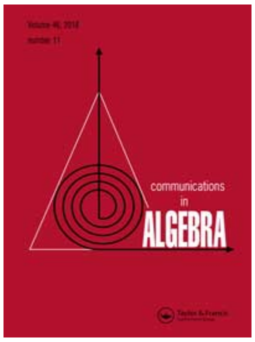

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c-capability of Lie algebras with the derived subalgebra of dimension two

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ABSTRACT
The current article is devoted to classify the c-capability of finite dimensional nilpotent Lie algebras with the derived subalgebra of dimension two.

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1. Introduction and motivation

From Ref. [8], a group G is called capable if there exists some group E such that $G \cong E/Z(E)$, where $Z(E)$ denotes the center of E. Several works has been done on the notion of capability. For example in Ref. [1, Corollary 4.16], it is well-known that the only capable extra-special p-groups are p-groups of order p^3 and exponent p, for odd p. Moreover, in the case $G' = Z(G)$ and $Z(G)$ is elementary abelian p-group of rank 2, we know from Ref. [9] that $p^5 \leq |G| \leq p^7$, for a capable p-group G. Since Lie algebras and groups have similar structures, many authors tried to define and prove similar concepts between them. But in this way not everything are the same and there are differences between groups and Lie algebras so that most of time the proofs are different. Similar to groups, a Lie algebra L is called capable provided that $L \cong H/Z(H)$ for a Lie algebra H. The epicenter of groups was defined in [1] while the analogs concept for Lie algebras can be found in Ref. [17]. The epicenter of Lie algebra L, $Z^*(L)$ is a useful instrument for detecting the capability. In fact, it is known L is capable if and only if $Z^*(L) = 0$.
Our approach is concerning the concept of the exterior center $Z^\wedge(L)$, the set of all elements l of L for which $l \wedge l' = 0_{L \wedge L}$ for all $l' \in L$, where $L \wedge L$ denotes the exterior square of L (see [6]). From Ref. [16], we know $Z^\wedge(L) = Z^*(L)$ for any finite dimensional Lie algebra L and also the structure of all capable nilpotent Lie algebras are given when their derived subalgebras are of dimension at most one. These results developed the results of Ref. [1, Corollary 4.16] for groups to the class of Lie algebras.
Capable special p-groups of rank 2 and exponent p were classified by Heineken in [9]. Similarly from Ref. [12], a Lie algebra H is called generalized Heisenberg of rank n if $H^2 = Z(H)$ and $\dim H^2 = n$, in which H^2 denotes the derived subalgebra of H. If $n = 1$, then H is called a

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Heisenberg Lie algebra. Recently in Ref. [12], we determine the structure of all capable nilpotent Lie algebras with the derived subalgebra of dimension 2 over an arbitrary filed. It helps to develop the result of Heineken [9] which is for groups, to the area of Lie algebras.

Similar to the case of groups, the c-capability of a Lie algebra can be defined as follows [19]. A Lie algebra L is c-capable if there exists some Lie algebra H such that $L \cong H/Z_c(H)$, where $Z_c(H)$ is the cth term of the upper central series of H defined inductively as $Z_1(H) = Z(H)$ and $Z_{c+1}(H)$ is the pre-image of $Z(H/Z_c(H))$ in H. Evidently, L is 1-capable if and only if it is an inner derivation Lie algebra, and L is c-capable ($c \geq 2$) if and only if it is an inner derivation Lie algebra of a $(c-1)$-capable Lie algebra. If L is c-capable, then $L \cong H/Z_c(H) \cong H/Z_{c-1}(H)/Z(H/Z_{c-1}(H))$ so it is $(c-1)$-capable, as well.

The last two authors in Ref. [14] proved that for extra-special p-groups, the notions "capable" and "2-capable" are equivalent. The same motivation allows us to ask about the structure of c-capable generalized Heisenberg Lie algebras of rank at most 2, while the c-capability of Heisenberg Lie algebras is determined in [15, 16, 18].

The current article is devoted to classify all c-capable finite dimensional nilpotent Lie algebras with the derived subalgebra of dimension two. In the same scene of research, we decide to show that the notions of "capability" and "c-capability" are equivalent for such class of Lie algebras.

