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Cubic B-splines collocation method for solving a partial integro-differential equation with a weakly singular kernel

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Abstract In this paper, we apply a numerical scheme for the solution of a second order partial integro-differential equation with a weakly singular kernel. In the time direction, the backward Euler method time-stepping is used to approximate the differential term and the cubic B-splines is applied to the space discretization. Detailed discrete schemes, the convergence and the stability of the method is demonstrated. Next, the computational efficiency and accuracy of the method are examined by the numerical results.

Keywords. Cubic B-splines, Partial integro-differential equation, Backward Euler method.

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1. Introduction

In this paper, we consider the following second order partial integro-differential equation with a weakly singular kernel

$$u_t(x,t) = \mu u_{xx}(x,t) + \int_0^t (t-s)^{-\frac{1}{2}} u_{xx}(x,s)ds, \quad x \in [a,b],\ t \geq 0, \tag{1.1}$$

where $\mu \geq 0$ and subject to the initial condition

$$u(x,0) = \nu(x), \tag{1.2}$$

with the boundary conditions

$$u(a,t) = 0, u(b,t) = 0, \quad t \geq 0. \tag{1.3}$$

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Such a problem can be appeared in mathematical modeling of some engineering and scientific systems that leads to a model by partial integro-differential equations(PIDEs). The equations (1.1)-(1.3) can be appeared in the applications such as heat conduction in materials with memory [10, 22], population dynamics and viscoelasticity [5, 27].

Of course, some numerical methods have been applied for partial integro- differential equations with weakly singular kernel. These methods are including finite-element methods [43]-[39], finite difference methods [33, 35], orthogonal spline collocation methods [1, 8], spectral collocation methods [9, 16], Galerkin methods [15] and quasi wavelet methods [17, 42].

Because of the singularity of the kernel, inducing sharp transitions in the solution, there is a challenge in developing accurate numerical methods for solving the partial integro-differential equations. Thus, applying the collocation method by using the B-splines functions in handling the sharp transitions is caused by the singularities of the kernel is an effective way. Because there are two useful and important features of B-splines in numerical work. One is that the continuity conditions are inherent. Compared with other piecewise polynomial interpolation functions, the B-spline functions are the smoothest interpolation functions. Second feature is each B-spline function is only non-zero over a few mesh subintervals, i.e, B-splines have small local support property. Therefore, the resulting matrix is sparse. Because of having smoothness and capability to handle local phenomena, B-splines have been offered with special advantage. In the case it combined with the collocation, these advantages can significantly simplify the solution procedure of the differential equations. For example, the cubic B-splines have been used to compute the numerical solution of the Klein-Gordon equation [13]. The Regularized Long Wave RLW equation [28] can be solved by quadratic B-splines and the quantic B-splines has been used to build up the numerical solution of the Burgers equation in [30], the KdVB equation [44], the RLW equation [6], the Kuramoto-Sivashinsky equation [23] and cubic spline quasi-interpolation and multi-node higher order expansion have been used to solve the Burgers equation in [41]. Caglar [2, 3] used the B-splines to solve the boundary value problems. RLW equation can be solved by B-splines in [28, 29]. Most recently, the quantic B-spline collocation method is applied to obtain the numerical solution of fourth order partial integro-differential equations in [45].

The rest of the paper is organized as follows. In section 2, a detailed description about the cubic B-Splines is explained. In section 3, a numerical scheme for solving the problem (1.1)-(1.3) is discussed. The convergence analysis of the method is described in section 4. The stability analysis is carried out via Von-Neumann stability as given in section 5. In section 6, numerical experiments are tested to demonstrate the viability of the proposed method and this paper ends with a conclusion in section 7.

2. The cubic B-Splines

Consider a mesh $a = x_0 < x_1 < \ldots < x_{N-1} < x_N = b$ as a uniform partition of the solution domain $a \leq x \leq b$ by the knots x_j with equally step length $h = x_{j+1} - x_j =$

$\frac{b-a}{N}, j = 0, 1, \dots, N-1$. The cubic B-splines $B_j(x)$ for $j = -1, 0, \cdots, N+1$ at the knots are given as follows [24, 25]:

$$B_j(x) = \frac{1}{h^3} \begin{cases} (x - x_{j-2})^3, & x \in [x_{j-2}, x_{j-1}) \\ (x - x_{j-2})^3 - 4(x - x_{j-1})^3, & x \in [x_{j-1}, x_j) \\ (x_{j+2} - x)^3 - 4(x_{j+1} - x)^3, & x \in [x_j, x_{j+1}) \\ (x_{j+2} - x)^3, & x \in [x_{j+1}, x_{j+2}) \\ 0, & otherwise, \end{cases} \tag{2.1}$$

where, $\{B_{-1}, B_0, B_1, \dots, B_{N-1}, B_N, B_{N+1}\}$ form a basis over the region $a \le x \le b$. Each cubic B-Spline covers four elements, so that each element is covered by four cubic B-splines. The values of $B_j(x)$ and its first and second derivatives are given as in Table 1.

