Applied Mathematical Modelling 77 (2020) 829–847

Contents lists available at ScienceDirect

Applied Mathematical Modelling

journal homepage: www.elsevier.com/locate/apm

First order control variates algorithm for reliability analysis of engineering structures

Mansour Ghalehnovi [a], Mohsen Rashki [b,*], Ala Ameryan [a]

[a] Department of Civil Engineering, Ferdowsi University of Mashhad, Mashhad, Iran
[b] Department of Architectural Engineering, University of Sistan and Baluchestan, Zahedan, Iran

ARTICLE INFO

Article history:
Received 5 November 2018
Revised 17 July 2019
Accepted 23 July 2019
Available online 29 July 2019

Keywords:
First order reliability
Control variates
Importance sampling
Most probable point

ABSTRACT

This study introduces an efficient method for safety evaluation of the structures with small failure probability. The proposed approach reformulates basic failure probability formula based on control variates method and suggests to firstly substitute the nonlinear limit state function with a linear limit state function for taking the advantages of linear problems during reliability process. Reliability analysis of linear problems would be fully accurate and dimension insensitive. Subsequently, the control variates method imposes the effect of limit state function nonlinearity on the obtained first-order failure probability by using a very small sampling size simulation. The result is a new reliability formulation that is highly suitable for solving nonlinear and high dimension problems. The capabilities of the method examined by solving several benchmarks numerical/engineering problems involving non-normal random variables, complex/noisy limit state functions, and nonlinear high dimension problems. Compared with mainstream reliability methods, for all solved problems, it is demonstrated that the proposed approach presents accuracy close to Monte Carlo simulation while the required number of performance function valuation is close to first-order reliability method.

1. Introduction

In recent years, accurate safety evaluation of existing structures and engineering systems become a topic with great importance [1]. A number of new methods have been suggested to estimate the structural reliability and the application of these approaches in safety factor calibration and design optimization has been highlighted in recent works of literature [2,3]. The structural reliability analysis approaches can be arranged into five types as follows: (1) Based on Most Probable Point (MPP) methods; (2) expansion methods; (3) approximate integration methods; (4) simulations methods; (5) surrogated based approximate methods. Among them, simulation methods are the most accurate that are suitably developed for analysis and design of systems [4,5]. Considering Monte Carlo Simulation (MCS) and Importance Sampling (IS) as the mainstream simulation methods in the last century, some other significant advances in simulation approaches may find in the later. New approaches such as Subset Simulation (SS) [6–8], Line Sampling (LS) [9], asymptotic simulation [10], and weighted average simulation [11–13] have been developed to reduce the computational cost of MCS and increase the robustness of IS. However, it should be mentioned that the applicability range of these methods is fewer than MCS. They involve parameters that should select properly. Otherwise, they may compute improper results. Furthermore, some of them are only suitable

* Corresponding author.
E-mail addresses: ghalehnovi@um.ac.ir (M. Ghalehnovi), mrashki@eng.ucb.ac.ir (M. Rashki), ala.ameryan@mail.um.ac.ir (A. Ameryan).

https://doi.org/10.1016/j.apm.2019.07.049

for small size problems and generally, the computational cost of these simulation methods is often huge for many complex reliability problems (i.e., nonlinear problems that their performance valuation requires finite element analysis).

As well as simulation methods, powerful computer-based prediction methods, surrogates, and Meta-models also developed and adapted for application in reliability analysis. Artificial Neural Network (ANN), Anfis, Kriging [14–16] and Support Vector Machine (SVM) [15] may consider as the newly developed computer-based prediction approaches that often present more efficiency compared to the formerly developed classical Polynomial Response Surface Method (PRSM) [17–19]. Proper use of these approaches highly improves the efficiency of reliability process. However, these methods are often facing difficulties for nonlinear and high dimension reliability problems. Moreover, the application of these approaches increases the operating parameters of the reliability process and employing them improperly may lead to erroneous results. Therefore, when the direct application of a reliability method is possible, the employment of such predictive methods in reliability process would be a risk.

By considering a few assumptions, Cornell in 1969 and accordingly Hasofer and Lind in 1974 have introduced an efficient method that still competes with the newly developed computer-based methods for reliability analysis [20,21]. The proposed approach that is named “First Order Reliability Method” (FORM) does not require complex calculations and, it computes reliability results without the usage of computers. Nonetheless, the application of computer-based methods in the FORM has been developed [16,22,23].

FORM maps the problem from original design space to standard normal space and assumes that the Limit State Function (LSF) is linear. The method then computes the so-called reliability index β as the minimum distance of LSF to the origin. In the reliability community, the obtained reliability index is known as Hasofer-Lind index (β_{HL}), and the point on LSF with the minimum distance from the origin is referred to as design point. Subsequently, FORM approximates the failure probability as $P_f = \Phi(-\beta)$, where $\Phi(.)$ is the Cumulative Distribution Function (CDF) of the standard normal random variable. When the LSF is linear, the proposed procedure computes the accurate failure probability with the minimum computational cost that is the major advantages of the method. This approach works well for many engineering problems, but it is not accurate for the problems with highly nonlinear LSFs. Some other approaches such as the hybrid chaos control [24], directional stability transformation method of chaos control for first-order reliability analysis [25] and target performance approach [26] have been proposed to reduce the number of function evaluations and also to overcome the convergence difficulties.

Second-order reliability methods (known as the main attempt to reduce the error of FORM) [27], newly developed First Order Saddle Point Approximation (FOSPA) [28] and Dimension Reduction Method (DRM) have also been developed to compute the reliability results with suitable precision [29,30]. These approaches may show more efficiency compared to the simulation methods, but the inaccuracy of estimation for highly nonlinear problems affected their application in many engineering problems.

The aim of this study is to provide a solution to solve the reliability problems with small failure probability and high dimension nonlinear LSF with the same accuracy as MCS and the efficiency close to FORM.

2. The basic idea of the proposed method

This study firstly suggests substituting a nonlinear LSF with a linear LSF, to take advantages of reliability analysis of linear problems. For a linear problem, the predicted failure probability would be exact. Furthermore, the number of random variables and also the value of failure probability do not affect the accuracy of the results. To consider the effect of the LSF nonlinearity on the obtained first-order failure probability, Control Variates Method (CVM) is introduced for application in structural reliability. CVM is a variance reduction technique that has been primarily presented for the approximation of integrals [31]. By assuming the index function of linearized LSF as the control variates of original index function and performing a small sampling size simulation, this study shows that CVM has the capability of imposing the LSF nonlinearity effects on the first-order failure probability to approximate the correct failure probability.

