Four-derivative couplings via the T-duality invariance constraint

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(Received 8 May 2019; published 12 June 2019)

We examine the proposal that the dimensional reduction of the effective action of perturbative string theory on a circle should be invariant under T-duality transformations. The T-duality transformations are the standard Buscher rules plus some higher covariant derivatives. By explicit calculations at order α' for metric, dilaton, and B-field, we show that the T-duality constraint can fix both the effective action and the higher derivative corrections to the Buscher rules up to an overall factor. The corrections depend on the scheme that one uses for the effective action. We have found the effective action and its corresponding T-duality transformations in an arbitrary scheme.

DOI: 10.1103/PhysRevD.99.126005

I. INTRODUCTION

One of the most exciting discoveries in perturbative string theory is T-duality [1,2]. This duality may be used to construct the D-dimensional effective field theory at any order of α'. One approach for constructing this effective action is the double field theory (DFT) [3–7] in which the effective action in $2D$-space is invariant under T-duality and a gauge transformation. The T-duality is the standard $O(D,D)$ transformation, whereas the gauge transformation is nonstandard and receives α' corrections [7–10]. Another proposal for constructing the D-dimensional effective action is to use the T-duality constraint on the reduction of the effective action on a circle [11]. In this approach one reduces the standard gauge invariance effective action on a circle to produce the corresponding $(D-1)$-dimensional effective action. Up to some boundary terms, this action should be invariant under the T-duality transformations which are the standard Buscher rules [12,13] plus some α' corrections [14–16]. Using this proposal, the known gravity and dilaton couplings in the effective actions at orders α', α'^2, α'^3 have been found up to some overall factors [17,18]. The corrections to the Buscher rules, however, could not be fixed in the case that B-field is zero. For the effective action at order α' that has been found in [19], the form of α' corrections to the Buscher rules have been found in [16] for the case that B-field is nonzero.

In this paper we speculate that in the presence of B-field, the T-duality constraint may fix both the effective action and the α' corrections to the Buscher rules. We have done this calculation explicitly at order α'. Using the Bianchi identities and the field redefinition freedom, one can write the most general D-dimensional covariant action at the four-derivative level in a specific scheme which has eight parameters [20]. We then reduce it on a circle to find its corresponding $(D-1)$-dimensional action which should be invariant under the T-duality transformations up to some boundary terms. Constraining this action to be invariant under the Buscher rules makes all parameters to be zero unless one adds some corrections to the Buscher rules. We then write the most general covariant corrections at the two-derivative level to the Buscher rules and impose the $(D-1)$-dimensional action to be invariant under this deformed T-duality transformations. Interestingly, the T-duality constraint fixes all parameters in the effective actions and in the deformed T-duality transformations, up to an overall factor. The effective action is exactly the standard action that has been found by the S-matrix calculation [20]. The T-duality transformations, however, are not the same as the T-duality transformations that have been found in [16] because the effective action that we have found and the effective action that has been used in [16] are in different schemes.

Since the T-duality transformations depend on the scheme that one uses for the effective action, it would be desirable to find the T-duality transformations for the effective action in an arbitrary scheme. We will show that the T-duality constraint can fix the effective action even if one does not use the field redefinition. If fact, using the Bianchi identity and removing total derivative terms, one finds that the most general D-dimensional effective action at order α' has 20 parameters [20]. Three of them are

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unambiguous as they are not changed under field redefinitions, and the other 17 parameters which are ambiguous are changed under the field redefinitions. However, there are five combinations of these parameters that remain invariant under the field redefinitions. To have the minimum number of couplings, one should keep five parameters which are called essential parameters and remove all other parameters [20]. In general, one may keep all ambiguous parameters. In this case, the S-matrix calculations should fix the three unambiguous and the five essential parameters. The other 12 parameters should remain arbitrary. We will show that the T-duality constraint on the most general effective action with the 20 parameters fixes the effective action up to 12 arbitrary parameters; one of them is an unambiguous parameter and all other 11 parameters are ambiguous parameters. The T-duality transformations are also found in terms of these parameters. Any choice for these 11 parameters gives the effective action and its corresponding T-duality transformations in a specific scheme. We will show that the effective action for a specific choice for these parameters becomes the action that has been found in [19] and the corresponding T-duality transformations are exactly the one that has been found in [16].

The outline of the paper is as follows: In Sec. II, we perform the calculations at order α'^0. In particular, we write the most general D-dimensional effective action at the two-derivative level which has three parameters. We then reduce it on a circle to find its corresponding $(D-1)$-dimensional effective action. Constraining it to be invariant under the Buscher rules up to some boundary terms, the three parameters are fixed up to an overall factor.

In Sec. III, we perform the calculations at order α'. In Sec. III. A, we consider the eight-parameter effective action in the specific field variables studied in [20]. We show that the reduction of this action on a circle is invariant under the Buscher rules when all parameters in the effective action are zero. To have nonzero effective action at order α', we then deform the Buscher rules by some terms at order α' with arbitrary parameters. Some relations between these parameters are found by the constraint that the T-duality transformations must form a Z_2-group. Constraining the reduction of the effective actions at orders α'^0 and α' to be invariant under the deformed T-duality transformations fixes all independent parameters in the deformed T-duality transformations and in the effective action. Up to an overall factor, the effective action is the one that has been found in [20] by the S-matrix method. In Sec. III. B, we consider the 20-parameter effective action in which the field redefinitions are not used. We then impose the T-duality constraint on this action. We find the effective action and the corresponding T-duality transformations in terms of one unambiguous parameter and 11 ambiguous parameters. A specific choice for these 11 parameters, gives the effective action and the T-duality transformations found in [16,19]. In this subsection, we have also shown that the Chern-Simons couplings in the heterotic theory which results from the nonstandard gauge transformation of B-field is also invariant under the T-duality and we found its corresponding T-duality transformations.

II. EFFECTIVE ACTION AT ORDER α'^0

We now construct the most general D-dimensional action at the two-derivative level which is invariant under the coordinate transformations and under the standard gauge transformation of B-field, i.e., $B_{ab} \to B_{ab} + \partial_{[a}\lambda_{b]}$. Up to total derivative terms, it has the following three terms:

$$\mathbf{S}_0 = -\frac{2}{\kappa^2}\int d^D x e^{-2\Phi}\sqrt{-G}(c_1 R + c_2 \nabla_a \Phi \nabla^a \Phi + c_3 H^2), \tag{1}$$

where the three-form H is field strength of the two-form B, i.e., $H_{abc} = \partial_a B_{bc} + \partial_c B_{ab} + \partial_b B_{ca}$, and c_1, c_2, c_3 are three constants.

To impose Abelian T-duality constraint on this action, we have to consider a background with $U(1)$ isometry. It is convenient to use the following background for metric and Kalb-Ramond field,

$$G_{ab} = \begin{pmatrix} \bar{g}_{\mu\nu} + e^{\varphi} g_\mu g_\nu & e^{\varphi} g_\mu \\ e^{\varphi} g_\nu & e^{\varphi} \end{pmatrix},$$
$$B_{ab} = \begin{pmatrix} \bar{b}_{\mu\nu} + \frac{1}{2} b_\mu g_\nu - \frac{1}{2} b_\nu g_\mu & b_\mu \\ -b_\nu & 0 \end{pmatrix}, \tag{2}$$

where $\bar{g}_{\mu\nu}$, $\bar{b}_{\mu\nu}$ are the metric and the B-field and g_μ, b_μ are two vectors in the $(D-1)$-dimensional base space. The inverse of the above D-dimensional metric is

$$G^{ab} = \begin{pmatrix} \bar{g}^{\mu\nu} & -g^\mu \\ -g^\nu & e^{-\varphi} + g_\alpha g^\alpha \end{pmatrix}, \tag{3}$$

where $\bar{g}^{\mu\nu}$ is the inverse of the $(D-1)$-dimensional metric which raises the index of the vectors. In this parametrization, the $(D-1)$-dimensional dilaton is $\bar{\phi} = \Phi - \varphi/4$. The Buscher rules [12,13] in this parametrization are the following linear transformations:

$$\varphi' = -\varphi, \qquad g'_\mu = b_\mu, \qquad b'_\mu = g_\mu,$$
$$\bar{g}'_{\alpha\beta} = \bar{g}_{\alpha\beta}, \qquad \bar{b}'_{\alpha\beta} = \bar{b}_{\alpha\beta}, \qquad \bar{\phi}' = \bar{\phi}. \tag{4}$$

They form a Z_2-group, i.e., $(x')' = x$ where x is any field in the base space.