2. Preliminaries

In Ref. [19], the c-epicenter of a Lie algebra L, $Z_c^*(L)$, is defined to be the smallest ideal M of L such that L/M is c-capable. For $c = 1$, the 1-epicenter of L is called the epicenter of L in [17] and is denoted by $Z^*(L)$. It is obvious that $Z_c^*(L)$ is a characteristic ideal of L contained in $Z_c(L)$, and $Z_c^*(L/Z_c^*(L)) = 0$. Clearly, L is c-capable if and only if $Z_c^*(L) = 0$.

The following proposition shows that a finite dimensional non-abelian nilpotent Lie algebra with the derived subalgebra of dimension two can be decomposed into a stem Lie algebra and an abelian Lie algebra. It gives the connection between the c-epicenter of such Lie algebras and the c-epicenter of the non-abelian direct summands.

Proposition 2.1. *Let L be a finite dimensional nilpotent non-abelian Lie algebra. Then $L = T \oplus A$ such that $Z(T) = L^2 \cap Z(L)$ and $Z_c^*(L) = Z_c^*(T)$, where A is an abelian Lie algebra.*

Proof. The result follows from [10, Corollary 4.3] and [11, Proposition 3.1]. □

The following proposition gives a criterion for detecting the capability of finite dimensional nilpotent Lie algebras with the derived subalgebra of dimension two.

Proposition 2.2. *Let L be a finite dimensional nilpotent Lie algebra such that $\dim L^2 = 2$. Then*

i. *If L is capable and $cl(L) = 2$, then $3 \leq \dim(L/Z(L)) \leq 5$.*
ii. *If L is capable and $cl(L) = 3$, then $3 \leq \dim(L/Z(L)) \leq 4$.*

Where $c(L)$ denotes the nilpotency class of a Lie algebra L.

Proof. The result follows from [12, Proposition 2.6] and [13, Corollary 5.4]. □

The Lie algebras in this article are given with multiplication tables with respect to fixed bases with trivial products of the form $[x_i, x_j] = 0$ omitted. We use the notation and terminology of Ref. [4, 5, 7]. First, for a field $\mathbb{F}$, let $\mathbb{F}^*$ denotes the multiplicative group of non-zero elements of $\mathbb{F}$. In the following, we list all capable generalized Heisenberg Lie algebras of rank two.

Theorem 2.3. [13, Theorem 3.6] *Let H be an n-dimensional generalized Heisenberg Lie algebra with $\dim H^2 = 2$. Then H is capable if and only if $n = 5, 6, 7$ and H is isomorphic to one the following Lie algebras*

$$L_{5,8} = \langle x_1, \ldots, x_5 | [x_1, x_2] = x_4, [x_1, x_3] = x_5 \rangle,$$
$$L_{6,22}(\epsilon) = \langle x_1, \ldots, x_6 | [x_1, x_2] = x_5 = [x_3, x_4], [x_1, x_3] = x_6, [x_2, x_4] = \epsilon x_6 \rangle,$$

where $\epsilon \in \mathbb{F}/(\overset{*}{\sim})$ *and* $\operatorname{char} \mathbb{F} \neq 2$,

$$L_{6,7}^{(2)}(\eta) = \langle x_1, \ldots, x_6 | [x_1, x_2] = x_5, [x_3, x_4] = x_5 + x_6, [x_1, x_3] = x_6, [x_2, x_4] = \eta x_6 \rangle,$$

where $\eta \in \{0, \omega\}$ *and* $\operatorname{char} \mathbb{F} = 2$, *or*

$$L_1 \cong \langle x_1, \ldots, x_7 | [x_1, x_2] = x_6 = [x_3, x_4], [x_1, x_5] = x_7 = [x_2, x_3] \rangle.$$

Recall that an n-dimensional nilpotent Lie algebra L is said to be nilpotent of maximal class if $L^n = 0$ and $L^{n-1} \neq 0$. In this case, we can see that $\dim(L^j/L^{j+1}) = 1$ for all j, $2 \leq j \leq n-1$ and $\dim(L/L^2) = 2$. Since the minimum number of elements required to generate a nilpotent Lie algebra is $\dim(L/L^2)$, Lie algebras of maximal class are two generated. If L is of maximal class, then $Z_i(L) = L^{n-i}$ for all i, $0 \leq i \leq n-1$ (see [2]). From Ref. [5], there is only a unique Lie algebra up to isomorphism of maximal class with dimension 4 which has the following presentation

$$L_{4,3} \cong \langle x_1, \ldots, x_4 | [x_1, x_2] = x_3, [x_1, x_3] = x_4 \rangle.$$

We say a Lie algebra L is a semidirect sum of an ideal I and a subalgebra K if $L = I + K$, $I \cap K = 0$ and K is not ideal. In this case L is denoted by $K \ltimes I$.