The idea of proposed First Order Control variates Method (FOCM) becomes more attractive knowing that FORM also uses a linear LSF in reliability process. It provides the opportunity of linking FORM to the proposed FOCM formulation to take the advantages of both FORM and simulations in a reliability process. Consequently, the mathematically exact first-order failure probability of the problem may compute by using Hasofer-Lind index and accordingly a small sample size simulation imposes the LSF nonlinearity effect to the proposed first-order probability.

3. FOCM procedure

In the reliability theory, the basic failure probability formula is expressed as follows:

$$P_f = \int_{\mathbb{X}} \mathbb{I}_{g\le 0}(\mathbf{x}) f_X(\mathbf{x}) \mathrm{d}\mathrm{x}, \tag{1}$$

where f_X is joint Probability Density Function (PDF) of random variables X and $\mathbb{I}_{g\le 0}$ defined as the index function as follows:

$$\mathbb{I}_{g\le 0}\left(\mathbf{x}^{(\mathrm{j})}\right) = \begin{cases} 0 \text{ if } g\left(\mathbf{x}^{(\mathrm{j})}\right) > 0 \\ 1 \text{ if } g\left(\mathbf{x}^{(\mathrm{j})}\right) \le 0 \end{cases}. \tag{2}$$

Considering g^L as the linearized LSF, CVM may use to rewrite the probability integral as follows:

$$P_f = \int_{\mathbb{X}} \mathbb{I}_{g^L\le 0}(\mathbf{x}) f_X(\mathbf{x})\mathrm{dx} + \int_{\mathbb{X}} \left(\mathbb{I}_{g\le 0}(\mathbf{x}) - \mathbb{I}_{g^L\le 0}(\mathbf{x})\right) f_X(\mathbf{x})\mathrm{dx}, \tag{3}$$

where $\mathbb{I}_{g^L\le 0}$ is the index function of the linearize function g^L:

$$\mathbb{I}_{g^L\le 0}\left(\mathbf{x}^{(\mathbf{j})}\right) = \begin{cases} 0 \text{ if } g^L\left(\mathbf{x}^{(\mathbf{j})}\right) > 0 \\ 1 \text{ if } g^L\left(\mathbf{x}^{(\mathbf{j})}\right) \le 0 \end{cases}. \tag{4}$$

In the proposed basic equation, the first term estimates the failure probability resulted from linearizing LSF, and the second integral term removes the errors raised from LSF linearization. It should be noticed that the proposed formulation accurately estimates the failure probability and is not involving any simplification and assumptions.

Two efficient frameworks are presented in this study for solving Eq. (3). Firstly, the sampling-based FOCM is discussed in the following subsections. Then, in the following section, the perceptions of Hasofer-Lind FORM are linked to the proposed FOCM formulation to take the advantages of both FORM and simulations in the reliability process.

3.1. Sampling process

According to the proposed approach, Eq. (3) may easily express as the following regressed form:

$$\begin{aligned} P_f &= \lambda.\int_{\mathbb{X}} \mathbb{I}_{g^L\le 0}(\mathbf{x}) f_X(\mathbf{x})\mathrm{dx} + \int_{\mathbb{X}} \left(\mathbb{I}_{g\le 0}(\mathbf{x}) - \lambda.\mathbb{I}_{g^L\le 0}(\mathbf{x})\right) f_X(\mathbf{x})\mathrm{dx}, \\ &\approx \lambda.\mathbb{E}_f\left(\mathbb{I}_{g^L\le 0}(\mathbf{x})\right) + \int_{\mathbb{X}} \left(\mathbb{I}_{g\le 0}(\mathbf{x}) - \lambda.\mathbb{I}_{g^L\le 0}(\mathbf{x})\right) f_X(\mathbf{x})\mathrm{dx}, \\ &= \lambda.P_f|^L + \int_{\mathbb{X}} \left(\mathbb{I}_{g\le 0}(\mathbf{x}) - \lambda.\mathbb{I}_{g^L\le 0}(\mathbf{x})\right) f_X(\mathbf{x})\mathrm{dx}, \end{aligned} \tag{5}$$

where $\mathbb{E}_f(\mathbb{I}_{g^L\le 0}(\mathbf{x})) = P_f|^L$ is representative of the failure probability resulted from linearizing g, and λ is the regression constant.

To simplify the proposed failure probability formulation, one may regulate λ such that the remained integral term of Eq. (5) becomes zero. To meet this aim, Eq. (5) reads:

$$P_f = \lambda.P_f|^L, \tag{6}$$

$$\int_{\mathbb{X}} \left(\mathbb{I}_{g\le 0}(\mathbf{x}) - \lambda.\mathbb{I}_{g^L\le 0}(\mathbf{x})\right) f_X(\mathbf{x})\mathrm{dx} = 0. \tag{7}$$

Estimation of the first-order failure probability $P_f|^L$ is discussed in the next subsection. To obtain λ based on Eq. (7), one may break the integral term and compute λ using the following formulation:

$$\int_{\mathbb{X}} \mathbb{I}_{g\le 0}(\mathbf{x}) f_X(\mathbf{x})\mathrm{dx} - \lambda \int_{\mathbb{X}} \mathbb{I}_{g^L\le 0}(\mathbf{u}) f_X(\mathbf{x})\mathrm{dx} = 0, \tag{8}$$

$$\lambda = \frac{\int_{\mathbb{X}} \mathbb{I}_{g\le 0}(\mathbf{x}) f_X(\mathbf{x})\mathrm{dx}}{\int_{\mathbb{X}} \mathbb{I}_{g^L\le 0}(\mathbf{x}) f_X(\mathbf{x})\mathrm{dx}}, \tag{9}$$

where sampling based on f and performing simulation would lead to estimate λ as follows:

$$\lambda \approx \frac{\mathbb{E}_{\mathrm{f}}\left(\mathbb{I}_{g\le 0}(\mathbf{x})\right)}{\mathbb{E}_{\mathrm{f}}\left(\mathbb{I}_{g^L\le 0}(\mathbf{x})\right)}. \tag{10}$$