To simplify the calculations, we assume that the base space is flat, i.e., $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$. As long as the T-duality constraint fixes all coefficients in the effective action in this case, we do not need to consider the general case of the curved base space. If the effective action contains terms with at most second derivative, i.e., $R, \nabla\nabla\Phi, \nabla H$, the covariant derivatives in the $(D-1)$-dimensional base space can be written as ordinary derivatives in the local frame in which $\Gamma_{\mu\nu}{}^{\alpha}=0$. However, the curvature terms in the base space are not zero in the local frame. One expects the coefficients of these terms appear in many other terms such as $\nabla\nabla\varphi$ which might be fixed by the T-duality constraint when the base space is flat.

In order to reduce R, one should write the curvature in terms of metric G_{ab} and then use the reductions (2) and (3). When the base space is flat, it becomes

$$R=-\partial^\mu\partial_\mu\varphi-\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi-\frac{1}{4}e^{\varphi}V^2, \tag{5}$$

where $V_{\mu\nu}$ is the field strength of the $U(1)$ gauge field g_μ, i.e., $V_{\mu\nu}=\partial_\mu g_\nu-\partial_\nu g_\mu$. For curved base space, the ordinary derivatives in (5) become covariant derivatives and there is also the scalar curvature of the base space. The reduction of the overall factor and the second term in (1) when the base space is flat are

$$\begin{aligned}&e^{-2\Phi}\sqrt{-G}=e^{-2\bar\phi},\\ &\nabla_a\Phi\nabla^a\Phi=\partial_\mu\bar\phi\partial^\mu\bar\phi+\frac{1}{2}\partial_\mu\bar\phi\partial^\mu\varphi+\frac{1}{16}\partial_\mu\varphi\partial^\mu\varphi.\end{aligned} \tag{6}$$

For the curved base space, there is a factor of $\sqrt{-\bar g}$ in the right-hand side of the first equation. The reduction of the third term in (1) is

$$H^2=\bar H_{\mu\nu\alpha}\bar H^{\mu\nu\alpha}+3e^{-\varphi}W^2, \tag{7}$$

where $W_{\mu\nu}$ is the field strength of the $U(1)$ gauge field b_μ, i.e., $W_{\mu\nu}=\partial_\mu b_\nu-\partial_\nu b_\mu$. The three-form $\bar H$ is defined as $\bar H_{\mu\nu\alpha}=\tilde H_{\mu\nu\alpha}-g_\mu W_{\nu\alpha}-g_\alpha W_{\mu\nu}-g_\nu W_{\alpha\mu}$ where the three-form $\tilde H$ is the field strength of the two-form $\bar b_{\mu\nu}+\frac{1}{2}b_\mu g_\nu-\frac{1}{2}b_\nu g_\nu$ in (2). The three-form $\bar H$ is not the field strength of a two-form. It satisfies the following Bianchi identity [16]:

$$\partial_{[\mu}\bar H_{\nu\alpha\beta]}=-\frac{3}{2}V_{[\mu\nu}W_{\alpha\beta]}. \tag{8}$$

To find the T-duality transformation of the three-form $\bar H$, one can rewrite it as

$$\begin{aligned}\bar H_{\mu\nu\alpha}=\hat H_{\mu\nu\alpha}&-\frac{1}{2}g_\mu W_{\nu\alpha}-\frac{1}{2}g_\alpha W_{\mu\nu}-\frac{1}{2}g_\nu W_{\alpha\mu}\\ &-\frac{1}{2}b_\mu V_{\nu\alpha}-\frac{1}{2}b_\alpha V_{\mu\nu}-\frac{1}{2}b_\nu V_{\alpha\mu},\end{aligned} \tag{9}$$

where $\hat H$ is the field strength of the T-duality invariance two-form $\bar b_{\mu\nu}$. It is evident that $\bar H$ is invariant under the T-duality transformations (4). Using the above relation, one may rewrite H^2 in (7) in terms of $\hat H$ which satisfies the standard Bianchi identity $d\hat H=0$, and some other terms that are not $U(1)\times U(1)$ gauge invariance. However, it is more convenient to write H^2 in terms of $\bar H$ which satisfies the anomalous Bianchi identity (8), and some gauge invariant terms as in (7).

The reduction of (1) when the base space is flat then becomes

$$\begin{aligned}S_0=-\frac{2}{\kappa^2}\int d^{D-1}xe^{-2\bar\phi}\bigg[&\left(-\frac{1}{2}c_1+\frac{1}{16}c_2\right)\partial_\mu\varphi\partial^\mu\varphi\\ &+c_2\partial_\mu\bar\phi\partial^\mu\bar\phi+c_3\bar H^2-c_1\partial^\mu\partial_\mu\varphi+\frac{1}{2}c_2\partial^\mu\bar\phi\partial_\mu\varphi\\ &-\frac{1}{4}c_1e^{\varphi}V^2+3c_3e^{-\varphi}W^2\bigg].\end{aligned} \tag{10}$$

For the curved base space, there is the factor $\sqrt{-\bar g}$ and the scalar curvature term $c_1\bar R$, and the partial derivatives become covariant derivatives. The terms in the first line are invariant under the Buscher rules.

The T-duality constraint is that the reduced action (10) must be invariant under T-duality up to some boundary terms, i.e.,

$$\begin{aligned}\delta S_0&\equiv S_0-S_0{}'\\ &=-\frac{2}{\kappa^2}\int d^{D-1}xe^{-2\bar\phi}\bigg[-2c_1\partial^\mu\partial_\mu\varphi+c_2\partial^\mu\bar\phi\partial_\mu\varphi\\ &\quad+\left(\frac{1}{4}c_1+3c_3\right)(e^{-\varphi}W^2-e^{\varphi}V^2)\bigg]\end{aligned} \tag{11}$$

must be a boundary term. Note that δS_0 is odd under the T-duality transformations and is invariant under the $U(1)\times U(1)$ gauge transformations. One can easily observe that δS_0 is a boundary term when

$$c_3=-\frac{1}{12}c_1;\qquad c_2=4c_1. \tag{12}$$

This fixes the D-dimensional effective action to be

$$\mathbf{S}_0=-\frac{2c_1}{\kappa^2}\int dxe^{-2\Phi}\sqrt{-G}\left(R+4\nabla_a\Phi\nabla^a\Phi-\frac{1}{12}H^2\right), \tag{13}$$

which is the standard effective action at order α'^0, up to an overall factor. The overall factor must be $c_1=1$ to be the effective action of string theory. In the next section, we extend these calculations to the order α'.

III. EFFECTIVE ACTION AT ORDER α'

The most general D-dimensional effective action at order α' which is invariant under the coordinate transformation and under the B-field gauge transformation has three classes. One class contains terms that are zero by Bianchi identities, one class contains terms that are total derivative terms, and all other terms belong to the third class. There are 20 such terms in which the field variables are arbitrary [20].

Using the field redefinition freedom, however, one may write the effective action in specific field variables. In this case there are eight independent couplings [20]. The T-duality constraint may be used for both specific field variables and for arbitrary field variables. In the next subsection we use the T-duality constraint for specific field variables.