The following results shows that the Lie algebra $L_{5,5} = \langle x_1, \ldots, x_5 | [x_1, x_2] = x_3, [x_1, x_3] = x_5, [x_2, x_4] = x_5 \rangle$ has an ideal of maximal class of dimension four by the classification of 5-dimensional nilpotent Lie algebras in Ref. [5].

Lemma 2.4. [13, Lemma 4.1] *Let* $L \cong L_{5,5}$. *Then* $L = I \rtimes \langle x_4 \rangle$, *where* $I = \langle x_1, x_2, x_3, x_5 | [x_1, x_2] = x_3, [x_1, x_3] = x_5 \rangle \cong L_{4,3}$, and $[I, \langle x_4 \rangle] = \langle x_5 \rangle$.

3. *c*-capable stem nilpotent Lie algebras with the derived subalgebra of dimension two

In this section, we are going to determine c-capable stem nilpotent Lie algebras with the derived subalgebra of dimension at most two. Since the nilpotent Lie algebras with the derived subalgebra of dimension at most two have the nilpotency class at most 3, we divide the classification into two subsections.

3.1. c-Capable stem nilpotent Lie algebras of class 2

We know that stem nilpotent Lie algebras of class two are generalized Heisenberg Lie algebras. We are going to show that "capability" and "c-capability" are equivalent for generalized Heisenberg Lie algebras of rank 2.

Here we recall the definition of standard fillform Lie algebras from [3].

Definition 3.1. For $c \geq 2$, Lie algebra L is called standard fillform of dimension $c+1$ provided that if

$$L = \langle x, y_1, \ldots, y_c | [x, y_i] = y_{i+1}, 1 \leq i \leq c-1 \rangle.$$

We denote this algebra by F_{c+1}.

Clearly, a standard filiform Lie algebra of dimension $c+1$ is of maximal class, so for $n \geq 3$, there is an n-dimensional Lie algebra of maximal class.

For every n-dimensional Lie algebra of maximal class L, the following proposition shows that $L/Z_{n-3}(L)$ is isomorphic to Heisenberg Lie algebra. Here we denote an abelian Lie algebra of dimension n and the Heisenberg Lie algebra of dimension $2m+1$ by $A(n)$ and $H(m)$, respectively.

Proposition 3.2. *Let L be an n-dimensional Lie algebra of maximal class for $n \geq 3$. Then $L/Z_{n-3}(L) = L/L^3 \cong H(1)$.*

Proof. Since $\dim(L/Z_{n-3}(L)) = 3$ and $L/Z_{n-3}(L)$ is a Lie algebra of maximal class, we have $L/Z_{n-3}(L) \cong H(1)$, as required. □

Corollary 3.3. *H(1) is c-capable.*

We give a necessary condition for detecting the c-capability of nilpotent Lie algebras of class 2 with the derived subalgebra of dimension two.

Lemma 3.4. *Let L be a c-capable finite dimensional nilpotent Lie algebra of class two and* $\dim L^2 = 2$. *Then*

$$3 \leq \dim L/Z(L) \leq 5.$$

Proof. Using Proposition 2.2, the result follows. □

We are going to construct a Lie algebra K of nilpotency class $c+2$ such that $K/Z_c(K)$ is a generalized Heisenberg Lie algebra of rank 2. The iterated commutator $[x, \underbrace{y, \ldots, y}_{c-\text{times}}]$ is denoted by $[x,_c y]$.

