\big|_N$ is the modular function of μ.*

Proof Assume that $\mathrm{d}\tilde{\mu}(x) = \xi^{-1}(x)\mathrm{d}x$. According to (2.3), we have

$$\begin{aligned}
\int_G f(kxh^{-1})\mathrm{d}\tilde{\mu}(x) &= \int_G f(kxh^{-1})\xi^{-1}(x)\mathrm{d}x\\
&= \int_G f(xh^{-1})\xi^{-1}(k^{-1}x)\mathrm{d}x
\end{aligned}$$

Author's personal copy

$$
\begin{aligned}
&= \int_G f(x)\xi^{-1}(k^{-1}xh)\Delta_G(h)\mathrm{d}x \\
&= \xi^{-1}(k^{-1})\xi^{-1}(h)\Delta_G(h)\int_G f(x)\xi^{-1}(x)\mathrm{d}x \\
&= \xi(k)\Delta_H(h)\int_G f(x)\mathrm{d}\tilde{\mu}(x) \\
&= \Delta_K(k)\Delta_H(h)\int_G f(x)\mathrm{d}\tilde{\mu}(x),
\end{aligned}
$$

where $f \in C_c(G)$ and $h \in H$ and $k \in K$. Now Theorem 2.1 guarantee the existence of a positive Radon measure μ on $K\backslash G/H$. Also, the measure μ is relatively invariant with modular function $\xi\big|_N$. Indeed,

$$
\begin{aligned}
\int_{K\backslash G/H} Q(f)(n\ddot{x})\mathrm{d}\mu(\ddot{x}) &= \int_{K\backslash G/H} Q(L_{n^{-1}}f)(\ddot{x})\mathrm{d}\mu(\ddot{x}) \\
&= \int_G (L_{n^{-1}}f)(x)\xi^{-1}(x)\mathrm{d}x \\
&= \int_G f(nx)\xi^{-1}(x)\mathrm{d}x \\
&= \int_G f(x)\xi^{-1}(n^{-1}x)\mathrm{d}x \\
&= \xi(n)\int_G f(x)\xi^{-1}(x)\mathrm{d}x \\
&= \xi(n)\int_G Q(f)(\ddot{x})\mathrm{d}\mu(\ddot{x}),
\end{aligned}
$$

and since the support of left Haar measure on G is G, therefore, $supp\ \mu = K\backslash G/H$. □

Corollary 3.2 *If Δ_H can be extended to a real character ξ on G such that $\dfrac{\Delta_G}{\xi|_K} = \Delta_K$, then there exists an N-relatively measure μ on $K\backslash G/H$ with supp $\mu = K\backslash G/H$.*

Proof If ξ is a real character on G such that $\xi\big|_H = \Delta_H$ then $\dfrac{\Delta_G}{\xi}$ is a real character on G and $\Delta_G(h) = (\frac{\Delta_G}{\xi}(h))\Delta_H(h)$ for all $h \in H$. The conclusion then follows from Theorem 3.1. □

Proposition 3.3 *Let μ be an N-strongly quasi invariant measure on $K\backslash G/H$ with function $\lambda : N \times K\backslash G/H \to (0, +\infty)$ such that the function λ is constant on double cosets. Then μ is N-relatively invariant measure.*

Proof Suppose that $n \in N$ and $f \in C_c(G)$. Then since for each $n \in N$, $\lambda_n(\ddot{x}) = c_n$ is constant, then we may define $\chi : N \to (0, +\infty)$ by $\chi(n) = c_n$. Therefore, we have

Author's personal copy

$$\int_{K\backslash G/H} Q(f)(n\ddot{x})\mathrm{d}\mu(\ddot{x}) = \int_{K\backslash G/H} Q(f)(\ddot{x})c_n \mathrm{d}\mu(\ddot{x})$$
$$= c_n \int_{K\backslash G/H} Q(f)(\ddot{x})\mathrm{d}\mu(\ddot{x})$$
$$= \chi(n) \int_{K\backslash G/H} Q(f)(\ddot{x})\mathrm{d}\mu(\ddot{x}).$$

□

In the next theorem, we show that there is a rho function homomorphism on G if and only if the measure μ is N-relatively invariant measure on double coset space.