However, measuring λ in the current form requires sampling based on f(x), and that is the main challenge of MCS for problems with small failure probability. For efficiently computing λ, it is proposed to perform a simulation by introducing sampling PDF k involving specifications of f but having a higher standard deviation $[\mu_k = \mu_f, \sigma_k = n.\sigma_f]$. Accordingly, employing k as a new sampling PDF would lead to the following results:

$$\begin{aligned} \lambda &= \frac{\int_{\mathbb{X}} \mathbb{I}_{g\le 0}(\mathbf{x}) f_X(\mathbf{x})\mathrm{dx}}{\int_{\mathbb{X}} \mathbb{I}_{g^L\le 0}(\mathbf{x}) f_X(\mathbf{x})\mathrm{dx}} \\ &= \frac{\int_{\mathbb{X}} \mathbb{I}_{g\le 0}(\mathbf{x}) f_X(\mathbf{x}) \frac{k_X(\mathbf{x})}{k_X(\mathbf{x})}\mathrm{dx}}{\int_{\mathbb{X}} \mathbb{I}_{g^L\le 0}(\mathbf{x}) f_X(\mathbf{x}) \frac{k_X(\mathbf{x})}{k_X}\mathrm{dx}} \approx \frac{\mathbb{E}_k\left(\mathbb{I}_{g\le 0}(\mathbf{x}) \frac{f_X(\mathbf{x})}{k_X(\mathbf{x})}\right)}{\mathbb{E}_k\left(\mathbb{I}_{g^L\le 0}(\mathbf{x}) \frac{f_X(\mathbf{x})}{k_X(\mathbf{x})}\right)}, \end{aligned} \tag{11}$$

In which using PDF k instead of f would lead to estimating λ requiring small sample size.

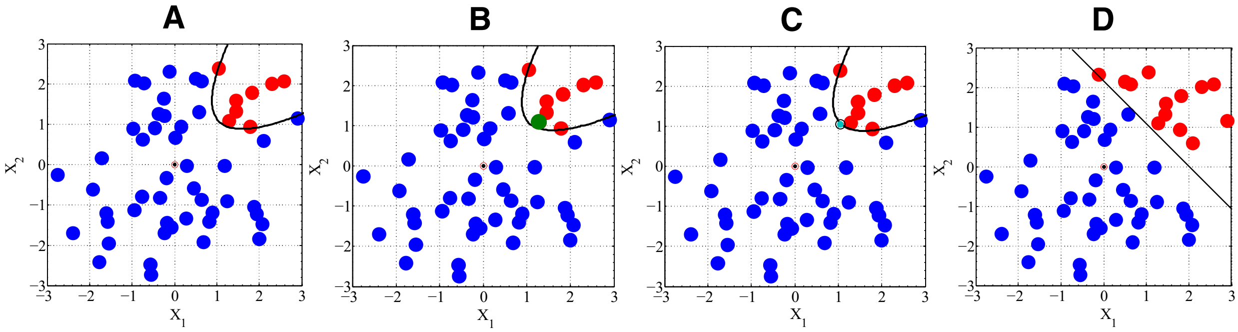

Fig. 1. Estimation of correction factor λ by using simulation: (A) generation of samples based on *k(x)* and separating failed/safe samples, (B) distinguishing MPP, (C) updating MPP and (D) linearizing LSF around MPP.

Remark 1. The simulation shows that the increase in the dimension of the problem would lead to a decrease in the correlation of the index functions and weighting term $\frac{f_X(\mathbf{x})}{k_X(\mathbf{x})}$ of Eq. (11). In such cases, by accepting very small errors and assuming two terms to be uncorrelated, Eq. (11) may rewrite using the mathematics of expectation:

$$\lambda_1 \approx \frac{\mathbb{E}_k\left(\mathbb{I}_{g\le 0}(\mathbf{x})\frac{f_X(\mathbf{x})}{k_X(\mathbf{x})}\right)}{\mathbb{E}_{\mathrm{ik}}\left(\mathbb{I}_{g^L\le 0}(\mathbf{x})\frac{f_X(\mathbf{x})}{k_X(\mathbf{x})}\right)} = \frac{\mathbb{E}_k\left(\mathbb{I}_{g\le 0}(\mathbf{x})\right)\mathbb{E}_k\left(\frac{f_X(\mathbf{x})}{k_X(\mathbf{x})}\right)}{\mathbb{E}_k\left(\mathbb{I}_{g^L\le 0}(\mathbf{x})\right)\mathbb{E}_k\left(\frac{f_X(\mathbf{x})}{k_X(\mathbf{x})}\right)} = \frac{\mathbb{E}_k\left(\mathbb{I}_{g\le 0}(\mathbf{x})\right)}{\mathbb{E}_k\left(\mathbb{I}_{g^L\le 0}(\mathbf{x})\right)}. \tag{12}$$

The proposed implementation means that in high dimensions, λ is almost insensitive to the standard deviation of sampling PDF (See Eq. (10)). Therefore, failure probability may accurately estimate using the following equation for small/moderate dimension problems:

$$P_f = P_f|^L.\frac{\mathbb{E}_k\left(\mathbb{I}_{g\le 0}(\mathbf{x})\frac{f_X(\mathbf{x})}{k_X(\mathbf{x})}\right)}{\mathbb{E}_k\left(\mathbb{I}_{g^L\le 0}(\mathbf{x})\frac{f_X(\mathbf{x})}{k_X(\mathbf{x})}\right)}, \tag{13}$$

And a high dimension problem may solve by the following equation:

$$P_f = P_f|^L.\frac{\mathbb{E}_k\left(\mathbb{I}_{g\le 0}(\mathbf{x})\right)}{\mathbb{E}_k\left(\mathbb{I}_{g^L\le 0}(\mathbf{x})\right)}. \tag{14}$$

3.2. First-order approximation of LSF and computing index functions

Estimation of index function $\mathbb{I}_{g^L\le 0}$ in the proposed formulation requires employing the first-order approximation of LSF. To linearize LSF, one may easily use the results of a simulation for determining Most Probable Point of failure (MPP) and consequently employ first-orders Taylor expansion around MPP. To obtain more accurate results, this study recommends considering the MPP of the simulation as the starting search point and improving the accuracy of MPP:

$$\begin{aligned}&\mathrm{Max}\left\| f_{(\mathrm{X})}\right\|,\\ &\text{Subject to}:\ g \le 0.\end{aligned} \tag{15}$$

As soon as the MPP is determined, a linear approximation for the original function $g(\mathbf{x})=g(x_1,x_2,\dots,x_n)$ may obtain by using the first-order term of Taylor polynomial around MPP $\mathbf{x}^*$:

$$g^L = g^L(\mathbf{x}) = g(\mathbf{x}^*) + \mathrm{D}g(\mathbf{x}^*)(\mathbf{x}-\mathbf{x}^*), \tag{16}$$

where $\mathrm{D}g(\mathbf{x}^*)$ is the matrix of partial derivatives. Accordingly, the index function $\mathbb{I}_{g^L\le 0}(\mathbf{x})$ (and subsequently correction factor λ) may easily compute.