A. Effective action in a specific scheme

Using the field redefinition freedom, one can write the 20-parameter effective action at order α' in terms of independent couplings. There are also choices for these minimal couplings. One may choose the couplings to be [20]

$$\mathbf{S}_1 = \frac{-2}{\kappa^2}\alpha' \int d^D x e^{-2\Phi}\sqrt{-G}[b_1 R_{abcd}R^{abcd} + b_2 R_{abcd}H^{abe}H^{cd}{}_e + b_3 H_{fgh}H^f{}_a{}^b H^g{}_b{}^c H^h{}_c{}^a + b_4 H_f{}^{ab}H_{gab}H^{fch}H^g{}_{ch} + b_5 H_{acd}H_b{}^{cd}\partial^a\Phi\partial^b\Phi + b_6 (H^2)^2 + b_7 H^2\partial_a\Phi\partial^a\Phi + b_8(\partial_a\Phi\partial^a\Phi)^2], \tag{14}$$

where $b_1, b_2, \ldots, b_8$ are eight parameters. The field redefinition freedom allows us to choose the eight arbitrary couplings in many different schemes. The above is one particular scheme. The above parameters have been found in [20] by the S-matrix method. We are going to show that the proposed T-duality constraint can fix these parameters up to an overall factor.

To impose the T-duality constraint on the couplings in (14), one should reduce it on the background with the $U(1)$ isometry as in the previous section. The reduction of the terms in the last line can easily be read from the reductions of the corresponding terms in (6) and (7). When the base space is flat, the reduction of the first, the second, the third, the fourth, and the fifth terms in (14) are

$$\begin{aligned}
\text{first} &= \partial_\mu\partial_\nu\varphi(\partial^\mu\partial^\nu\varphi + \partial^\mu\varphi\partial^\nu\varphi)\frac{1}{4}(\partial_\mu\varphi\partial^\mu\varphi)^2 + e^{2\varphi}\left(\frac{5}{8}V_\mu{}^\nu V^{\mu\alpha}V_\alpha{}^\beta V_{\nu\beta} + \frac{3}{8}(V^2)^2\right) \\
&\quad + e^{\varphi}\left[\partial_\mu V_{\nu\alpha}(\partial^\mu V^{\nu\alpha} + 3\partial^\mu\varphi V^{\nu\alpha}) - (\partial_\mu\partial_\nu\varphi - \partial_\mu\varphi\partial_\nu\varphi)V^{\mu\alpha}V^\nu{}_\alpha + \frac{3}{2}\partial_\mu\varphi\partial^\mu\varphi V^2\right], \\
\text{second} &= -e^{-\varphi}(2\partial_\mu\partial_\nu\varphi + \partial_\mu\varphi\partial_\nu\varphi)W^{\mu\alpha}W^\nu{}_\alpha - \frac{1}{2}e^{\varphi}(\bar{H}_{\mu\alpha}{}^\gamma\bar{H}_{\nu\beta\gamma} + \bar{H}_{\mu\nu}{}^\gamma\bar{H}_{\alpha\beta\gamma})V^{\mu\nu}V^{\alpha\beta} \\
&\quad + 2\bar{H}_{\nu\alpha\beta}(\partial_\mu V^{\nu\alpha}W^{\mu\beta} + \partial_\mu\varphi V^{\nu\alpha}W^{\mu\beta} + \partial^\nu\varphi V^{\mu\alpha}W_\mu{}^\beta) - \frac{1}{2}V_{\mu\nu}V_{\alpha\beta}(W^{\mu\alpha}W^{\nu\beta} + W^{\mu\nu}W^{\alpha\beta}) + V_{\mu\alpha}V^{\mu\beta}W_\beta{}^\nu W^\alpha{}_\nu, \\
\text{third} &= \bar{H}_\mu{}^{\beta\gamma}\bar{H}^{\mu\nu\alpha}\bar{H}_{\nu\beta}{}^\lambda\bar{H}_{\alpha\gamma\lambda} + 3e^{-2\varphi}W_\mu{}^\alpha W^{\mu\nu}W_\nu{}^\beta W_{\alpha\beta} + 6e^{-\varphi}\bar{H}_{\mu\alpha}{}^\gamma\bar{H}_{\nu\beta\gamma}W^{\mu\nu}W^{\alpha\beta}, \\
\text{fourth} &= \bar{H}_{\mu\nu}{}^\lambda\bar{H}^{\mu\alpha\gamma}\bar{H}_\gamma{}^{\alpha\beta}\bar{H}_{\lambda\alpha\beta} + 2e^{-\varphi}[\bar{H}_{\mu\nu}{}^\gamma\bar{H}_{\alpha\beta\gamma}W^{\mu\nu}W^{\alpha\beta} + 2\bar{H}_\gamma{}^{\mu\nu}\bar{H}_{\beta\mu\nu}W_\alpha{}^\beta W^{\alpha\gamma}] + e^{-2\varphi}[4W_\mu{}^\beta W^{\mu\nu}W_\nu{}^\alpha W_{\beta\alpha} + (W^2)^2], \\
\text{fifth} &= \left(\partial_\mu\bar{\phi}\partial_\nu\bar{\phi} + \frac{1}{2}\partial_\mu\bar{\phi}\partial_\nu\varphi + \frac{1}{16}\partial_\mu\varphi\partial_\nu\varphi\right)(\bar{H}^{\mu\alpha\beta}\bar{H}^\nu{}_{\alpha\beta} + 2e^{-\varphi}W^{\mu\alpha}W^\nu{}_\alpha).
\end{aligned} \tag{15}$$

As expected, all terms on the right-hand side are invariant under $U(1)\times U(1)$ gauge transformations. Under parity $\bar{H}$ and W are odd and all other fields are even. All above terms are even under the parity because the original terms in (14) are even. Note that each V has a factor of $e^{\varphi/2}$ and each W has a factor of $e^{-\varphi/2}$.

The T-duality constraint is that the reduction of the effective action (14) must be invariant under T-duality transformation up to some boundary terms. If the T-duality transformations at order α' are only the Buscher rules (4), then one finds

$$\begin{aligned}
\delta S_1 \equiv S_1 - S_1{}' = -\frac{2}{\kappa^2}\int d^{D-1}x e^{-2\bar{\phi}}\Bigg(\Bigg[& b_1\partial_\mu\partial_\nu\varphi\partial^\mu\varphi\partial^\nu\varphi + b_8\partial_\mu\bar{\phi}\partial^\mu\bar{\phi}\partial_\nu\bar{\phi}\partial^\nu\varphi + \frac{1}{16}b_8\partial_\mu\varphi\partial^\mu\varphi\partial_\nu\varphi\partial^\nu\bar{\phi} \\
&+ \frac{1}{2}b_7\partial_\mu\bar{\phi}\partial^\mu\varphi\bar{H}^2 + \frac{1}{2}b_5\partial_\mu\bar{\phi}\partial_\nu\varphi\bar{H}^{\mu\alpha\beta}\bar{H}^\nu{}_{\alpha\beta} + b_1 e^{\varphi}\partial_\mu V_{\nu\alpha}\partial^\mu V^{\nu\alpha} - 2b_2\partial_\mu W_{\nu\alpha}\bar{H}^{\nu\alpha\beta}V^\mu{}_\beta \\
&- 6b_6 e^{\varphi}\bar{H}^2V^2 - 4b_4 e^{\varphi}\bar{H}_{\alpha\beta\gamma}\bar{H}_\nu{}^{\beta\gamma}V_\mu{}^\nu V^{\mu\alpha} + 3b_1 e^{\varphi}\partial_\mu V_{\nu\alpha}\partial^\mu\varphi V^{\nu\alpha} + e^{\varphi}\Big((-b_1 - 2b_2)\partial_\mu\partial_\nu\varphi - 2b_5\partial_\mu\bar{\phi}\partial_\nu\bar{\phi} + b_5\partial_\mu\bar{\phi}\partial_\nu\varphi \\
&+ \left(b_1 + b_2 - \frac{b_5}{8}\right)\partial_\mu\varphi\partial_\nu\varphi\Big)V^{\mu\alpha}V^\nu{}_\alpha + e^{\varphi}\left(-3b_7\partial_\mu\bar{\phi}\partial^\mu\bar{\phi} + \frac{3}{2}b_7\partial_\mu\bar{\phi}\partial^\mu\varphi + \frac{3}{16}(8b_1 - b_7)\partial_\mu\varphi\partial^\mu\varphi\right)V^2 \\
&+ 2b_2\partial_\mu\varphi\bar{H}_{\alpha\beta\nu}V^{\alpha\beta}W^{\mu\nu} + e^{2\varphi}\left(\left(\frac{5}{8}b_1 - 3b_3 - 4b_4\right)V_\mu{}^\beta V^{\mu\nu}V_\nu{}^\alpha V_{\beta\alpha} + \left(\frac{3}{8}b_1 - 9b_6 - b_4\right)(V^2)^2\right) \\
&- e^{\varphi}\left(\left(\frac{b_2}{2} + 6b_3\right)\bar{H}_{\mu\alpha}{}^\gamma\bar{H}_{\nu\beta\gamma} + \left(\frac{b_2}{2} + 2b_4\right)\bar{H}_{\mu\nu}{}^\gamma\bar{H}_{\alpha\beta\gamma}\right)V^{\mu\nu}V^{\alpha\beta}\Bigg] - [\varphi \to -\varphi, V \leftrightarrow W]\Bigg).
\end{aligned} \tag{16}$$