Theorem 3.5. *Consider the following nilpotent Lie algebras*

$$K_1 = \langle y_1, \ldots, y_{c+5} | [y_1, y_j] = y_{j+2}, [y_1, y_j,_r y_1] = y_{r+5}, 2 \leq j \leq 3, 1 \leq r \leq c \rangle,$$

$$K_2 = \langle y_1, \ldots, y_{c+6} | [y_1, y_2] = [y_3, y_4] = y_5, [y_1, y_3] = y_6, [y_2, y_4] = \epsilon y_6,$$
$$[y_1, y_2,_r y_1] = [y_1, y_3,_r y_1] = y_{r+6}, 1 \leq r \leq c \rangle,$$

where $\epsilon \in \mathbb{F}/(\overset{*}{\sim})$ and $\operatorname{char} \mathbb{F} \neq 2$.

$$K_3 = \langle y_1, \ldots, y_6, y_{r+6}, y'_{r+6} | [y_1, y_2] = y_5, [y_3, y_4] = y_5 + y_6, [y_1, y_3] = y_6,$$
$$[y_2, y_4] = \eta y_6, [y_1, y_2,_r y_1] = y'_{r+6}, [y_1, y_3,_r y_1] = y_{r+6}, 1 \leq r \leq c \rangle,$$

where $\eta \in \{0, \omega\}$ and $\operatorname{char} \mathbb{F} = 2$,

$$K_4 = \langle y_1, \ldots, y_{c+7} | [y_1, y_2] = [y_3, y_4] = y_6,$$
$$[y_4, y_5] = [y_2, y_3] = y_7, [y_1, y_2,_r y_1] = [y_2, y_3,_r y_2] = y_{r+7}, 1 \leq r \leq c \rangle.$$

Then $\dim K_1 = c+5, \dim K_2 = c+6, \dim K_3 = 2c+6, \dim K_4 = c+7$, $cl(K_i) = c+2$, $K_i^3 = Z_c(K_i)$ *and $K_i/Z_c(K_i)$ is a generalized Heisenberg Lie algebra of rank 2 for all i, $1 \leq i \leq 4$, and they are isomorphic to $L_{5,8}$, $L_{6,22}(\epsilon), L_{6,7}^{(2)}(\eta)$ and L_1, respectively.*

Proof. Clearly $cl(K_i) = c+2$. Also we can easily check that K_i/K_i^3 is isomorphic to $L_{5,8}$, $L_{6,22}(\epsilon), L_{6,7}^{(2)}(\eta)$ and L_1, for all i, $1 \leq i \leq 4$, respectively, by Theorem 2.3. Since K_i/K_i^3 is a generalized Heisenberg Lie algebra of rank two, we have

$$\left(Z(K_i) + K_i^3\right)/K_i^3 \subseteq Z\left(K_i/K_i^3\right) = K_i^2/K_i^3,$$

so $Z(K_i) \subseteq K_i^2$ for all i, $1 \leq i \leq 4$. We claim that $Z(K_i) \subseteq K_i^3$ for all i, $1 \leq i \leq 4$. We may assume that $K = K_1$. By contrary, let $Z(K) \not\subseteq K^3$. We have

$$\left(K/\left(K^3+Z(K)\right)\right)^{ab} \cong K/K^2 \cong A(3) \text{ and } K/K^3 \cong L_{5,8}.$$

By using the natural epimorphism $\eta : K/K^3 \to K/(K^3+Z(K))$ and considering the fact that $\dim(K/K^3)=5$, we have $1 \le \dim(K/(K^3+Z(K))) \le 5$. Since $Z(K)\nsubseteq K^3$, we have $\dim(K/(K^3+Z(K))) \neq 5$. Since $\dim(K/(K^3+Z(K))^{ab}=3$, we get $3 \le \dim(K/(K^3+Z(K))) \le 4$. If $\dim(K/(K^3+Z(K)))=3$, then $\dim(K/(K^3+Z(K)))=3=\dim(K/(K^3+Z(K)))^{ab}$. Thus $K/Z(K)/(K/Z(K))^3$ is abelian and $K^2/Z(K)=(K/Z(K))^3$, which is a contradiction.