Table 1. Coefficients of cubic B-splines and its derivatives at knots x_j.

x_j	x_{j-2}	x_{j-1}	x_j	x_{j+1}	x_{j+2}
$B_j(x)$	0	1	4	1	0
$B'_j(x_j)$	0	$\frac{3}{h}$	0	$-\frac{3}{h}$	0
$B''_j(x_j)$	0	$\frac{6}{h^2}$	$-\frac{12}{h^2}$	$\frac{6}{h^2}$	0

Our numerical scheme for problem (1.1)-(1.3) using the collocation method with the cubic B-splines is to compute an approximate solution $U_N(x,t)$ to the exact solution $u(x,t)$ in the following form

$$U_N(x,t) = \sum_{j=-1}^{N+1} C_j(t) B_j(x), \tag{2.2}$$

where, $B_j(x)$ are the cubic B-splines in our proposed method, and $C_j(t)$ are time dependent quantities to be determined by the boundary conditions and collocation form of the partial integro-differential equation.

3. Numerical scheme

In this section, we propose the numerical scheme for the solution of the problem Eqs. (1.1)-(1.3) in which the time derivative is dealt with the second order backward finite difference method and cubic B-splines are applied to the spacial derivative. First, we let the time level is denoted by $t_n = n\Delta t, n = 0, 1, \dots$, where Δt is the time step. To apply the proposed method on Eq. (1.1) at time point $t = t_{n+1}$, the first expression in the left of Eq. (1.1) is approximated by

$$u_t(x, t_{n+1}) \approx \frac{u^{n+1}(x) - u^n(x)}{\Delta t}, \quad a \le x \le b, \ n \ge 1. \tag{3.1}$$

Therefore, for every $x \in [a, b]$, we have

$$\frac{u^{n+1}(x) - u^n(x)}{\Delta t} = \mu u_{xx} + \int_0^{t_{n+1}} (t_{n+1} - s)^{-\frac{1}{2}} u_{xx}(x, s) ds. \tag{3.2}$$

For approximating the second expression in the right side of Eq. (3.2), we get

$$\begin{aligned}\int_0^{t_{n+1}}(t_{n+1}-s)^{-\frac{1}{2}}u_{xx}(x,s)ds &= \int_0^{t_{n+1}} s^{-\frac{1}{2}}u_{xx}(x,t_{n+1}-s)ds\\ &= \sum_{j=0}^{n}\int_{t_j}^{t_{j+1}} s^{-\frac{1}{2}}u_{xx}(x,t_{n+1}-s)ds\\ &\approx \sum_{j=0}^{n}u_{xx}(x,t_{n-j+1})\int_{t_j}^{t_{j+1}} s^{-\frac{1}{2}}ds\\ &= 2\Delta t^{\frac{1}{2}}\sum_{j=0}^{n}u_{xx}(x,t_{n-j+1})\left[(j+1)^{\frac{1}{2}}-j^{\frac{1}{2}}\right].\end{aligned} \tag{3.3}$$

By substituting Eq. (3.3) into Eq. (3.2) and rearranging, the Eq.(3.2) is become as follows

$$\begin{aligned}u^{n+1}(x)-\mu\Delta t u_{xx}(x,t_{n+1})-2\Delta t^{\frac{3}{2}}u_{xx}(x,t_{n+1})&\\ = u^n(x)+2\Delta t^{\frac{3}{2}}\sum_{j=1}^{n}\nu_j u_{xx}(x,t_{n-j+1}),&\end{aligned} \tag{3.4}$$

where $\nu_j=(j+1)^{\frac{1}{2}}-j^{\frac{1}{2}}$, $j=0,1,\dots,n$.
Next, the spacial discretization of Eq. (3.4) is carried out by using Eq. (2.2) and the collocation method is implemented by identifying the collocation points as nodes. Therefore, for $i=0,1,\dots,N$, yields