Theorem 3.4 *The existence of a homomorphism rho-function $\rho : G \to (0, +\infty)$, for the triple (G, H, K) is necessary and sufficient for the existence of an N-relatively invariant measure on $K\backslash G/H$.*

Proof Let μ be the N-strongly quasi invariant measure which arises from a rho-function ρ ([6]). If ρ is a homomorphism then by (2.1) we get $\mathrm{d}\mu_n = \rho(n)\mathrm{d}\mu$, for all $n \in N$. That is μ is N-relatively invariant. Conversely, if μ is an N-relatively invariant measure, then there is a continuous homomorphism $\chi : G \to (0, +\infty)$ such that $\mathrm{d}\mu_n = \chi(n)\mathrm{d}\mu$ for all $n \in N$. So for all $n \in N$ and $f \in C_c(G)$ we can write

$$\int_G f(y)\rho(ny)\mathrm{d}y = \int_G f(n^{-1}y)\rho(y)\mathrm{d}y$$
$$= \int_{K\backslash G/H} Q(f)(Kn^{-1}yH)\mathrm{d}\mu(\ddot{y})$$
$$= \int_{K\backslash G/H} Q(f)(KyH)\mathrm{d}\mu_n(\ddot{y})$$
$$= \int_{K\backslash G/H} Q(f)(\ddot{y})\chi(n)\mathrm{d}\mu(\ddot{y})$$
$$= \chi(n)\int_{K\backslash G/H} Q(f)(\ddot{y})\mathrm{d}\mu(\ddot{y})$$
$$= \chi(n)\int_G f(y)\rho(y)\mathrm{d}y.$$

Thus for a fixed $n \in N$ we have

$$\int_G f(y)(\rho(ny) - \chi(n)\rho(y))\mathrm{d}y = 0,$$

for all $f \in C_c(G)$. This leads to

$$\frac{\rho(ny)}{\rho(y)} = \chi(n) \quad (n \in N, y \in G).$$

Author's personal copy

Now define $\rho_0 : G \to (0, \infty)$ by $\rho_0(x) = \frac{\rho(x)}{\rho(e)}$. Then ρ_0 is a homomorphism rho-function for (G, K, H) since

$$\begin{aligned}\rho_0(kxh) &= \frac{\rho(kxh)}{\rho(e)} = \frac{\Delta_H(h)\Delta_K(k)}{\Delta_G(h)}\frac{\rho(x)}{\rho(e)}\\ &= \frac{\Delta_H(h)\Delta_K(k)}{\Delta_G(h)}\rho_0(x),\end{aligned}$$

and

$$\begin{aligned}\rho_0(xy) &= \frac{\rho(xy)}{\rho(e)}\\ &= \frac{\rho(xy)}{\rho(y)}\frac{\rho(y)}{\rho(e)}\\ &= \rho_0(x)\rho_0(y).\end{aligned}$$

□

It is worthwhile to mention that if μ is the N-relatively invariant Radon measure which arises from a rho-function ρ, then using (2.1) we get

$$\rho(xy) = \frac{\rho(x)\rho(y)}{\rho(e)} \quad (x, y \in G). \tag{3.1}$$

In [10], the following Theorems 3.5, 3.6 are proven when K is a compact subgroup of G. Here, the compactness of K is replaced as an IN-group but the proofs are almost the same, so we omit the proofs.