If $\dim(K/(K^3+Z(K)))=4$, since $\dim(K/(K^3+Z(K)))^{ab}=3$, we have

$$A(1) \cong K^2/\left(K^3+Z(K)\right) \subseteq Z\left(K/\left(K^3+Z(K)\right)\right).$$

Thus $\dim(K/(K^3+Z(K)))^2=1$ and so by [16, Theorem 3.6], we have $K/(K^3+Z(K)) \cong H(1)\oplus A(1)$. Let

$$\langle y_1+K^3+Z(K), y_2+K^3+Z(K)\rangle \cong H(1) \text{ and } \langle y_3+K^3+Z(K)\rangle \cong A(1).$$

It follows that

$$K^2/\left(K^3+Z(K)\right)=\langle\left[y_1+K^3+Z(K), y_2+K^3+Z(K)\right]\rangle.$$

Hence $K^2=\langle[y_1,y_2]\rangle+K^3+Z(K)$. On the other hand,

$$\begin{aligned} K^2/K^3 &= \frac{\langle[y_1,y_2]\rangle+K^3}{K^3} \oplus \frac{\langle[y_1,y_3]\rangle+K^3}{K^3} \\ &= \frac{\langle[y_1,y_2]\rangle+K^3}{K^3} \oplus \frac{K^3+Z(K)}{K^3}. \end{aligned}$$

Since $[y_1,y_3] \in K^3+Z(K)$, we have $(\langle[y_1,y_3]\rangle+K^3)/K^3 \cong (K^3+Z(K))/K^3 \cong A(1)$. Hence $K^3+Z(K)=\langle[y_1,y_3]\rangle+K^3$. Let A be the subalgebra of K generated by the set $Y\setminus\{y_2,y_4\}$, where $Y=\{y_1,y_2,\ldots,y_{c+5}\}$. Since $[y_1,y_3]=y_5, [y_1,y_{3},_r y_1]=y_{5+r}$, for all r, $1\le r\le c$, by looking Definition 3.1, it is clear to see that A is a standard fillform Lie algebra of maximal class of dimension $c+3$. So $A^2=\langle[y_1,y_3]\rangle+K^3=K^3+Z(K)$. Thus $A(1)\cong Z(K)=Z(A)\subseteq A^3=K^3$ and so $Z(K)\subseteq K^3$. It contradicts our assumption. Thus $Z(K)\subseteq K^3$. Therefore

$$\begin{gathered} K/Z(K)=\langle y_1+Z(K),\ldots,y_{c+5}+Z(K) | [y_1,y_2]+Z(K)=y_4+Z(K), \\ [y_1,y_3]+Z(K)=y_5+Z(K), [y_1,y_{3},_r y_1]+Z(K)=[y_1,y_{2},_r y_1]+Z(K)=y_{5+r}+Z(K), 1\le r\le c-1\rangle \end{gathered}$$

and $K/Z(K)/(K/Z(K))^3 \cong L_{5,8}$. We conclude $K/K^3\cong L_{5,8}\cong K/(K^3+Z(K))$. Using induction on $cl(K)$, we are going to prove that $Z_c(K)=K^3$. Assume that $c=1$. So $cl(K)=3$ and we get $Z(K)=K^3$. Using the induction hypothesis,

$$\left(K/Z(K)\right)^3=Z_{c-1}\left(K/Z(K)\right)=Z_c(K)/Z(K).$$

Hence $Z_c(K)=K^3$. By a similar method, we may obtain $Z_c(K_i)=K_i^3$ for $i=2,3,4$. The result follows.

The following corollary is an immediate consequence of Theorem 3.5. □

Corollary 3.6. $L_{5,8}$, $L_{6,22}(\epsilon), L_{6,7}^{(2)}(\eta)$ *and* L_1 *are c-capable.*

The capability of generalized Heisenberg Lie algebras of rank 2 is equivalent to the c-capability of them by the following theorem.

Theorem 3.7. *Let H be an n-dimensional generalized Heisenberg Lie algebra of rank 2. Then H is c-capable if and only if H is isomorphic to one of Lie algebras $L_{5,8}$, $L_{6,22}(\epsilon)$, $L_{6,7}^{(2)}(\eta)$, or L_1.*

Proof. Let H be a c-capable. Then by Theorem 2.3 and Lemma 3.4, H should be one of Lie algebras $L_{5,8}$, $L_{6,22}(\epsilon)$, $L_{6,7}^{(2)}(\eta)$, or L_1. The converse is held by Corollary 3.6. The result follows. □

3.2. c-Capable stem nilpotent Lie algebras of class 3

We are going to show capable stem nilpotent Lie algebras of class 3 with the derived subalgebra of dimension two are c-capable.