$$\begin{aligned}\sum_{j=-1}^{N+1}C_j^{n+1}B_j(x_i)-\mu\Delta t\sum_{j=-1}^{N+1}C_j^{n+1}B_j''(x_i)-2\Delta t^{\frac{3}{2}}\sum_{j=-1}^{N+1}C_j^{n+1}B_j''(x_i)&\\ = u_i^n+2\Delta t^{\frac{3}{2}}\sum_{k=1}^{n}\nu_k\sum_{j=-1}^{N+1}C_j^{n-k+1}B_j''(x_i),&\end{aligned} \tag{3.5}$$

where $C_j^{n+1}=C_j(t_{n+1})$, U_i^{n+1} is the approximate solution of $u(x_i,t_{n+1})$ in Eq. (3.2) and $B_j''(x_i)$ is the second order partial derivative with respect to the space variable x of B_j at x_i. Let

$$D_i=u_i^n+2\Delta t^{\frac{3}{2}}\sum_{k=1}^{n}\nu_k\sum_{j=-1}^{N+1}C_j^{n-k+1}B_j''(x_i),\quad i=0,1,\dots,N. \tag{3.6}$$

Therefore, we can rewrite Eq. (3.5) as follows

$$\sum_{j=-1}^{N+1}\left[B_j(x_i)-(\mu\Delta t+2\Delta t^{\frac{3}{2}})B_j''(x_i)\right]C_j^{n+1}=D_i, i=0,1,\dots,N. \tag{3.7}$$

The system (3.7) consists of $N+1$ linear equations with $N+3$ unknowns, C_{-1}^{n+1},C_0^{n+1}, $\dots,C_{N+1}^{n+1}$. To compute the unique solution to this system, the parameters C_{-1}^{n+1} and C_{N+1}^{n+1} are eliminated by imposing the boundary conditions. From Eq. (1.3), we expand u in terms of the cubic B-splines formula Eq. (2.2) at $x_0 = a$ and $x_N = b$, yield

$$\begin{cases} C_{-1} + 4C_0 + C_1 = 0, \\ C_{N-1} + 4C_N + C_{N+1} = 0. \end{cases} \tag{3.8}$$

By solving the Eq. (3.8), we get the values of C_{-1} in terms of C_0 and C_1 and similarly C_{N+1} in terms of C_{N-1} and C_N. Thus, the system (3.7) is reduced to a tridiagonal system of $N+1$ linear equations and $N+1$ unknowns. For the sake of simplification, the system (3.7) is denoted by the following matrix form

$$\mathbf{A}\boldsymbol{C} = \mathbf{D}, \tag{3.9}$$

where, the matrices $\mathbf{A}$, $\boldsymbol{C}$ and $\mathbf{D}$ read as follow

$$\mathbf{A} = \begin{bmatrix} \gamma - 4\beta & 0 & 0 & 0 & \cdots & 0 \\ \beta & \gamma & \beta & 0 & \cdots & 0 \\ 0 & \beta & \gamma & \beta & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \beta & \gamma & \beta \\ 0 & \cdots & 0 & 0 & 0 & \gamma - 4\beta \end{bmatrix},$$

$$\boldsymbol{C} = [C_0^{n+1}, C_1^{n+1}, \ldots, C_N^{n+1}]^T, \quad n = 0, 1, 2, \ldots, \quad \mathbf{D} = [D_0, D_1, \ldots, D_N]^T,$$

where,

$$\beta = 1 - \frac{6\Delta t}{h^2}(\mu + 2\Delta t^{\frac{1}{2}}), \quad \gamma = 4(1 + \frac{3\Delta t}{h^2}(\mu + 2\Delta t^{\frac{1}{2}})). \tag{3.10}$$

Next, from the initial condition given in Eq. (1.2) and the collocation form (2.2), we deduce

$$u(x, 0) = U_N(x, 0) = \sum_{j=-1}^{N+1} C_j^0 B_j(x), \quad x \in [a, b].$$

By partitioning $[a, b]$ into $N + 1$ points, namely $x_0, x_1, \cdots$ and x_N, the initial vector $\boldsymbol{C^0} = [C_{-1}^0, C_0^0, \cdots, C_N^0, C_{N+1}^0]$ can be obtained and next the initial numerical solution $U_N^0 = [U_N(x_0), U_N(x_1), \cdots, U_N(x_N)]$ results at the first step $t = 0$, easily.

4. Convergence analysis

In this section, we prove the convergency of the proposed numerical scheme in the spatial and temporal directions, respectively.