Theorem 3.5 *Suppose that N is an open subgroup of G. If μ is the N-relatively invariant measure on $K\backslash G/H$ with modular function χ such that $\operatorname{supp}\mu \cap q(N) \neq \emptyset$, then $\operatorname{supp}\mu \supset q(N)$ and $\Delta_G(t) = \chi(t)\Delta_H(t)$ for all $t \in N \cap H$.*

Theorem 3.6 *Let $H \subseteq N$ and μ be the N-relatively invariant measure on $K\backslash G/H$ with modular function χ such that $\mu\big|_{q(N)} \neq 0$. Then $\Delta_N(h) = \chi(h)\Delta_H(h)$ for all $h \in H$. Conversely if χ is a real character on N such that $\Delta_N(h) = \chi(h)\Delta_H(h)$ for all $h \in H$, then there exists an N-relatively invariant measure μ on $K\backslash G/H$ with χ as its modular function such that $\mu\big|_{q(N)} \neq 0$.*

4 *N*-Invariant Measure on *K**G*/*H*

In this section, we give some results about N-invariant measure on $K \backslash G/H$. Note that if μ is an N-strongly quasi invariant measure on $K\backslash G/H$ with the associate function $\lambda \equiv 1$ on $N \times K\backslash G/H$, then it is clear that μ is N-invariant measure.
In the following, we investigate that a necessary and sufficient condition for the existence of the N-invariant measure on double coset space $K\backslash G/H$.

Author's personal copy

Theorem 4.1 *With the notations as above, μ is N-invariant measure on $K\backslash G/H$ if and only if $\tilde{\mu}$ has the following property.*

$$\int_G f(nxh^{-1})\mathrm{d}\tilde{\mu}(x) = \Delta_H(h)\int_G f(x)\mathrm{d}\tilde{\mu}(x) \quad \text{for all } f \in C_c(G) \tag{4.1}$$

Proof Since K is an IN-group so $\Delta_K = 1$. Consider μ as N-invariant measure. Hence $\chi \equiv 1$. We claim $\tilde{\mu}$ defined by

$$\int_G f(x)\mathrm{d}\tilde{\mu}(x) = \int_{K\backslash G/H} Q(f)(\ddot{x})\mathrm{d}\mu(\ddot{x}),$$

is a positive Radon measure on G satisfying (4.1). Using Theorem 2.1 we get,

$$\begin{aligned}
\int_G f(nxh^{-1})\mathrm{d}\tilde{\mu}(x) &= \Delta_H(h)\int_G f(nx)\mathrm{d}\tilde{\mu}(x)\\
&= \Delta_H(h)\int_G L_{n^{-1}}f(x)\mathrm{d}\tilde{\mu}(x)\\
&= \Delta_H(h)\int_{K\backslash G/H} Q(L_{n^{-1}}f)(\ddot{x})\mathrm{d}\mu(\ddot{x})\\
&= \Delta_H(h)\int_{K\backslash G/H} L_{n^{-1}}Q(f)(\ddot{x})\mathrm{d}\mu(\ddot{x})\\
&= \Delta_H(h)\int_G Q(f)(n\ddot{x})\mathrm{d}\mu(\ddot{x})\\
&= \Delta_H(h)\int_G Q(f)(\ddot{x})\mathrm{d}\mu(\ddot{x})\\
&= \Delta_H(h)\int_G f(x)\mathrm{d}\tilde{\mu}(x).
\end{aligned}$$

Conversely, suppose that $\tilde{\mu}$ satisfying (4.1). Using Theorem 2.1 and the fact $\Delta_K = 1$, there is positive Radon measure μ on $K\backslash G/H$. Moreover, μ is N-invariant. In fact we have,

$$\begin{aligned}
\int_{K\backslash G/H} Q(f)(n\ddot{x})\mathrm{d}\mu(\ddot{x}) &= \int_{K\backslash G/H} L_{n-1}Q(f)(\ddot{x})\mathrm{d}\mu(\ddot{x})\\
&= \int_{K\backslash G/H} Q(L_{n^{-1}}f)(\ddot{x})\mathrm{d}\mu(\ddot{x})\\
&= \int_G L_{n^{-1}}f(x)\mathrm{d}\tilde{\mu}(x)\\
&= \int_G f(nx)d\tilde{\mu}(x)\\
&= \int_G f(x)\tilde{\mu}(x)
\end{aligned}$$

Author's personal copy

$$= \int_{K\backslash G/H} Q(f)(\ddot{x})\mathrm{d}\mu(\ddot{x}).$$

□

In the following theorem, it is shown that there is an N-invariant measure μ on $K\backslash G/H$ if and only if $\Delta_G|_H = \Delta_H$.