Note that δS_1 is odd under the Buscher rules. One observes that constraining the integrand to be a total derivative term would constrain all coefficients in (14) to be zero. To clarify this point, we note that the total derivative terms in $(D-1)$-spacetime must have the following structure:

$$J = \int d^{D-1}x \partial_\mu (e^{-2\bar{\phi}} J^\mu), \tag{17}$$

where the vector J^μ should be a parity invariance, it should be odd under the Buscher rules, and it should be invariant under the $U(1)\times U(1)$ gauge transformations. This vector, which is at three-derivative order, is made of $\partial\varphi, \partial\bar{\phi}, \bar{H}, e^{\varphi/2}V, e^{-\varphi/2}W$, and their derivatives. Hence, the four-derivative terms in (16), which contain only $\bar{H}, e^{\varphi/2}V, e^{-\varphi/2}W$, have no contribution from the total derivative terms. The coefficients of these terms must be zero. Moreover, since there is no term with the derivative of $\bar{H}$ in (16), the total derivative terms cannot produce the terms with $\bar{H}$; hence, the coefficients of the term in (16) which have $\bar{H}$ must be zero. These two constraints force the coefficients $b_1, \ldots, b_7$ to be zero. Removing these coefficients from (16), there remains two terms with coefficient b_8 which contains only first derivatives $\partial\varphi$ and $\partial\bar{\phi}$. They cannot be written as a total derivative term because total derivative terms must include at least one term with a second derivative. Hence, b_8 is also zero.

Therefore, to have nonzero effective action, one has to assume the T-duality transformations (4) to receive higher-derivative corrections [14–16]. At order α', the T-duality transformations should be

$$\begin{aligned}
\varphi' &= -\varphi + \alpha'\Delta\varphi, \qquad g'_\mu = b_\mu + \alpha' e^{\varphi/2}\Delta g_\mu,\\
b'_\mu &= g_\mu + \alpha' e^{-\varphi/2}\Delta b_\mu, \qquad \bar{g}'_{\alpha\beta} = \bar{g}_{\alpha\beta} + \alpha'\Delta\bar{g}_{\alpha\beta},\\
\bar{H}'_{\alpha\beta\gamma} &= \bar{H}_{\alpha\beta\gamma} + \alpha'\Delta\bar{H}_{\alpha\beta\gamma}, \qquad \bar{\phi}' = \bar{\phi} + \alpha'\Delta\bar{\phi},
\end{aligned} \tag{18}$$

where $\Delta\varphi, \ldots, \Delta\bar{\phi}$ contains some contractions of $\partial\varphi, \partial\bar{\phi}, e^{\varphi/2}V, e^{-\varphi/2}W, \bar{H}$, and their derivatives at order α'. We have multiplied the factors of $e^{\varphi/2}$ and $e^{-\varphi/2}$ to Δg_μ and Δb_μ, respectively. As we will see, this makes it explicit to have a factor of $e^{\varphi/2}$ in front of each V and a factor of $e^{-\varphi/2}$ in front of each W in the T-duality transformation of (10).

Since $\bar{H}$ is not the field strength of a two-form, it is convenient to consider the T-duality transformation of the three-form $\bar{H}$. The deformation $\Delta\bar{H}_{\mu\nu\alpha}$, however, is not independent of the deformations Δg_μ and Δb_μ [16]. The T-dual field $\bar{H}'$ must satisfy the same Bianchi identity as $\bar{H}$, i.e.,

$$\partial_{[\mu}\bar{H}'_{\nu\alpha\beta]} = -\frac{3}{2}V'_{[\mu\nu}W'_{\alpha\beta]}. \tag{19}$$

Using the T-duality transformations (18), one finds at order α' the corrections $\Delta\bar{H}, \Delta g, \Delta b$ satisfy the following differential equation:

$$\partial_{[\mu}\Delta\bar{H}_{\nu\alpha\beta]} = -3\partial_{[\mu}(V_{\nu\alpha}e^{\varphi/2}\Delta g_{\beta]}) - 3\partial_{[\mu}(W_{\nu\alpha}e^{-\varphi/2}\Delta b_{\beta]}), \tag{20}$$

where we have used the fact that the exterior derivative of V and W is zero. This leads to the following relation between $\Delta\bar{H}$ and $\Delta g, \Delta b$:

$$\begin{aligned}
\Delta\bar{H}_{\mu\nu\alpha} &= \alpha_{19}\partial_{[\mu}(W_\nu{}^\beta V_{\alpha\beta]}) - 3e^{\varphi/2}V_{[\mu\nu}\Delta g_{\alpha]}\\
&\quad - 3e^{-\varphi/2}W_{[\mu\nu}\Delta b_{\alpha]},
\end{aligned} \tag{21}$$

where α_{19} is an arbitrary parameter.

The T-duality transformations (18) should form a Z_2-group [17]. This indicates that the corrections $\Delta\varphi, \Delta\bar{\phi}, \Delta\bar{g}, \Delta g, \Delta b, \Delta\bar{H}$ must satisfy the following constraints:

$$\begin{aligned}
\Delta\varphi - \Delta\varphi|_{\varphi\to-\varphi,V\to W,W\to V} &= 0,\\
\Delta\bar{\phi} + \Delta\bar{\phi}|_{\varphi\to-\varphi,V\to W,W\to V} &= 0,\\
\Delta\bar{g} + \Delta\bar{g}|_{\varphi\to-\varphi,V\to W,W\to V} &= 0,\\
\Delta b + \Delta g|_{\varphi\to-\varphi,V\to W,W\to V} &= 0,\\
\Delta g + \Delta b|_{\varphi\to-\varphi,V\to W,W\to V} &= 0,\\
\Delta\bar{H} + \Delta\bar{H}|_{\varphi\to-\varphi,V\to W,W\to V} &= 0.
\end{aligned} \tag{22}$$