4.1. **The convergence in the spacial direction.** Let $\widehat{U}_{exact}(x)$ be the exact solution of Eq. (1.1)-(1.3) and $\widehat{S}(x)$ be the cubic B-splines collocation approximation to $\widehat{U}(x)$. Then

$$\widehat{S}(x) \approx \widehat{U}_{exact}(x) = \sum_{j=-1}^{N+1} \widehat{C}_j(t) B_j(x), \tag{4.1}$$

where $\widehat{C} = (\widehat{C}_{-1}, \widehat{C}_0, \ldots, \widehat{C}_{N+1})$.
Also, suppose that $\widetilde{S}(x)$ be the computed cubic B-splines approximation to $\widehat{S}(x)$, namely

$$\widetilde{S}(x) = \sum_{j=-1}^{N+1} \widetilde{C}_j(t) B_j(x), \tag{4.2}$$

where $\widetilde{C} = (\widetilde{C}_{-1}, \widetilde{C}_0, \ldots, \widetilde{C}_{N+1})$. To estimate the error $\parallel \widehat{U}_{exact}(x) - \widetilde{S}(x) \parallel_\infty$, we must approximate the error $\parallel \widehat{U}_{exact}(x) - \widehat{S}(x) \parallel_\infty$ and $\parallel \widetilde{s}(x) - \widehat{s}(x) \parallel_\infty$ separately. According to Eq. (3.7), to compute $\widetilde{S}(x)$ and $\widehat{S}(x)$, we have to obtain the values of vectors $\widehat{\boldsymbol{C}}$ and $\widetilde{\boldsymbol{C}}$ from two systems of linear equations as follows

$$\mathbf{A}\widehat{\boldsymbol{C}} = \widehat{\mathbf{D}}, \tag{4.3}$$

and

$$\mathbf{A}\widetilde{\boldsymbol{C}} = \widetilde{\mathbf{D}}. \tag{4.4}$$

By subtracting Eqs. (4.3) and (4.4), we obtain

$$\mathbf{A}(\widetilde{\boldsymbol{C}} - \widehat{\boldsymbol{C}}) = \widetilde{\mathbf{D}} - \widehat{\mathbf{D}}. \tag{4.5}$$

Furthermore, $\mathbf{A}$ is a strictly diagonally dominant matrix. Thus, it is nonsingular and we have

$$\widetilde{\boldsymbol{C}} - \widehat{\boldsymbol{C}} = \mathbf{A}^{-1}(\widetilde{\mathbf{D}} - \widehat{\mathbf{D}}). \tag{4.6}$$

Taking infinity norm from both sides of Eq. (4.6), we obtain

$$\parallel \widetilde{\boldsymbol{C}} - \widehat{\boldsymbol{C}} \parallel_\infty \leq \parallel \mathbf{A}^{-1} \parallel_\infty \parallel \widetilde{\mathbf{D}} - \widehat{\mathbf{D}} \parallel_\infty . \tag{4.7}$$

Now consider that $\lambda_i (0 \leq i \leq N)$ is the summation of the ith row of matrix $\mathbf{A} = [a_{ij}]_{(N+1)(N+1)}$. Therefore, we get

$$\lambda_0 = \sum_{j=0}^{N} a_{0j} = \gamma - 4\beta, \tag{4.8}$$

$$\lambda_i = \sum_{j=0}^{N} a_{ij} = \gamma + 2\beta, \quad i = 1, \ldots, N-1, \tag{4.9}$$

$$\lambda_N = \sum_{j=0}^{N} a_{Nj} = \gamma - 4\beta. \tag{4.10}$$

Due to the theory of matrices

$$\sum_{i=0}^{N} a_{ki}^{-1}\lambda_i = 1, \quad k = 0, 1, \dots, N, \tag{4.11}$$

where a_{ki}^{-1} are the entries of $\mathbf{A}^{-1}$. Thus,

$$\parallel \mathbf{A}^{-1} \parallel_\infty = \sum_{i=0}^{N} |a_{ki}^{-1}| \le \frac{1}{\lambda}, \tag{4.12}$$

where $\lambda = \min_{0\le i\le N}\lambda_i = \min\{\gamma - 4\beta, \gamma + 2\beta\} = \min\{\eta, 6\}$ and $\eta = \frac{36\Delta t}{h^2}(\mu + 2\Delta t^{\frac{1}{2}})$. Substituting Eq. (4.12) into Eq. (4.7) we can find

$$\parallel \widetilde{\boldsymbol{C}} - \widehat{\boldsymbol{C}} \parallel_\infty \le \frac{1}{\lambda} \parallel \widetilde{\mathbf{D}} - \widehat{\mathbf{D}} \parallel_\infty . \tag{4.13}$$