Theorem 4.2 *With the above notation, $\Delta_G(h) = \Delta_H(h)$, for all $h \in H$ if and only if there exists an N-invariant measure μ on $K\backslash G/H$ with supp $\mu = K\backslash G/H$.*

Proof If $\Delta_G(h) = \Delta_H(h)$ for all $h \in H$ then the left Haar measure $\mathrm{d}x$ on G satisfies (4.1) in Theorem 4.1. That is,

$$\int_G f(nxh^{-1})\mathrm{d}x = \Delta_H(h)\int_G f(x)\mathrm{d}x.$$

Therefore, μ defined by $\int_{K\backslash G/H} Q(f)(\ddot{x})\mathrm{d}\mu(\ddot{x}) = \int_G f(x)\mathrm{d}x$ is an N-invariant measure on $K\backslash G/H$. Conversely, if $K\backslash G/H$ has N-invariant measure then we have,

$$\begin{aligned}\Delta_H(h)\int_G f(x)\mathrm{d}\tilde{\mu}(x) &= \int_G f(xh^{-1})\mathrm{d}\tilde{\mu}(x)\\ &= \Delta_G(h)\int_G f(x)\mathrm{d}\tilde{\mu}(x).\end{aligned}$$

Therefore, $\Delta_G(h) = \Delta_H(h)$ for all $h \in H$. □

Corollary 4.3 *If G is unimodular and H is an IN-group then, there exists an N-invariant measure μ on $K\backslash G/H$ with supp $\mu = K\backslash G/H$.*

Corollary 4.4 *If $K = \{e\}$ and if $\Delta_H(h) = \Delta_G(h)$ for all $h \in H$. Then there exists a G-invariant measure μ on G/H with supp $\mu = G/H$.*

Remark 4.5 Suppose that H and K are subgroups of locally compact group G. If H is compact or normal or open in G, then there is an N-invariant measure μ on $K\backslash G/H$ with $supp\ \mu = K\backslash G/H$.

References

1. Amini, M., Medghalchi, A.: Harmonic analysis on double coset spaces. arXiv:math/0205259 [math.FA] (2002)
2. Bourbaki, N.: Elements of Mathematics Integration (I), (II) (2004)
3. Bruhat, F.: Sur les reprsentations induites des groupes de Lie. Bull. Soc. Math. France. **81**, 97–205 (1956)
4. Derikvand, T., Kamyabi Gol, R.A., Janfada, M.: On the generalized quotient integrals on homogeneous spaces. J. Appl. Anal. **23**(2), 59–65 (2017)
5. Esmaeelzadeh, F., Gol, RAKamyabi: Homogeneous spaces and square-integrable representations. Ann. Funct. Anal. **7**(1), 9–16 (2016)

Author's personal copy

6. Fahimian, F., Gol, R. A. Kamyabi, Esmaeelzadeh, F.: N-strongly quasi invariant measure on double coset spaces. arXiv:math/1807.00132 [math.RT] (2017)
7. Folland, B. G.: A Course in Abstract Harmonic Analysis. CRC press, Inc (1995)
8. Farashahi, A.Ghaani: A class of abstract linear representations for convolution function algebras over homogeneous spaces of compact groups. Can. J. Math. **70**(1), 97–116 (2018)
9. Jewett, R.I.: Spaces with an abstract convolution of measures. Adv. Math. **18** (1975)
10. Liu, T.S.: Invariant measure on double coset spaces. J. Aust. Math. Soc. **5**, 495–505 (1965)
11. Loomis, L.H.: Positive definite functions and induced representations. Duke Math. J. **27**, 569–579 (1960)
12. Mackey, G.W.: Induced representations of locally compact groups I. Ann. Math. **55**, 101–139 (1959)
13. Reiter, H., Stegeman, J.: Classical Harmonic Analysis and Locally Compact Groups. Oxford University Press, Oxford (2000)