Theorem 3.8. *Let T be a c-capable n-dimensional Lie algebra of class 3 and $\dim T^2 = 2$. Then $3 \leq \dim(T/Z(T)) \leq 4$.*

Proof. Since T is capable, the result follows from Proposition 2.2 (ii). □

For every n-dimensional Lie algebra L of maximal class, the following proposition shows that $L/Z_{n-4}(L)$ is isomorphic to $L_{4,3}$.

Proposition 3.9. *Let L be an n-dimensional Lie algebra of maximal class for $n \geq 4$. Then $L/Z_{n-4}(L) = L/L^4 \cong L_{4,3}$.*

Proof. Since $L^2/Z_{n-4}(L) \cong A(2)$ and $\dim(L/Z_{n-4}(L)) = 4$, we have $L/Z_{n-4}(L)$ is a Lie algebra of maximal class of dimension 4 and so $L/Z_{n-4}(L) \cong L_{4,3}$, as required. □

Corollary 3.10. *$L_{4,3}$ is c-capable.*

Proof. Let L be a standard fillform Lie algebra of maximal class of dimension $c+4$. By Proposition 3.9, we have $L/Z_c(L) \cong L_{4,3}$, as required. □

Proposition 3.11. *$L_{5,5}$ is c-capable.*

Proof. By Lemma 2.4, we have $L_{5,5} = I \rtimes \langle x_4 \rangle$, in which $I \cong L = 3, 4$. Let H be the standard fillform Lie algebra of nilpotency class $c+3$ with the following presentation

$$H = \langle x, y_1, \ldots, y_{c+3} | [x, y_1] = y_2, [y_{i+1}, y_1] = y_{i+2}, 1 \leq i \leq c+1 \rangle.$$

By Proposition 3.9 we have $H/Z_c(H) \cong L_{3,4}$.

$$\text{Put} \quad H_1 = \langle x, y_1, \ldots, y_{c+3}, b | [x, y_1] = y_2, [y_{i+1}, y_1] = y_{i+2}, [b,_{j+1} y_1] = y_{3+j}, \\ 0 \leq j \leq c, 1 \leq i \leq c+1 \rangle.$$

It is obviously seen that $H_1 = H + \langle b \rangle$ such that $b \notin H, H^2 = H_1^2, cl(H) = cl(H_1) = c+3$ and $\dim\ (H_1) = c+5$. We claim that $Z_c(H_1) = Z_c(H) = H^4$. Since $[b,_{c+1} y_1] = y_{c+3}$, we have $b \notin Z_c(H_1)$ and $Z_c(H) = H^4 = H_1^4 \subseteq Z_c(H_1)$. It is sufficient to show $Z_c(H_1) \subseteq Z_c(H)$. By contrary, let $Z_c(H_1) \neq Z_c(H)$. We have $H_1/Z_c(H) = H/Z_c(H) \rtimes (\langle b \rangle + Z_c(H)/Z_c(H))$ and $H/Z_c(H) \cong L_{4,3}$. Thus

$$\begin{aligned} H_1/Z_c(H) = \langle & x + Z_c(H), y_1 + Z_c(H), y_2 + Z_c(H), y_3 + Z_c(H), b + Z_c(H) | \\ & [y_2, y_1] + Z_c(H) = y_3 + Z_c(H), [x, y_1] + Z_c(H) = y_2 + Z_c(H), \\ & [b, y_1] + Z_c(H) = y_3 + Z_c(H) \rangle \cong L_{4,3} \rtimes \langle b \rangle \cong L_{5,5}. \end{aligned}$$

There exists a natural epimorphism $\eta : H_1/Z_c(H) \to H_1/Z_c(H_1)$. Since $cl(H_1/Z_c(H_1)) = 3$ and $\dim H_1/Z_c(H) = 5$, we have $\dim H_1/Z_c(H_1) = 4$. So $H_1/Z_c(H_1) \cong L_{4,3}$. Therefore

$$H_1/Z_c(H_1) = \langle x + Z_c(H_1), y_1 + Z_c(H_1), y_2 + Z_c(H_1), y_3 + Z_c(H_1) |$$
$$[y_2, y_1] + Z_c(H_1) = y_3 + Z_c(H_1), [x, y_1] + Z_c(H_1) = y_2 + Z_c(H_1) \rangle.$$

We conclude $\ker\eta = Z_c(H_1)/Z_c(H) = (\langle b \rangle + Z_c(H))/Z_c(H)$. Therefore $b \in Z_c(H_1)$. It is a contradiction. Thus $Z_c(H_1) = Z_c(H)$ and so $H_1/Z_c(H_1) = H/Z_c(H) \rtimes (\langle b \rangle + Z_c(H)/Z_c(H)) \cong L_{5,5}$. The proof is complete. □

Now, we are in a position to show the c-capability of stem Lie algebras T of class 3 and $\dim T^2 = 2$.