For computing the upper bound of $\parallel \widetilde{\mathbf{D}} - \widehat{\mathbf{D}} \parallel_\infty$, from Eq. (3.6) for all values of $0 \le i \le N$, we conclude

$$\begin{aligned}|\widetilde{D}_i - \widehat{D}_i| \le |\widetilde{U}_i - \widehat{U}_i| &\\ + \frac{12\Delta t^{\frac{3}{2}}}{h^2}\sum_{k=1}^{n}|\nu_k|(|\widetilde{C}_{i-1}^{n-k+1} - \widehat{C}_{i-1}^{n-k+1}| &\\ + 2|\widetilde{C}_{i}^{n-k+1} - \widehat{C}_{i}^{n-k+1}| + |\widetilde{C}_{i+1}^{n-k+1} - \widehat{C}_{i+1}^{n-k+1}|).\end{aligned} \tag{4.14}$$

Now, we need to recall the following theorem.

Theorem 4.1. *If $f(x) \in c^4[a,b]$, $|f^{(4)}(x)| \le L$, $\forall x \in [a,b]$ and*

$$\Delta = \{a = x_0 < x_1 < \dots < x_N = b\}$$

be the equally spaced partition of $[a,b]$ with step size h and $s(x)$ is the unique spline function interpolate $f(x)$ at knots $x_0, x_1, \dots, x_N$, then there exits a constant λ_j such that,

$$||f^{(j)} - s^{(j)}||_\infty \le \lambda_j L h^{4-j}, \quad j = 0, 1, 2, 3.$$

Proof. See [7]-[11].

According to the above theorem, we have

$$|\widetilde{U}_i - \widehat{U}_i| = |\widetilde{S}(x_i) - \widehat{S}(x_i)| \le \lambda_0 L h^4. \tag{4.15}$$

In addition, $\{\nu_k\}_{k=1}^{n}$ is a sequence of positive terms descending and $\nu_k \le 1$ for $1 \le k \le n$. Thus, from Eq. (4.15) we can rewrite Eq. (4.14) as follows

$$||\widetilde{\mathbf{D}} - \widehat{\mathbf{D}}||_\infty \le \lambda_0 L h^4 + \frac{12\Delta t^{\frac{3}{2}}}{h^2}\sum_{k=1}^{n} m_k, \tag{4.16}$$

where

$$(|\widetilde{C}_{i-1}^{n-k+1} - \widehat{C}_{i-1}^{n-k+1}| + 2|\widetilde{C}_{i}^{n-k+1} - \widehat{C}_{i}^{n-k+1}| + |\widetilde{C}_{i+1}^{n-k+1} - \widehat{C}_{i+1}^{n-k+1}|) \le m_k.$$

By assuming $\sum_{k=1}^{n} m_k = M_n$ and $\lambda_0 L h^4 + \frac{12\Delta t^{\frac{3}{2}}}{h^2}M_n = K_n$, we get

$$||\widetilde{\mathbf{D}} - \widehat{\mathbf{D}}||_\infty \le K_n. \tag{4.17}$$

Using Eq. (4.17), from Eq. (4.13) we come up with

$$\| \widetilde{\boldsymbol{C}} - \widehat{\boldsymbol{C}} \|_\infty \leq Kh^2, \tag{4.18}$$

where $Kh^2 = \frac{1}{\lambda} K_n = \max(\frac{1}{6}, \frac{1}{\eta}) K_n$.
To proceed the rest, we note the following theorem.

Theorem 4.2. *The B-splines $\{B_{-1}, B_0, B_1, \ldots, B_{N-1}, B_N, B_{N+1}\}$ satisfy the following inequality*

$$|\sum_{j=-1}^{N+1} B_j(x)| \leq 1, \quad 0 \leq x \leq 1. \tag{4.19}$$

Proof. See [26].
Now, by subtracting (4.2) from Eq. (4.1), we have

$$\widetilde{S}(x) - \widehat{S}(x) = \sum_{j=-1}^{N+1} (\widetilde{C}_j - \widehat{C}_j) B_j(x). \tag{4.20}$$