For illustrative purpose, Fig. 1 shows the four required steps for computing λ. By considering $\lambda_1 = \frac{\mathbb{E}_k(\mathbb{I}_{g\le 0}(\mathbf{x}))}{\mathbb{E}_k(\mathbb{I}_{g^L\le 0}(\mathbf{x}))}$ as the correction factor, it may simply compute as $\lambda_1 = \frac{8}{13}$ for regulating the failure probability (as previously stated, for low dimension problems, $\lambda = \frac{\mathbb{E}_k(\mathbb{I}_{g\le 0}(\mathbf{x})\frac{f_X(\mathbf{x})}{k_X(\mathbf{x})})}{\mathbb{E}_k(\mathbb{I}_{g^L\le 0}(\mathbf{x})\frac{f_X(\mathbf{x})}{k_X(\mathbf{x})})}$ should be used to obtain accurate results).

3.3. First-order failure probability approximation

Estimation of the first integral term of Eq. (3) by MCS requires sampling based on $f(\mathrm{x})$ as follows:

$$P_f|^L_{MCS} = \int_{\mathbb{X}} \mathbb{I}_{g^L\le 0}(\mathbf{x}) f_X(\mathbf{x})\mathrm{d}\mathrm{x} = \mathbb{E}_f\left(\mathbb{I}_{g^L\le 0}(\mathbf{x})\right). \tag{17}$$

However it does not need additional performance function evaluation, but in the current form, it requires huge computational time for small values of $P_f|^L$.

To approximate the proposed integral, especially for high dimension problems, this study offers to use an n-Sigma approach.

Suppose a one-dimension reliability problem involving a normal random variable and linear LSF. According to the basic first-order perceptions proposed by Cornell, the reliability index and failure probability of the problem may accurately compute as:

$$\beta = \frac{\mu}{\sigma}, \tag{18}$$

$$P_f = \Phi(-\beta), \tag{19}$$

where μ and σ are the mean value and standard deviation of performance function, respectively. Now, suppose that the problem is mapped to a so-called n-sigma space with the standard deviation of $\sigma^{(n)} = n.\sigma$ (similar to that was performed in simulation using $\sigma_k = n.\sigma_f$). Since the mean value has no change, the relation between the original and n-sigma reliability index (β and $\beta^{(n)}$, respectively) may express as follows by using Eq. (18):

$$\begin{cases} \mu = \beta.\sigma \\ \mu = \beta^{(n)}.\sigma^{(n)} \end{cases} \xrightarrow{\beta.\sigma=\beta^{(n)}.\sigma^{(n)}} \beta = \frac{\sigma^{(n)}}{\sigma}\beta^{(n)} = \frac{n.\sigma}{\sigma}\beta^{(n)} = n.\beta^{(n)}, \tag{20}$$

And the corresponding failure probability may present as $P_f = \Phi(-\beta) = \Phi(-n.\beta^{(n)})$.

The proposed implementation may *easily* extend to high dimension linear LSFs involving M random variables by linearity properties.

According to the proposed implementations and for the case of FOCM, the reliability index $\beta^{(n)}$ may be simply approximated by using the samples generated by simulation and linearized performance function g^L as:

$$\beta^{(n)} = -\Phi^{-1}\left(\mathbb{E}_k\left(\mathbb{I}_{g^L\le 0}(\mathbf{x})\right)\right). \tag{21}$$

Consequently, since $P_f|^L = \Phi(-\beta)$, an estimation for first-order failure probability may present as follows:

$$P_f|^L_{n-Sigma} = \Phi\left(n.\Phi^{-1}\left(\mathbb{E}_k\left(\mathbb{I}_{g^L\le 0}(\mathbf{x})\right)\right)\right). \tag{22}$$

The main advantage of the proposed approach is that the proposed estimation does not require evaluating the original performance function g and the accuracy of this procedure would be insensitive to the dimension of the problem.

3.4. Original FOCM formulation

Once the first-order failure probability and correction factor being estimated by simulation, FOCM provides an estimation of failure probability as follows:

$$P_{f\,FOCM} = \lambda.P_f|^L = \lambda.\Phi\left(n.\Phi^{-1}\left(\mathbb{E}_k\left(\mathbb{I}_{g^L\le 0}(\mathbf{x})\right)\right)\right). \tag{23}$$

In the proposed FOCM, the number of performance function evaluation would be equal to the required samples for performing simulation in addition to the required function valuation for updating MPP.

The proposed procedure should be performed in normal space. For the problems involved in non-normal random variables, a sample may be generated in standard normal space (U) and then be transferred from U- Space into the original X-Space. By this approach, the index function corresponds to each sample may simply compute in original space.

4. Linking FORM to FOCM

In the proposed approach, the first-order failure probability may compute by sampling noting that the estimation does not require any additional g call. However, thanks to suitable FORM perceptions introduced by Hasofer and Lind, the exact value of first-order failure probability may compute by Hasofer-Lind reliability index. Rewriting the basic failure probability as Eq. (3) provides the opportunity to determine the first-order failure probability accurately as follows:

$$\begin{aligned} P_f|^L = P_f|^L_{HL} &= \int_{\mathbb{X}} \mathbb{I}_{g^L\le 0}(\mathbf{x}) f_X(\mathbf{x})\mathrm{d}x \\ &= \Phi(-\beta_{HL}), \end{aligned} \tag{24}$$

where β_{HL} may compute by employing the following familiar FORM formulation:

$$\begin{aligned} &Min\ \beta_{HL} = \|u^*\|, \\ &\text{Subject to} : g = 0. \end{aligned} \tag{25}$$

Accordingly, the failure probability of the problems with small/moderate dimensions may be approximated as:

$$P_{f\,FOCM} \cong \Phi(-\beta_{HL}).\frac{\mathbb{E}_k\left(\mathbb{I}_{g\le 0}(\mathbf{x})\frac{f_X(\mathbf{x})}{k_X(\mathbf{x})}\right)}{\mathbb{E}_k\left(\mathbb{I}_{g^L\le 0}(\mathbf{x})\frac{f_X(\mathbf{x})}{k_X(\mathbf{x})}\right)} \tag{26}$$

And the following equation may be used for high dimension problems:

$$P_{f\,FOCM} \cong \Phi(-\beta_{HL}).\frac{\mathbb{E}_k\left(\mathbb{I}_{g\le 0}(\mathbf{x})\right)}{\mathbb{E}_k\left(\mathbb{I}_{g^L\le 0}(\mathbf{x})\right)}. \tag{27}$$

The proposed approach would show to be highly efficient and accurate for reliability analysis of structures.