Now, we consider the T-duality constraint on the effective actions (1) and (14) using the T-duality transformations (18). The T-duality transformation of action (1) is now

$$\begin{aligned}
\delta S_0 \equiv S_0 - S_0{}' = -\frac{2\alpha'}{\kappa^2}\int d^{D-1}x e^{-2\bar{\phi}}\Big[&-\Big(-2\partial^\mu\partial^\nu\bar{\phi} + \frac{1}{4}\partial^\mu\varphi\partial^\nu\varphi + \frac{1}{4}\bar{H}^{\mu\alpha\beta}\bar{H}^\nu{}_{\alpha\beta} + \frac{1}{2}e^\varphi V^{\mu\alpha}V^\nu{}_\alpha + \frac{1}{2}e^{-\varphi}W^{\mu\alpha}W^\nu{}_\alpha\Big)\Delta\bar{g}_{\mu\nu}\\
&-\Big(2\partial_\alpha\partial^\alpha\bar{\phi} - 2\partial_\alpha\bar{\phi}\partial^\alpha\bar{\phi} - \frac{1}{8}\partial_\alpha\varphi\partial^\alpha\varphi - \frac{1}{24}\bar{H}^2 - \frac{1}{8}e^\varphi V^2 - \frac{1}{8}e^{-\varphi}W^2\Big)(\eta^{\mu\nu}\Delta\bar{g}_{\mu\nu} - 4\Delta\bar{\phi})\\
&+\Big(\frac{1}{2}\partial_\mu\partial^\mu\varphi - \partial_\mu\bar{\phi}\partial^\mu\varphi - \frac{1}{4}e^\varphi V^2 + \frac{1}{4}e^{-\varphi}W^2\Big)\Delta\varphi - e^{-\varphi/2}(-\partial_\nu W^{\mu\nu} + 2\partial_\nu\bar{\phi}W^{\mu\nu} + \partial_\nu\varphi W^{\mu\nu})\Delta g_\mu\\
&- e^{\varphi/2}(-\partial_\nu V^{\mu\nu} + 2\partial_\nu\bar{\phi}V^{\mu\nu} - \partial_\nu\varphi V^{\mu\nu})\Delta b_\mu + \frac{1}{6}\bar{H}^{\mu\nu\alpha}\Delta\bar{H}_{\mu\nu\alpha}\Big],
\end{aligned} \tag{23}$$

where we have used the relations (22), removed some total derivative terms, and used the leading order T-duality constraint (12). We have also absorbed the overall coefficient c_1 in the arbitrary parameters in $\Delta\varphi, \Delta\bar{\phi}, \Delta\bar{g}, \Delta g, \Delta b, \Delta\bar{H}$. In finding the above result for $\Delta\bar{g}_{\mu\nu}$ we assumed the metric of the base space is $\bar{g}_{\mu\nu}$ and then use the T-duality transformations (18). At the end we set $\bar{g}_{\mu\nu}=\eta_{\mu\nu}$. Note that the extra factors of $e^{\varphi/2}$ and $e^{-\varphi/2}$ in (18) make it possible to have a factor of $e^{\varphi/2}$ in front of each V and a factor of $e^{-\varphi/2}$ in front of each W. Note that δS_0 is odd under the Buscher rules.

Since the terms in (16) are all invariant under the parity, the most general forms of the corrections $\Delta\varphi, \Delta\bar{\phi}, \Delta\bar{g}, \Delta g, \Delta b$ satisfying the constraints (22) and making the terms in (23) to be even under the parity are

$$\begin{aligned}
\Delta\varphi &= \alpha_1\partial_\mu\partial^\mu\bar{\phi}+\alpha_2\partial_\mu\bar{\phi}\partial^\mu\bar{\phi}+\alpha_3\partial_\mu\varphi\partial^\mu\varphi+\alpha_4\bar{H}^2+\alpha_5(e^\varphi V^2+e^{-\varphi}W^2),\\
\Delta\bar{\phi} &= \alpha_6\partial_\mu\partial^\mu\varphi+\alpha_7\partial_\mu\varphi\partial^\mu\bar{\phi}+\alpha_8(e^\varphi V^2+e^{-\varphi}W^2),\\
\Delta\bar{g}_{\mu\nu} &= \alpha_9\partial_\mu\partial_\nu\varphi+\alpha_{10}(\partial_\mu\varphi\partial_\nu\bar{\phi}+\partial_\nu\varphi\partial_\mu\bar{\phi})+\alpha_{11}(e^\varphi V_\mu{}^\alpha V_{\nu\alpha}-e^{-\varphi}W_\mu{}^\alpha W_{\nu\alpha})+\eta_{\mu\nu}[\alpha_{12}\partial_\alpha\partial^\alpha\varphi+\alpha_{13}\partial_\alpha\varphi\partial^\alpha\bar{\phi}\\
&\quad+\alpha_{14}(e^\varphi V^2-e^{-\varphi}W^2)],\\
\Delta g_\mu &= \alpha_{15}e^{-\varphi/2}\partial^\nu W_{\mu\nu}+\alpha_{16}e^{\varphi/2}\bar{H}_{\mu\nu\alpha}V^{\nu\alpha}+\alpha_{17}e^{-\varphi/2}\partial^\nu\bar{\phi}W_{\mu\nu}+\alpha_{18}e^{-\varphi/2}\partial^\nu\varphi W_{\mu\nu},\\
\Delta b_\mu &= -\alpha_{15}e^{\varphi/2}\partial^\nu V_{\mu\nu}-\alpha_{16}e^{-\varphi/2}\bar{H}_{\mu\nu\alpha}W^{\nu\alpha}-\alpha_{17}e^{\varphi/2}\partial^\nu\bar{\phi}V_{\mu\nu}+\alpha_{18}e^{\varphi/2}\partial^\nu\varphi V_{\mu\nu},
\end{aligned}\tag{24}$$

where $\alpha_1,\ldots,\alpha_{18}$ are arbitrary parameters. If δS_1 were odd under the parity, then the corrections $\Delta\varphi, \Delta\bar{\phi}, \Delta\bar{g}, \Delta g, \Delta b$ would contain terms that have opposite parity.

When one replaces (24) into (23), one would find that for some specific relations between the parameters, δS_0 becomes zero. That indicates that not all the parameters in (24) produce nonzero δS_0. We are not going to write (24) in terms of the parameters that produce nonzero δS_0 and then impose the T-duality constraint. Instead, we first impose the T-duality constraint on all parameters and then remove the terms that produce zero δS_0.

The T-duality transformation of action (14) under (18) produce the same terms as in (16) plus some terms at a higher order of α' in which we are not interested. Hence, the T-duality constraint at order α' requires $\delta S_0+\delta S_1$, where δS_0 is given in (23) and δS_1 is given in (16), to be a boundary term. This constraint produces some algebraic equations that their solution fixes the coefficients of both the effective action (14) and the corrections to the Buscher rules. We have found that for the following parameters:

$$\begin{aligned}
&b_2=-\frac{b_1}{2},\qquad b_3=\frac{b_1}{24},\qquad b_4=-\frac{b_1}{8},\qquad b_5=b_6=b_7=b_8=0,\\
&\alpha_2=8\alpha_{14}-\alpha_1,\qquad \alpha_3=2b_1+\frac{-\alpha_{14}}{2}-\frac{\alpha_1}{16},\qquad \alpha_4=-\frac{5\alpha_{14}}{6}-\frac{\alpha_1}{48},\qquad \alpha_5=2b_1-\frac{3\alpha_{14}}{2}-\frac{\alpha_1}{16},\\
&\alpha_6=-12\alpha_{14}-\frac{\alpha_1}{16},\qquad \alpha_7=24\alpha_{14}+\frac{\alpha_1}{8},\qquad \alpha_8=\frac{b_1}{2}+6\alpha_{14}+\frac{\alpha_1}{32},\\
&\alpha_9=0,\qquad \alpha_{10}=0,\qquad \alpha_{11}=2b_1,\qquad \alpha_{12}=-2\alpha_{14},\qquad \alpha_{13}=4\alpha_{14},\\
&\alpha_{15}=2b_1,\qquad \alpha_{16}=b_1,\qquad \alpha_{17}=-4b_1,\qquad \alpha_{18}=0,\qquad \alpha_{19}=-12b_1,
\end{aligned}\tag{25}$$

the T-duality transformation $\delta S_0+\delta S_1$ is a total derivative (17) with the following vector:

$$\begin{aligned}
J^\mu &= -b_1\partial^\mu\varphi\partial_\nu\varphi\partial^\nu\varphi+b_1e^\varphi(2\partial^\alpha V_{\alpha\beta}V^{\mu\beta}-2\partial^\alpha V^{\mu\beta}V_{\alpha\beta}-4\partial^\alpha\bar{\phi}V^{\mu\beta}V_{\alpha\beta}-\partial^\mu\varphi V^2)\\
&\quad+b_1e^{-\varphi}(-2\partial^\alpha W_{\alpha\beta}W^{\mu\beta}+2\partial^\alpha W^{\mu\beta}W_{\alpha\beta}+4\partial^\alpha\bar{\phi}W^{\mu\beta}W_{\alpha\beta}-\partial^\mu\varphi W^2),
\end{aligned}\tag{26}$$

which is odd under the Buscher rules and is even under the parity, as expected because $\delta S_0+\delta S_1$ is also odd under the Buscher rules and is even under the parity.