Theorem 3.12. *Let T be an n-dimensional stem Lie algebra of class 3 and $\dim T^2 = 2$. Then T is c-capable if and only if $T \cong L_{4,3}$ or $T \cong L_{5,5}$.*

Proof. Let T be c-capable. Since T is capable, [13, Theorem 4.9] implies $T \cong L_{4,3}$ or $T \cong L_{5,5}$. The converse holds by using Corollary 3.10 and Proposition 3.11. □

4. *c*-capable nilpotent Lie algebras with the derived subalgebra of dimension two

In this section, we are going to determine all c-capable nilpotent Lie algebras with the derived subalgebra of dimension two.

In the following corollary, all c-capable nilpotent Lie algebras of class 2 with the derived subalgebra of dimension 2 are classified.

Theorem 4.1. *Let L be an n-dimensional nilpotent Lie algebra of nilpotency class 2 and $\dim L^2 = 2$. Then L is c-capable if and only if L is isomorphic to one of Lie algebra $L_{5,8} \oplus A(n-5)$, $L_{6,22}(\epsilon) \oplus A(n-6)$, $L_{6,7}^{(2)}(\eta) \oplus A(n-6)$ or $L_1 \oplus A(n-7)$.*

Proof. This is an immediate consequence of Proposition 2.1, [13, Theorem 5.2] and Theorem 3.7.

Now we are ready to determine all c-capable Lie algebras L of class 3 and $\dim L^2 = 2$. □

Theorem 4.2. *Let L be an n-dimensional Lie algebra of class 3 such that $\dim L^2 = 2$. Then L is c-capable if and only if $L \cong L_{4,3} \oplus A(n-4)$ or $L \cong L_{5,5} \oplus A(n-5)$.*

Proof. This is immediate from Proposition 2.1, [13, Theorem 5.3] and Theorem 3.12.

Finally the following theorem gives the classification of all c-capable finite dimensional nilpotent Lie algebras with the derived subalgebra of dimension at most two.

Theorem 4.3. *Let L be an n-dimensional nilpotent Lie algebra such that $\dim L^2 \leq 2$. Then L is c-capable is if and only if L is isomorphic to one of the following Lie algebras*

i. *if $\dim L^2 = 0$, then $L \cong A(n)$ and $n \geq 2$,*
ii. *if $\dim L^2 = 1$, then $L \cong H(1) \oplus A(n-3)$,*
iii. *if $\dim L^2 = 2$ and $cl(L) = 2$, then $L \cong L_{5,8} \oplus A(n-5)$, $L \cong L_{6,7}^{(2)}(\eta) \oplus A(n-6)$, $L \cong L_{6,22}(\epsilon) \oplus A(n-6)$, or $L \cong L_1 \oplus A(n-7)$.*
iv. *if $\dim L^2 = 2$ and $cl(L) = 3$, then $L \cong L_{4,3} \oplus A(n-4)$ or $L \cong L_{5,5} \oplus A(n-5)$.*

Proof. The result follows from Ref. [18, Corollary 3.2 and Proposition 3.6], Theorems 4.1 and 4.2. □

Corollary 4.4. *Let L be an n-dimensional nilpotent Lie algebra such that $\dim L^2 = 2$. Then $Z_c^*(L) = Z^*(L)$ for all $c \geq 1$.*

Proof. If L is capable by the above we have $Z_c^*(L) = Z^*(L) = 0$. Now let L be a non-capable Lie algebra, we have $Z^*(L) \subseteq L^2$. Hence $L/Z^*(L)$ is abelian or $\dim(L/Z^*(L))^2 = 1$. Theorem 4.3 implies that $L/Z^*(L)$ is c-capable and so $Z_c^*(L) \subseteq Z^*(L)$. The converse always holds so $Z_c^*(L) = Z^*(L)$ for all $c \geq 1$. □

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