Using the above theorem and taking norm from (4.20), we obtain

$$\begin{aligned} \| \widetilde{S}(x) - \widehat{S}(x) \|_\infty &= \Big\| \sum_{j=-1}^{N+1} (\widetilde{C}_j - \widehat{C}_j) B_j(x) \Big\|_\infty \\ &\leq \Big| \sum_{j=-1}^{N+1} B_j(x) \Big| \, \| \widetilde{C}_j - \widehat{C}_j \|_\infty \\ &\leq Kh^2. \end{aligned} \tag{4.21}$$

Theorem 4.3. *Let $\widehat{U}(x)$ be the exact solution of Eq. (1.1)-(1.3) and $\widehat{S}(x)$ be the B-spline collocation approximation to $\widehat{U}(x)$, then the method has second order convergence and*

$$\|\widehat{U}(x) - \widehat{S}(x)\|_\infty \leq \omega h^2, \tag{4.22}$$

where $\omega = \lambda_0 L h^2 + K$ is finite constant.

Proof. From Theorem 4.2 we have

$$\| \widehat{U}(x) - \widetilde{S}(x) \|_\infty \leq \lambda_0 L h^4. \tag{4.23}$$

Therefore, from Eqs. (4.21) and (4.23) we get

$$\begin{aligned} \| \widehat{U}(x) - \widehat{S}(x) \|_\infty &\leq \| \widehat{U}(x) - \widetilde{S}(x) \|_\infty + \| \widetilde{S}(x) - \widehat{S}(x) \|_\infty \\ &\leq \lambda_0 L h^4 + Kh^2 \\ &= \omega h^2, \end{aligned} \tag{4.24}$$

where $\omega = \lambda_0 L h^2 + K$.

4.2. **The convergence in the temporal direction.** To get the estimate the convergence in temporal direction, we applied Taylor expansion to Eq. (3.4). Therefore,

$$\begin{aligned}&\left(u^n(x)+\Delta t u_t^n+\frac{\Delta t^2}{2!}u_{tt}^n+\cdots\right)\\&\qquad-(\mu\Delta t+2\Delta t^{\frac{3}{2}})\left(u_{xx}(x,t_n)+\Delta t u_{xxt}(x,t_n)+\frac{\Delta t^2}{2!}u_{xxtt}(x,t_n)+\cdots\right)\\&\qquad\qquad=u^n(x)+2\Delta t^{\frac{3}{2}}\sum_{j=1}^{n}b_j u_{xx}(x,t_{n-j+1}),\end{aligned}\tag{4.25}$$

and from rearranging Eq. (4.25), we get

$$\begin{aligned}&\Delta t\left(u_t^n-\mu u_{xx}(x,t_n)\right)\\&-2\Delta t^{\frac{3}{2}}\left(u_{xx}(x,t_n)+\sum_{j=1}^{n}b_j u_{xx}(x,t_{n-j+1})\right)+\frac{\Delta t^2}{2!}u_{tt}^n+\cdots=O(\Delta t).\end{aligned}\tag{4.26}$$

Finally, if we let $u(x,t)$ is the exact solution of the Eq. (1.1)-(1.3) and $u^N(x,t)$ is the numerical approximation to this solution by applying the numerical method, we will have

$$\parallel u(x,t)-u^N(x,t)\parallel\leq\rho(k+h^2),\tag{4.27}$$

where ρ is finite constant.

5. The stability of the method

By Von-Neumann stability, we prove the stability of the proposed method. Using Table 1 and Eq. (3.7), for any x_i, $i=0,1,\ldots,N$, we get

$$\beta C_{i-1}^{n+1}+\gamma C_i^{n+1}+\beta C_{i+1}^{n+1}=D_i,\tag{5.1}$$

where

$$D_i=(C_{i-1}^n+4C_i^n+C_{i+1}^n)+\frac{12\Delta t^{\frac{3}{2}}}{h^2}\sum_{l=1}^{n}\nu_l(C_{i-1}^{n-l+1}-2C_i^{n-l+1}+C_{i+1}^{n-l+1}).$$

We suppose that the solution of Eq. (3.7) is presented as

$$C_i^n=\xi^n e^{ki\eta h},$$

where ξ represents the time dependence of the solution, the exponential function shows the spatial dependence such that ηh represents the position along the grid and k is $\sqrt{-1}$. By substituting C_i^n into Eq. (5.1), we have

$$p\xi^{n+1}e^{ki\eta h}=q\xi^n e^{ki\eta h}+\frac{12\Delta t^{\frac{3}{2}}}{h^2}\sum_{l=1}^{n}\nu_l r\xi^{n-l+1}e^{ki\eta h},\tag{5.2}$$

where

$$\begin{cases} p = \beta(e^{k\eta h} + e^{-k\eta h}) + \gamma, \\ q = e^{k\eta h} + e^{-k\eta h} + 4, \\ r = e^{k\eta h} + e^{-k\eta h} - 2. \end{cases} \tag{5.3}$$