The most important part of the results (25) is that they fix uniquely all eight parameters in the D-dimensional action (14) in terms of b_1, i.e.,

$$\mathbf{S}_1=\frac{-2b_1}{\kappa^2}\alpha'\int d^Dxe^{-2\Phi}\sqrt{-G}\left(R_{abcd}R^{abcd}-\frac{1}{2}R_{abcd}H^{abe}H^{cd}{}_e+\frac{1}{24}H_{fhg}H^f{}_a{}^bH^h{}_b{}^cH^g{}_c{}^a-\frac{1}{8}H_f{}^{ab}H_{hab}H^{fcg}H^h{}_{cg}\right).\tag{27}$$

Up to the overall factor b_1, the above couplings are the standard effective action of the bosonic string theory which has been found in [20] by the S-matrix calculations. This action now is invariant under T-duality.

When replacing the relations (25) into (18), one finds the following corrections to the Buscher rules:

$$\begin{aligned}\Delta\bar{g}_{\mu\nu} &= 2b_1(e^{\varphi}V_\mu{}^\alpha V_{\nu\alpha} - e^{-\varphi}W_\mu{}^\alpha W_{\nu\alpha}),\\ \Delta\bar{\phi} &= \frac{b_1}{2}(e^{\varphi}V^2 - e^{-\varphi}W^2),\\ \Delta\varphi &= 2b_1(\partial_\mu\varphi\partial^\mu\varphi + e^{\varphi}V^2 + e^{-\varphi}W^2),\\ \Delta g_\mu &= b_1(2e^{-\varphi/2}\partial^\nu W_{\mu\nu} + e^{\varphi/2}\bar{H}_{\mu\nu\alpha}V^{\nu\alpha} - 4e^{-\varphi/2}\partial^\nu\bar{\phi}W_{\mu\nu}),\\ \Delta b_\mu &= -b_1(2e^{\varphi/2}\partial^\nu V_{\mu\nu} + e^{-\varphi/2}\bar{H}_{\mu\nu\alpha}W^{\nu\alpha} - 4e^{\varphi/2}\partial^\nu\bar{\phi}V_{\mu\nu}),\\ \Delta\bar{H}_{\mu\nu\alpha} &= 12b_1\partial_{[\mu}(W_\nu{}^\beta V_{\alpha\beta]})\\ &\quad - 3e^{\varphi/2}V_{[\mu\nu}\Delta g_{\alpha]} - 3e^{-\varphi/2}W_{[\mu\nu}\Delta b_{\alpha]}.\end{aligned} \tag{28}$$

The corrections $\Delta\bar{g}_{\mu\nu}, \Delta\bar{\phi}, \Delta\varphi$ have also some terms that depend on α_1, α_{14}. They are

$$\begin{aligned}\tilde{\Delta}\bar{g}_{\mu\nu} &= \alpha_{14}(-2\partial_\alpha\partial^\alpha\varphi + 4\partial_\alpha\bar{\phi}\partial^\alpha\varphi + e^{\varphi}V^2 - e^{-\varphi}W^2)\eta_{\mu\nu},\\ \tilde{\Delta}\bar{\phi} &= \left(6\alpha_{14} + \frac{1}{32}\alpha_1\right)(-2\partial_\mu\partial^\mu\varphi + 4\partial_\mu\bar{\phi}\partial^\mu\varphi\\ &\quad + e^{\varphi}V^2 - e^{-\varphi}W^2),\\ \tilde{\Delta}\varphi &= \alpha_{14}\left(8\partial_\mu\bar{\phi}\partial^\mu\bar{\phi} - \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{5}{6}\bar{H}^2\right.\\ &\quad \left. - \frac{3}{2}e^{\varphi}V^2 - \frac{3}{2}e^{-\varphi}W^2\right) + \alpha_1\left(\partial_\mu\partial^\mu\bar{\phi} - \partial_\mu\bar{\phi}\partial^\mu\bar{\phi}\right.\\ &\quad \left. - \frac{1}{16}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{48}\bar{H}^2 - \frac{1}{16}e^{\varphi}V^2 - \frac{1}{16}e^{-\varphi}W^2\right).\end{aligned} \tag{29}$$

However, replacing $\tilde{\Delta}\bar{g}_{\mu\nu}, \tilde{\Delta}\bar{\phi}, \tilde{\Delta}\varphi$ into (23), one would find δS_0 becomes zero. That is the reflection of the fact that the parameters $\alpha_1, \ldots, \alpha_{18}$ in (23) do not all produce nonzero δS_0. To consider the parameters that produce nonzero δS_0, one has to set these two parameters to zero. Hence,

$$\tilde{\Delta}\bar{g}_{\mu\nu} = \tilde{\Delta}\bar{\phi} = \tilde{\Delta}\varphi = 0. \tag{30}$$

This ends our illustration that the T-duality constraint on the effective action (14) can fix both the effective action and the corresponding corrections to the Buscher rules up to the overall factor of b_1.

A similar T-duality constraint has been used in [21] by reducing the effective action (14) to one dimension. In that approach, however, not all parameters in (14) are fixed up to an overall factor because some of the terms in (14) become zero when reducing them to one dimension [21].

B. Effective action in arbitrary scheme

The corrections to the Buscher rules depend on the scheme that one uses for the effective action. The corrections (28) correspond to the effective action (27). If we had started with the effective action (14) in a different scheme, then the T-duality constraint would fix the eight arbitrary parameters in the action and the corresponding corrections to the Buscher rules up to an overall factor.

The field redefinitions have been used to write the effective action (14) in terms of only eight parameters. If one does not use the field redefinition to reduce the independent couplings, then the effective action would have the following 20 terms [20]:

$$\begin{aligned}\mathbf{S}_1 = \frac{-2\alpha'}{\kappa^2}\int d^Dxe^{-2\Phi}\sqrt{-G}[&a_1R_{abcd}R^{abcd} + a_2(H^2)^2 + a_3H_{fgh}H^f{}_a{}^bH^g{}_b{}^cH^h{}_c{}^a + a_4R_{ab}H^{acd}H^b{}_{cd}\\ &+ a_5R_{ab}R^{ab} + a_6RH^2 + a_7R^2 + a_8R_{abcd}H^{abe}H^{cd}{}_e + a_9H_{acd}H_b{}^{cd}\partial^a\Phi\partial^b\Phi + a_{10}R_{ab}\partial^a\Phi\partial^b\Phi\\ &+ a_{11}R\partial_a\Phi\partial^a\Phi + a_{12}H^2\partial_a\Phi\partial^a\Phi + a_{13}\nabla_a\nabla^a\Phi\partial_b\Phi\partial^b\Phi + a_{14}(\partial_a\Phi\partial^a\Phi)^2 + a_{15}H^2\nabla_a\nabla^a\Phi\\ &+ a_{16}H^{abc}\nabla^dH_{dab}\partial_c\Phi + a_{17}\nabla^aH_{abc}\nabla_dH^{bcd} + a_{18}R\nabla_a\nabla^a\Phi + a_{19}H_{acd}H_b{}^{cd}\nabla^a\nabla^b\Phi\\ &+ a_{20}H_f{}^{ab}H_{gab}H^{fch}H^g{}_{ch}].\end{aligned} \tag{31}$$

Apart from the unambiguous coefficients a_1, a_3, a_8 which are not changed under field redefinitions, all other coefficients are ambiguous because they are changed under field redefinitions. There are five parameters in the ambiguous parameters which are essential and all others are arbitrary parameters. If one does not use the field redefinitions, one would not be able to distinguish between the essential and the arbitrary parameters. This distinction, however, can be found by imposing the T-duality constraint on (31). We find that the T-duality constraint can fix the three unambiguous parameters in terms of one of them, and the 17 ambiguous parameters in terms of 11 arbitrary parameters.