4.1. Variance of estimation

Linking FORM and FOCM not only increases the accuracy of results (by precise approximation of first-order failure probability) but also reduces the variance of original FOCM. The reason is that in contrast with the failure probability estimated by simulation ($P_f|^L = \Phi(n.\Phi^{-1}(\mathbb{E}_k(\mathbb{I}_{g^L\le 0}(\mathbf{x}))))$), employing $P_f|^L = \Phi(-\beta_{HL})$ has no variance. Therefore, once the failure probability is approximated by the proposed approach, the variance of estimation may present as:

$$Var\left(P_{f\,FOCM}\right) = \Phi(-\beta_{HL})^2.Var(\lambda). \tag{28}$$

Supposing $\mathbb{E}_k(\mathbb{I}_{g\le 0}(\mathbf{x}))$ and $\mathbb{E}_k(\mathbb{I}_{g^L\le 0}(\mathbf{x}))$ as a and b, respectively, the variance of correction factor $Var(\lambda)$ may be calculated as:

$$Var(\lambda) = Var\left(\frac{a}{b}\right) = \frac{(\mu_a)^2}{(\mu_b)^2}\left[\frac{\sigma_a^2}{(\mu_a)^2} + \frac{\sigma_b^2}{(\mu_b)^2} - 2.\frac{Cov(a,b)}{\mu_a.\mu_b}\right], \tag{29}$$

where $\sigma_a^2 = \frac{Var(a)}{N}$ and $\sigma_b^2 = \frac{Var(b)}{N}$.

Accordingly, the coefficient of variation of estimation ($\delta_{PF_{FOCM}}$) may be approximated as:

$$\begin{aligned}\delta_{PF_{FOCM}} &= \frac{\sqrt{Var\left(P_f\right)}}{\mathbb{E}\left(P_f\right)} = \frac{\Phi(-\beta_{HL})\frac{(\mu_a)}{(\mu_b)}\sqrt{\left[\frac{\sigma_a^2}{(\mu_a)^2} + \frac{\sigma_b^2}{(\mu_b)^2} - 2.\frac{Cov(a,b)}{\mu_a.\mu_b}\right]}}{\mathbb{E}\left(\Phi(-\beta_{HL})\frac{(\mu_a)}{(\mu_b)}\right)} \\ &= \sqrt{\left[\frac{\sigma_a^2}{(\mu_a)^2} + \frac{\sigma_b^2}{(\mu_b)^2} - 2.\frac{Cov(a,b)}{\mu_a.\mu_b}\right]} \\ &= \sqrt{\left[\delta_a^2 + \delta_b^2 - 2.\frac{Cov(a,b)}{\mu_a.\mu_b}\right]} < \sqrt{\delta_a^2 + \delta_b^2}.\end{aligned} \tag{30}$$

4.2. Implementations procedure

For a given reliability problem, the steps to perform the FOCM is shown in Fig. 2. Herein, the minimum required failed samples for determining initial MPP is set to 15 to ensure that determined MPP is a good approximation of original MPP. For high dimension problems, $\lambda = \frac{\mathbb{E}_k(\mathbb{I}_{g\le 0}(\boldsymbol{u}))}{\mathbb{E}_k(\mathbb{I}_{g^L\le 0}(\boldsymbol{u}))}$ should be used instead of $\lambda = \frac{\mathbb{E}_k(\mathbb{I}_{g\le 0}(\boldsymbol{u})\frac{f_U(\boldsymbol{u})}{k_U(\boldsymbol{u})})}{\mathbb{E}_k(\mathbb{I}_{g^L\le 0}(\boldsymbol{u})\frac{f_U(\boldsymbol{u})}{k_U(\boldsymbol{u})})}$.

5. Illustrative examples

5.1. Numerical/engineering reliability test problems

Case 1. Each of current reliability approaches is suitable for just some groups of problems. So from the perspective of accuracy and time, to compare the efficiency of some famous methods with the proposed one, a set of widely used reliability test problems representing a broad range of possible limit states that can occur in practice are gathered from the literature and tabulated in Appendix. The engineering states for some of the proposed problems and their respective LSF are illustrated in Fig. 3. The effect of nonlinearity properties of engineering systems has been considered in the LSF of these problems. Problems are solved by common reliability methods. The results presented in Table 1, demonstrate the inaccuracy of FORM and LS for the case of problems with nonlinear/non-normal LSFs. However, these methods presented high efficiency for solving linear and moderate-nonlinear LSF. Because of the huge computational cost of MCS and SS compared to FORM, application of these approaches would be insensible for reliability analysis.

1. Introduce $k(\boldsymbol{u})$ as the sampling PDF by n times increase in the standard deviation of random variables, $\sigma_k = n.\sigma_f$ (n may set to 3, 4 or 5).
2. Generate MCS samples based on k(u) and evaluate g for the generated samples.
3. If the number of failed samples is less than 15, go to step 2. Else, go to step 4.
4. Compute the design point u^*, and determine $\beta_{HL} = \|u^*\|$.
5. Linearize the LSF around the design point.
6. Estimate index functions and compute $\lambda = \dfrac{\mathbb{E}_k\left(\mathbb{I}_{g\le 0}(u)\frac{f_X(u)}{k_X(u)}\right)}{\mathbb{E}_k\left(\mathbb{I}_{g^L\le 0}(u)\frac{f_X(u)}{k_X(u)}\right)}$.
7. Approximate the failure probability as $P_f = \lambda.\Phi(-\beta_{HL})$.
8. Compute δ_{PF} using Eq. (34). If δ_{PF} meet the convergence criteria, stop. Else:

   Generate a new sample $\boldsymbol{u}^{New}$ based on k and evaluate g:

   If the sample is in the failure domain and $\|\boldsymbol{u}^{New}\| < \beta_{HL}$, go to step 4.

   Else: go to step 6.

Fig. 2. The proposed FOCM steps.

In contrast with FORM, IS, LS and SS, the proposed FOCM computed the accurate result for all presented LSFs with the number of g call close to FORM.

Besides the efficiency, FOCM presents another superiority to SS. Similar to MCS, the accuracy level of FOCM is in the control of the researcher. According to the proposed approach, the researcher may perform sampling until a target coefficient of variation (δ_{PF}) being satisfied. However, the variance of SS may calculate after performing the simulation. Therefore, there is no control over the accuracy of SS.

If δ_{PF} is large, a new SS with a new initial sample size should perform for ensuring the accuracy of the result. Therefore, the results of former SS sampling would be useless.