Furthermore, by substituting the values of γ and β from Eq. (3.10) into Eq. (5.3) and using $e^{k\eta h} + e^{-k\eta h} = 2cos\eta h$, Eq. (5.3) becomes as follows:

$$\begin{cases} p = 2(1 - \frac{6\Delta t}{h^2}(\mu + 2\Delta t^{\frac{1}{2}}))cos\eta h + 4(1 + \frac{3\Delta t}{h^2}(\mu + 2\Delta t^{\frac{1}{2}})), \\ q = 2cos\eta h + 4, \\ r = 2cos\eta h - 2. \end{cases} \tag{5.4}$$

Now, dividing both sides of Eq. (5.2) by $P\xi e^{ik\eta h}$ and after rearranging the results equation, we have

$$\xi^n - (\frac{q}{p} + \frac{12\Delta t^{\frac{3}{2}}}{h^2}\frac{r}{p}\nu_1)\xi^{n-1} - \frac{12\Delta t^{\frac{3}{2}}}{h^2}\frac{r}{p}\sum_{l=2}^{n}\nu_l\xi^{n-l} = 0. \tag{5.5}$$

We choose

$$\begin{cases} a_1 = -(\frac{q}{p} + \frac{12\Delta t^{\frac{3}{2}}}{h^2}\frac{r}{p}\nu_1), \\ a_l = -\frac{12\Delta t^{\frac{3}{2}}}{h^2}\frac{r}{p}\nu_l, \quad l = 2, \dots, n. \end{cases} \tag{5.6}$$

Using Eq. (5.6), Eq. (5.5) becomes as follows

$$\xi^n + a_1\xi^{n-1} + a_2\xi^{n-2} + \dots + a_{n-1}\xi + a_n = 0. \tag{5.7}$$

It is easy to see that in Eq. (5.4) $q > 0$ and $r > 0 (h \neq 0)$. If $\beta \geq 0$, we have $p > 0$. In the rest of the procedure, we also need to use the following theorem:

Theorem 5.1. *For all values of roots x_i of an arbitrary polynomial as*

$$P(x) = a_0x^n + a_1x^{n-1} + \dots + a_n,$$

we have

$$|\xi_i| \leq \max\{1, \sum_{j=1}^{n}|\frac{a_j}{a_0}|\}. \tag{5.8}$$

Proof. For the proof see [32].

For the stability, it should be proved that all roots ξ_i of the Eq. (5.7) satisfy $|\xi_i| \leq 1$. According to the Theorem 5.1, we have

$$\sum_{j=1}^{n}|\frac{a_j}{a_0}| = \frac{q + \frac{12\Delta t^{\frac{3}{2}}}{h^2}r\sum_{j=1}^{n}\nu_j}{p} = \frac{q + \frac{12\Delta t^{\frac{3}{2}}}{h^2}r[(n+1)^{\frac{1}{2}} - 1]}{p}, \tag{5.9}$$

where

$$\sum_{j=1}^{n}\nu_j = \sum_{j=1}^{n}[(j+1)^{\frac{1}{2}} - j^{\frac{1}{2}}] = (n+1)^{\frac{1}{2}} - 1. \tag{5.10}$$

Now, by using Theorem 5.1, from Eq. (5.9) we obtain

$$q + \frac{12\Delta t^{\frac{3}{2}}}{h^2} r[(n+1)^{\frac{1}{2}} - 1] < p. \tag{5.11}$$

By using Eq. (5.4), we get

$$cos\eta h < \frac{\gamma h^2 + 24\Delta t^{\frac{3}{2}}[(n+1)^{\frac{1}{2}} - 1]}{12\Delta t(3\Delta t^{\frac{1}{2}} + \mu)}. \tag{5.12}$$

Therefore, for conditional stability of the method, Eq. (5.12) must be valid.