Using the same steps as in the previous subsection, one finds that the T-duality constraint on the above action produces the following relations:

$$
\begin{aligned}
&a_{17}=-\frac{a_{16}}{4},\qquad a_3=\frac{a_1}{24},\qquad a_4=-\frac{a_{10}}{16}+\frac{3a_{12}}{4}+\frac{3a_{13}}{32}+\frac{5a_{14}}{64}+\frac{9a_{15}}{10}+\frac{36a_2}{5}-\frac{24a_{20}}{5}-\frac{3a_1}{5}-\frac{a_{19}}{10},\\
&a_5=\frac{a_{10}}{4}-3a_{12}+\frac{2a_{19}}{5}+\frac{16a_{20}}{5}-\frac{18a_{15}}{5}-\frac{144a_2}{5}+\frac{2a_1}{5}-\frac{3a_{13}}{8}-\frac{5a_{14}}{16},\\
&a_6=-\frac{a_{10}}{48}+\frac{a_{12}}{4}+\frac{a_{13}}{12}+\frac{5a_{14}}{96}+\frac{a_{15}}{2}-\frac{a_{18}}{8}-\frac{5a_{11}}{48},\qquad a_7=\frac{a_{11}}{4}+\frac{a_{18}}{2}-\frac{a_{13}}{8}-\frac{a_{14}}{16},\\
&a_8=-\frac{a_1}{2},\qquad a_9=-3a_{12}-a_{16}+\frac{2a_{19}}{5}+\frac{16a_{20}}{5}-\frac{18a_{15}}{5}-\frac{144a_2}{5}+\frac{2a_1}{5}-\frac{3a_{13}}{8}-\frac{5a_{14}}{16},\\
&\alpha_{15}=\frac{a_{10}}{8}+\frac{a_{16}}{2}+\frac{a_{19}}{5}+\frac{8a_{20}}{5}-\frac{3a_{12}}{2}-\frac{9a_{15}}{5}-\frac{72a_2}{5}+\frac{11a_1}{5}-\frac{3a_{13}}{16}-\frac{5a_{14}}{32},\\
&\alpha_{16}=-\frac{a_{16}}{4}-4a_{20}+\frac{a_1}{2},\qquad \alpha_{17}=-a_{16}+2a_{19}+16a_{20}-2a_1,\\
&\alpha_{18}=-\frac{3a_{10}}{16}+\frac{9a_{12}}{4}+\frac{9a_{13}}{32}+\frac{15a_{14}}{64}+\frac{27a_{15}}{10}+\frac{108a_2}{5}-\frac{a_{16}}{2}-\frac{4a_{19}}{5}-\frac{32a_{20}}{5}-\frac{4a_1}{5},\\
&\alpha_{11}=\frac{a_{10}}{8}+\frac{a_{19}}{5}-\frac{3a_{12}}{2}-\frac{9a_{15}}{5}-\frac{72a_2}{5}-\frac{32a_{20}}{5}+\frac{6a_1}{5}-\frac{3a_{13}}{16}-\frac{5a_{14}}{32},\\
&\alpha_9=\frac{a_{10}}{4}-3a_{12}+\frac{12a_{19}}{5}+\frac{96a_{20}}{5}-\frac{18a_{15}}{5}-\frac{144a_2}{5}+\frac{12a_1}{5}-\frac{3a_{13}}{8}-\frac{5a_{14}}{16},\\
&\alpha_{12}=\frac{5a_{10}}{16}+\frac{5a_{11}}{4}+\frac{5a_{18}}{4}+\frac{2a_{19}}{5}+\frac{16a_{20}}{5}-\frac{15a_{12}}{4}-\frac{324a_2}{5}+\frac{2a_1}{5}-\frac{51a_{15}}{10}-\frac{35a_{13}}{32}-\frac{45a_{14}}{64},\\
&\alpha_{10}=-6a_{12}+\frac{4a_{19}}{5}+\frac{32a_{20}}{5}-\frac{3a_{13}}{4}-\frac{36a_{15}}{5}-\frac{288a_2}{5}+\frac{4a_1}{5}-\frac{5a_{14}}{8},\\
&\alpha_{13}=-\frac{a_{10}}{8}+\frac{15a_{12}}{2}+\frac{11a_{13}}{16}+\frac{13a_{14}}{32}+\frac{51a_{15}}{5}+\frac{648a_2}{5}-\frac{a_{11}}{2}-\frac{a_{18}}{2}-\frac{4a_{19}}{5}-\frac{32a_{20}}{5}-\frac{4a_1}{5},\\
&\alpha_6=\frac{15a_{10}}{8}+\frac{15a_{11}}{2}+\frac{31a_{18}}{4}+\frac{12a_{19}}{5}+\frac{96a_{20}}{5}-\frac{45a_{12}}{2}-\frac{153a_{15}}{5}-\frac{1944a_2}{5}\\
&\quad+\frac{12a_1}{5}-\frac{105a_{13}}{16}-\frac{135a_{14}}{32},\qquad \alpha_7=-\frac{3a_{10}}{4}-3a_{11}+\frac{81a_{12}}{2}+\frac{55a_{13}}{16}+\frac{63a_{14}}{32}+\frac{279a_{15}}{5}\\
&\quad-3a_{18}+\frac{3672a_2}{5}-\frac{16a_{19}}{5}-\frac{128a_{20}}{5}-\frac{16a_1}{5},\\
&\alpha_2=-\frac{a_{10}}{4}-a_{11}-9a_{12}+\frac{3a_{13}}{8}+\frac{5a_{14}}{16}-a_{18}+\frac{8a_{19}}{5}+\frac{144a_2}{5}+\frac{64a_{20}}{5}-\frac{42a_{15}}{5}+\frac{8a_1}{5},\\
&\alpha_3=-\frac{15a_{10}}{64}+\frac{33a_{12}}{16}+\frac{69a_{13}}{128}+\frac{99a_{14}}{256}+\frac{93a_{15}}{40}+\frac{63a_2}{5}-\frac{52a_{20}}{5}-\frac{13a_{19}}{10}+\frac{7a_1}{10}-\frac{7a_{11}}{16}-\frac{7a_{18}}{16},\\
&\alpha_4=\frac{5a_{10}}{192}+\frac{5a_{11}}{48}+\frac{5a_{18}}{48}-\frac{123a_2}{5}-\frac{4a_{20}}{15}-\frac{5a_{12}}{16}-\frac{a_{19}}{30}-\frac{a_1}{30}-\frac{33a_{15}}{40}-\frac{15a_{14}}{256}-\frac{35a_{13}}{384},\\
&\alpha_5=\frac{5a_{10}}{64}+\frac{a_{11}}{16}+\frac{a_{18}}{16}-45a_2-4a_{20}+\frac{3a_1}{2}-\frac{15a_{15}}{8}-\frac{15a_{12}}{16}-\frac{19a_{13}}{128}-\frac{29a_{14}}{256},\\
&\alpha_8=\frac{3a_{10}}{128}+\frac{3a_{15}}{80}+\frac{a_{18}}{32}+\frac{a_{19}}{10}-\frac{6a_{20}}{5}-\frac{27a_2}{10}+\frac{7a_1}{20}-\frac{9a_{12}}{32}-\frac{9a_{13}}{256}-\frac{15a_{14}}{512},\qquad \alpha_{19}=-12a_1,
\end{aligned}
\tag{32}
$$