Case 2. The effect of the statistical parameters of random variables on the performance of reliability methods is examined by analysis the LSF# 12 as follows:

$$g(X) = X_1^3 + 2X_1^2X_2 + X_1^3 - 18, \tag{31}$$

where, X_1 and X_2 are normal variables with random parameters with the mean and standard deviations of [10.0, 9.9] and [5.0, 5.0], respectively.

The standard deviation of the first random variable has changed in the interval of [0.0, 11.0] and the obtained result is illustrated in Fig. 4. The result shows that FORM and LS present instability in results when the standard deviation of the first example is in the range of [3.8, 10.0]. However, the proposed FOCM, SS and IS computed results with the same accuracy as MCS.

Case 3. Reliability analysis of the performance function #1 with an island failure region is examined (See Appendix and Fig. 3, Case 1):

$$g(X) = 7 - \left(8.\exp(-((X_1+1)^2 + (X_2+1)^2)) + 2.\exp(-((X_1-5)^2 + (X_2-4)^2)) + 1 + (X_1X_2)/10\right) \tag{32}$$

For this problem, the mean value of the first random variable is changed in the interval of [0.0, 4.0] and the corresponded reliability results are presented in Fig. 5. Results show that both FORM and SS present sensitivity to the mean value of random variables. As illustrated in Figs. 3 (Case 1) and 6, the nonlinearity of LSF would led to improper results for the

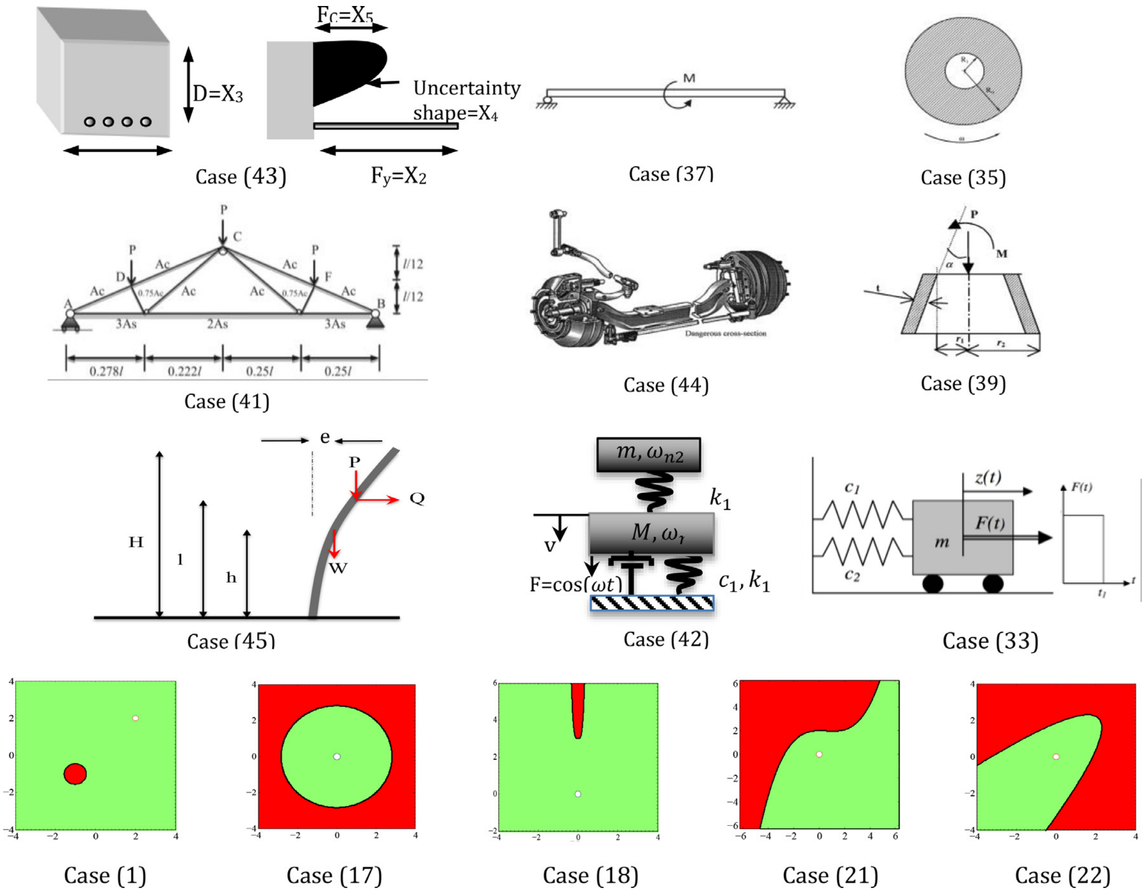

Fig. 3. Engineering states and LSF representations for some of the problems in Section 5.1.

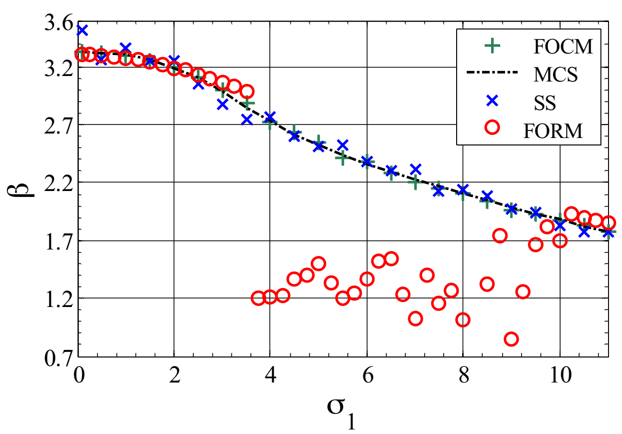

Fig. 4. Reliability indices with respect to various standard deviations of the variable x_1.

FORM/LS and, the geometry of performance function also misconducts SS samples to find proper failure region when the mean value of random variable X_1 is higher than 2.5. For this problem, SS requires a sample size close to crude MCS for properly solving the problem. In contrast with SS and FORM that present sensitivity to the values of statistical parameters of PDFs, the proposed FOCM -similar to the former example- presents efficiency for the problem.

Case 4. This problem investigates the reliability analysis of road-holding safety of a passive vehicle suspension that was appeared as the probabilistic constraint of a design optimization problem in Ref. [55]. The mechanical system is modeled

Table 1
Results of benchmark reliability problems.