6. Numerical experiments

In this section, we present some numerical results to demonstrate the efficiency and accuracy of the proposed method. All calculations are run with Matlab R2014a software on a Pentium PC Laptop with Core i3-350M Processor 2.26 GHz of CPU and 4G RAM. We have solved the problem based on a variety of temporal and spatial divisions. In numerical experiments, we have used the variables M and N for temporal and spatial divisions, respectively.
Furthermore, because of conditionally stability, we have applied the stability condition for temporal and spatial divisions and obtained the Errors of computations in L_2 and L_∞ error norms as follows

$$L_2 = \| u^{exact} - u^{num} \|_2 = \left(\sum_{i=0}^{N} | u_i^{exact} - u_i^{num} |^2 \right)^{1/2},$$
$$L_\infty = \| u^{exact} - u^{num} \|_\infty = \max_{0 \le i \le N} | u_i^{exact} - u_i^{num} |,$$

Example 6.1. *Consider the following weakly singular partial integro-differential equation in case of $\mu = 0$ in Eq. (1.1) [19, 34].*

$$u_t(x,t) = \int_0^t (t-s)^{-\frac{1}{2}} u_{xx}(x,s) ds, \quad x \in [0,1],\ t \ge 0,$$

with the boundary and initial conditions

$$u(0,t) = u(1,t) = 0, \quad 0 \le t \le 1,$$
$$u(x,0) = \sin(\pi x), \quad 0 \le x \le 1.$$

The exact solution is $u(x,t) = M(\pi^{\frac{5}{2}} t^{\frac{3}{2}}) \sin(\pi x)$, where M denotes the series $M(z) = \sum_0^{+\infty} (-1)^n \Gamma(\frac{3}{2}n+1)^{-1} z^n$.

We have solved the problem based on a variety of temporal and spatial divisions and applied the stability condition for temporal and spatial divisions. The errors of computations are obtained in L_2 and L_∞. The numerical and exact solutions are shown for a case of divisions as $M = 50$ and $N = 100$ in Figures 1 and 2.

In Figure 3, the error norm is shown for the case of $M = 100$ and $N = 50$.

In Table 2, the L_∞ error norm is shown for some cases of divisions of N and M as follows

Figure 1. The numerical solution for $M = 100$ and $N = 50$.

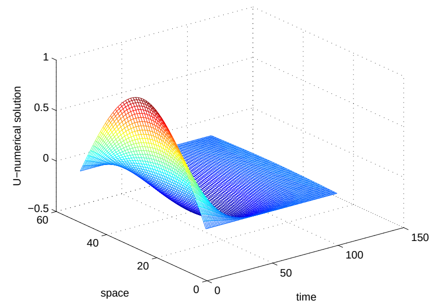


Figure 2. The exact solution for $M = 100$ and $N = 50$.

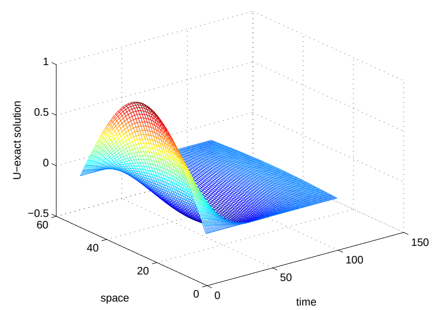
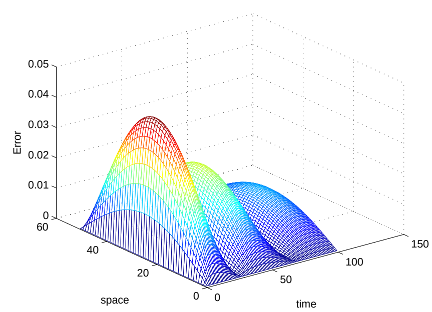


Figure 3. The error for $M = 100$ and $N = 50$.



7. Conclusion

In this article, the cubic B-splines collocation is implemented for computing the numerical solution of a PIDE with a weakly singular kernel successfully. The results which obtained in this research indicated that the discrete schemes are developed in this study and the convergence and stability of the method is confirmed by the analysis. The numerical results have indicated the accuracy of the method. The

TABLE 2. The L_∞ error norm for some cases of divisions of N and M.

	$N = 10$	$N = 25$	$N = 50$	$N = 100$
$M = 50$	0.0795	0.0768	0.0766	0.0765
$M = 100$	0.0475	0.0452	0.0450	0.0449
$M = 200$	0.0274	0.0252	0.0250	0.0249
$M = 400$	0.0159	0.0137	0.0133	0.0133
$M = 800$	0.0097	0.0073	0.0070	0.0070

results have also demonstrated that the proposed computational method is efficient for these type of problems.

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