where the unambiguous parameters a_1 are not fixed, and the 11 ambiguous parameters a_2, a_{10}, a_{11}, a_{12}, a_{13}, a_{14}, a_{15}, a_{16}, a_{18}, a_{19}, a_{20} remain arbitrary. Any specific value for these parameters, gives the effective action in one particular scheme. For the special case that the parameters are

$$a_2 = \frac{a_1}{144}, \quad a_{10} = -16a_1, \quad a_{11} = 8a_1, \quad a_{12} = \frac{2a_1}{3}, \quad a_{13} = 16a_1, \quad a_{14} = -16a_1,$$
$$a_{15} = -\frac{2a_1}{3}, \quad a_{16} = 0, \quad a_{18} = 0, \quad a_{19} = 2a_1, \quad a_{20} = -\frac{a_1}{8}, \tag{33}$$

one finds the effective action (31) becomes

$$\mathbf{S}_1 = \frac{2\alpha' a_1}{\kappa^2}\int d^D x e^{-2\phi}\sqrt{-G}\Big[-R^2_{GB} + 16\Big(R^{ab} - \frac{1}{2}g^{ab}R\Big)\partial_a\phi\partial_b\phi - 16\nabla^2\phi(\partial\phi)^2 + 16(\partial\phi)^4$$
$$+\frac{1}{2}\Big(R_{abcd}H^{abe}H^{cd}{}_e - 2R^{ab}H^2_{ab} + \frac{1}{3}RH^2\Big) - 2\Big(\nabla^a\partial^b\phi H^2_{ab} - \frac{1}{3}\nabla^2\phi H^2\Big) - \frac{2}{3}(\partial\phi)^2H^2$$
$$-\frac{1}{24}H_{fgh}H^f{}_a{}^bH^g{}_b{}^cH^h{}_c{}^a + \frac{1}{8}H^2_{ab}H^{2ab} - \frac{1}{144}(H^2)^2\Big], \tag{34}$$

where $R^2_{GB} = R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2$ and $H^2_{ab} = H_a{}^{cd}H_{bcd}$. This action has been found in [19]. Replacing (33) into (32), one finds the corresponding T-duality transformations to be

$$\begin{aligned}
\Delta\bar{g}_{\mu\nu} &= 0,\\
\Delta\bar{\phi} &= 0,\\
\Delta\varphi &= a_1(\partial_\mu\varphi\partial^\mu\varphi + e^{\varphi}V^2 + e^{-\varphi}W^2),\\
\Delta g_\mu &= a_1(2e^{-\varphi/2}\partial^\nu\varphi W_{\mu\nu} + e^{\varphi/2}\bar{H}_{\mu\nu\alpha}V^{\nu\alpha}),\\
\Delta b_\mu &= a_1(2e^{\varphi/2}\partial^\nu\varphi V_{\mu\nu} - e^{-\varphi/2}\bar{H}_{\mu\nu\alpha}W^{\nu\alpha}),\\
\Delta\bar{H}_{\mu\nu\alpha} &= 12a_1\partial_{[\mu}(W_\nu{}^\beta V_{\alpha\beta]}) - 3e^{\varphi/2}V_{[\mu\nu}\Delta g_{\alpha]}\\
&\quad - 3e^{-\varphi/2}W_{[\mu\nu}\Delta b_{\alpha]}.
\end{aligned} \tag{35}$$

These transformations are exactly those that have been found in [16].

We have seen that the T-duality constraint cannot fix the overall factor b_1 in (27) or a_1 in (34). This is as expected because the bosonic, the heterotic, and the superstring theories all have the same T-duality but they have different overall factors. In fact, $a_1 = b_1 = -1/16$ for bosonic theory, $a_1 = b_1 = -1/32$ for heterotic theory, and $a_1 = b_1 = 0$ for superstring theory. If the T-duality constraint could fix the overall factor, then the effective action that the T-duality constraint generated would not be the correct effective action of all bosonic, heterotic, and superstring theories at order α'.

The heterotic theory has another term at the four-derivative level, i.e.,

$$\mathbf{S}_1 \supset -\frac{2c_1\alpha'}{\kappa^2}\int d^{10}xe^{-2\Phi}\sqrt{-G}\Big(-\frac{1}{6}H_{abc}\Omega^{abc}\Big). \tag{36}$$

This term results from the Green-Schwarz anomaly cancellation mechanism [22] which requires the nonstandard gauge transformation of the B-field, i.e.,

$$B_{ab} \to B_{ab} + \partial_{[a}\lambda_{b]} + \alpha'\partial_{[a}\Lambda_i{}^j\omega_{b]j}{}^i, \tag{37}$$

where $\Lambda_i{}^j$ is the matrix of the Lorentz transformations and $\omega_{bi}{}^j$ is the spin connection. Under this transformation the three-form $H_{abc} + \alpha'\Omega_{abc}$ is invariant, i.e., $H_{abc} + \alpha'\Omega_{abc} \to H_{abc} + \alpha'\Omega_{abc}$. The Chern-Simons three-form Ω is

$$\begin{aligned}
\Omega_{abc} &= \omega_{[ai}{}^j\partial_b\omega_{c]j}{}^i + \frac{2}{3}\omega_{[ai}{}^j\omega_{bj}{}^k\omega_{c]k}{}^i;\\
\omega_{ai}{}^j &= \partial_a e_b{}^j e^b{}_i - \Gamma_{ab}{}^c e_c{}^j e^b{}_i,
\end{aligned} \tag{38}$$

where $e_a{}^ie_b{}^j\eta_{ij} = G_{ab}$. We have imposed the T-duality constraint on this action and found that it is invariant under the T-duality transformation (18) provided that $\Delta\bar{g}_{\mu\nu} = \Delta\bar{\varphi} = 0$ and

$$\begin{aligned}
\Delta\varphi &= \frac{c_1}{6}V_{\mu\nu}W^{\mu\nu},\\
\Delta g_\mu &= -\frac{c_1}{12}\Big(e^{\varphi/2}\partial^\nu\varphi V_{\mu\nu} - \frac{1}{2}e^{-\varphi/2}\bar{H}_{\mu\nu\alpha}W^{\nu\alpha}\\
&\quad + e^{\varphi/2}\bar{\omega}_{\mu\nu\alpha}V^{\nu\alpha}\Big),\\
\Delta b_\mu &= -\frac{c_1}{12}\Big(e^{-\varphi/2}\partial^\nu\varphi W_{\mu\nu} + \frac{1}{2}e^{\varphi/2}\bar{H}_{\mu\nu\alpha}V^{\nu\alpha}\\
&\quad - e^{-\varphi/2}\bar{\omega}_{\mu\nu\alpha}W^{\nu\alpha}\Big),\\
\Delta\bar{H}_{\mu\nu\alpha} &= -3e^{\varphi/2}V_{[\mu\nu}\Delta g_{\alpha]} - 3e^{-\varphi/2}W_{[\mu\nu}\Delta b_{\alpha]},
\end{aligned} \tag{39}$$

where $\bar{\omega}_{\mu\nu\alpha}$ is a nine-dimensional spin connection. In finding the above result, we did not assume that the base space is flat. As expected, the parity of the above terms are different from the corresponding terms in (28) because the parity of their corresponding actions is different.

We have shown in this paper that the T-duality constraint when the B-field is not zero can be used to find both the effective action and its corresponding T-duality transformations at order α'. It would be interesting to extend these calculations to the orders α'^2, α'^3 as their effective actions are not known in the literature. When B-field is zero, it has been shown in [17,18] that the T-duality constraint reproduces the known couplings in the literature.

ACKNOWLEDGMENTS

This work is supported by Ferdowsi University of Mashhad under Grant No. 1/49736(1398/02/17).

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