No.	MCS		SS		IS		LS		FORM		FOCM	
	β	g call	β	g call	β	g call	β	g call	β	g call	β	g call
1	4.01	10^7	3.98	23,891	3.91	9.0×10^3	Failed	500	Failed	28	4.02	23+1796
2	2.704	10^5	2.69	2139	2.66	8.0×10^3	2.72	100	2.688	134	2.69	134+62
3	2.24	10^4	2.21	3514	2.18	3.0×10^3	2.24	150	2.348	85	2.27	85+69
4	1.39	10^4	1.41	1295	1.32	7.0×10^3	Failed	50	Failed	50	1.40	50+671
5	3.99	10^7	21.38	305,628	4.06	4.0×10^3	Failed	460	79.17	45	4.01	529
6	2.07	10^4	2.12	1849	2.02	2.0×10^3	2.16	160	2.226	24	2.09	24+28
7	2.76	10^5	2.73	4017	2.81	4.0×10^3	2.75	110	2.50	35	2.76	35+1047
8	2.53	10^5	2.58	4319	2.58	8.0×10^3	2.53	120	2.226	33	2.53	33+395
9	2.52	10^6	2.45	4611	2.56	2.0×10^3	2.56	90	3.41	16	2.52	16+248
10	3.73	10^7	3.71	10,039	3.67	3.0×10^3	3.74	100	4.6	67	3.94	67+1984
11	2.23	10^4	2.21	5081	2.19	7.0×10^3	2.24	80	1.90	27	2.27	27+296
12	2.52	10^4	2.55	7237	2.55	1.0×10^4	2.51	85	2.30	30	2.55	30+314
13	2.90	10^5	2.85	7319	2.88	8.0×10^3	2.78	160	2.36	42	2.96	42+796
14	5.14	3×10^8	5.31	13,841	5.14	7.0×10^3	5.24	100	5.428	200	5.16	200+1052
15	5.22	10^9	5.28	23,519	5.35	9.0×10^3	5.21	100	5.21	21	5.21	21+29
16	2.183	10^4	2.16	7633	2.16	1.1×10^4	2.17	110	2.19	44	2.19	44+36
17	2.28	10^4	2.27	7353	2.30	6.5×10^3	3.82	115	3.0	19	2.28	19+407
18	3.57	10^6	3.55	13,811	3.62	8.8×10^3	1.74	410	3.0	38	3.54	38+671
19	2.13	10^5	2.12	1846	2.30	1.4×10^4	Failed	90	Failed	27	2.11	89+185
20	1.44	10^5	1.43	3785	1.43	1.3×10^4	2.89	190	0.48	18	1.42	18+2017
21	1.82	10^4	1.91	2918	1.77	1.0×10^3	1.91	100	2.0	46	1.80	46+305
22	1.25	10^4	1.32	2611	1.31	2.2×10^3	1.43	300	3.0	9	1.29	9+527
23	1.71	10^5	1.73	5184	1.70	8.0×10^3	3.82	500	2.5	21	1.74	21+1174
24	3.15	10^5	3.19	8647	3.20	5.0×10^3	3.16	100	3.22	26	3.13	26+93
25	4.15	10^7	4.17	15,730	4.11	1.1×10^4	4.12	110	4.03	15	4.11	5+316
26	3.42	10^7	3.35	11,984	3.36	9.0×10^3	3.41	100	3.0	36	3.44	36+3613
27	2.686	10^5	2.72	7414	2.67	7.0×10^3	2.68	90	2.71	27	2.70	27+124
28	3.381	10^6	3.36	11,883	3.38	8.4×10^3	3.39	120	3.35	27	3.38	27+142
29	3.711	10^6	3.71	14,102	3.72	1.8×10^4	3.74	110	3.705	53	3.71	53+682
30	4.44	10^7	4.46	26,971	4.53	9.0×10^3	4.51	120	4.32	18	4.46	18+894
31	0.849	10^3	0.891	1420	0.86	1.0×10^4	1.32	420	1.683	178	0.831	178+618
32	2.35	10^5	2.33	4715	2.38	1.1×10^4	2.34	110	2.35	48	2.35	48+37
33	1.85	10^5	1.87	2459	1.81	1.0×10^4	1.84	120	1.83	78	1.835	78+63
34	3.62	10^6	3.62	6940	3.57	7.5×10^4	3.65	130	3.65	127	3.63	127+83
35	3.045	10^6	3.62	6940	3.13	1.0×10^4	3.09	150	3.03	11,726	3.09	105+658
36	2.323	10^6	2.35	4816	2.30	1.2×10^4	Failed	130	Failed	89	2.32	46+162
37	2.731	10^6	2.69	4682	2.76	9.0×10^3	2.73	140	2.73	37	2.73	37+49
38	2.74	10^6	2.71	9417	2.77	6.0×10^3	2.74	80	2.73	56	2.74	35+61
39	4.78	10^8	4.79	16,838	5.00	1.1×10^4	4.79	120	4.79	120	4.79	120+148
40	2.41	10^4	2.41	3715	2.43	1.1×10^4	2.42	110	2.41	27	2.41	27+39
41	2.59	10^5	0.65	9638	2.63	2.2×10^4	2.61	90	2.34	89	2.61	59+1071
42	2.29	10^5	2.30	3792	2.28	3.0×10^4	2.66	300	3.41	74	2.31	21+1573
43	3.50	10^6	3.61	20,728	3.66	1.15×10^4	Failed	500	Failed	89	3.56	106+12,470
44	2.05	10^5	2.02	7418	2.00	1.1×10^4	2.06	100	2.06	49	2.06	49+74
45	3.49	10^6	3.50	13,471	3.53	1.2×10^4	3.49	110	3.50	75	3.50	75+136

*g call (FOCM)= g call #1 (Design point search)+g call #2 (simulation).

as presented in Fig. 7 and the performance function is formulated as in Eq. (33). For the proposed problem, c (kg/cm), tire stiffness c_k (kg/cm) and damping coefficient k (kg/cm s) were considered as random variables with the mean and standard deviation of [424.0,1480.0,47.0] and [10.0,10.0,10.0], respectively. The following parameter values were selected as well:

$A = 1.0\ \text{cm}^2/\text{cycle m}$, $b_0 = 0.27$, $V = 10.0\,\text{m/s}$, $M = 3.2633\,\text{kg s}^2\ /\text{cm}$, $G = 981\,\text{cm/s}^2$ and $m = 0.8158\,\text{kg s}^2\ /\text{cm}$.

$$g = \left(\frac{\pi A V m}{b_0 G^2}\right)\left(\left(\frac{c_k}{M+m} - \frac{c}{M}\right)^2 + \frac{c^2}{Mm} + \frac{c_k k^2}{m M^2}\right) - 1, \tag{33}$$

The problem is solved by four reliability methods, and results are presented in Table 2. It could be found that similar to the previous example, the geometry of performance function misconducts SS, LS and FORM to find the important failure domain of the problem. The SS failed to approximate the proper failure probability even by using 7.5 $\times$ 10^4 samples. Common FORM algorithms also were unable to find the accurate design point. However, by employing the implementations presented in Section 3.2, the proposed FOCM efficiently approximated the failure probability with high accuracy by linearizing performance function around the MPP obtained by the simulation.

In contrast with SS that presents a performance similar to an optimization problem, as illustrated in Fig. 8, the proposed simulation approach would generate samples at the vicinity of LSF (g=0.0) and therefore when FORM fails to determine

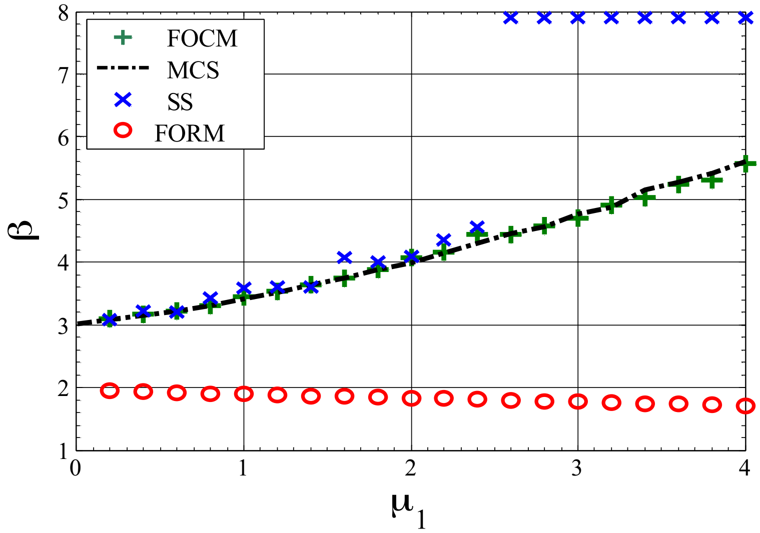

Fig. 5. Reliability indices with respect to various mean values of variable X_1.

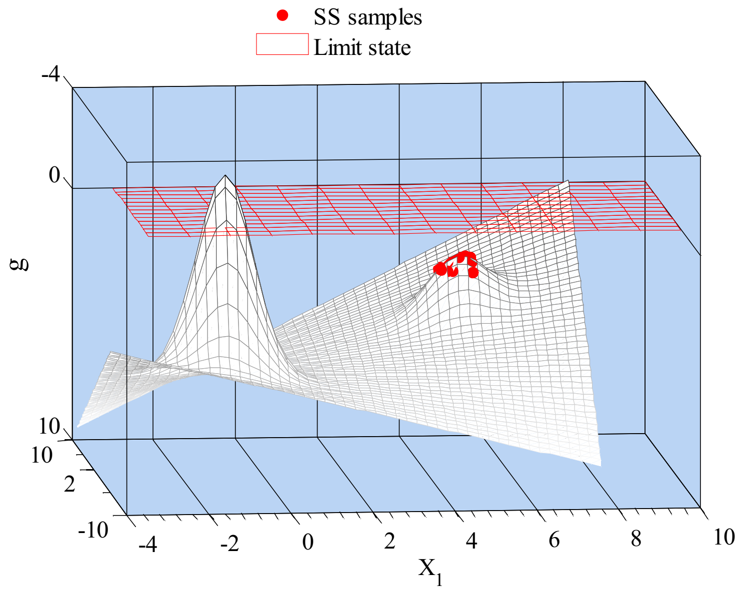

Fig. 6. Representation of performance function #1 and the convergence of SS samples.

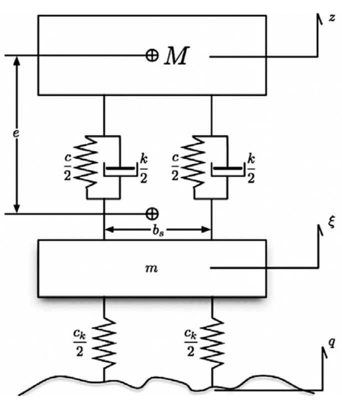

Fig. 7. Passive vehicle suspension model [55].

Table 2
Reliability results of passive vehicle suspension model.

	FORM	MCS	SS	IS	LS	FOCM
P_f	4.38×10^{-11}	1.45×10^{-6}	4.91×10^{-10}	1.05×10^{-6}	Failed	1.17×10^{-6}
δ_{PF}	–	0.16	0.10	0.10	–	0.15
g call	4994	2×10^7	7.5×10^4	5×10^4	–	4617

Table 3
Reliability results of the high dimension linear reliability problem.

$\beta(P_F)$		FORM	MCS	CW- SS	IS	LS	FOCM
2.0 (2.28×10^{-2})	P_F	2.28×10^{-2}	2.44×10^{-2}	2.28×10^{-2}	Failed	2.28×10^{-2}	2.28×10^{-2}
	δ_{PF}	–	0.09	0.09	–	0	0.12
	g call	202	5×10^3	4×10^3	–	221	202+24
3.0 (1.35×10^{-3})	P_F	1.35×10^{-3}	1.42×10^{-3}	1.13×10^{-3}	Failed	1.35×10^{-3}	1.35×10^{-3}
	δ_{PF}	–	0.11	0.1	–	0	0.09
	g call	202	5×10^4	9×10^3	–	218	202+38
4.0 (3.17×10^{-5})	P_F	3.17×10^{-5}	3.19×10^{-5}	3.49×10^{-5}	Failed	3.17×10^{-5}	3.17×10^{-5}
	δ_{PF}	–	0.08	0.11	–	0	0.11
	g call	202	5×10^7	16×10^3	–	225	202+29
5.0 (2.87×10^{-7})	P_F	2.87×10^{-7}	3.48×10^{-7}	2.03×10^{-7}	Failed	2.87×10^{-7}	2.87×10^{-7}
	δ_{PF}	–	0.12	0.12	–	0	0.06
	g call	202	5×10^8	27.5×10^3	–	228	202+76
6.0 (9.87×10^{-10})	P_F	9.87×10^{-10}	Infeasible by g call less than 10^{10}	2.03×10^{-10}	Failed	9.87×10^{-10}	9.87×10^{-10}
	δ_{PF}	–		0.12	–	0	0.07
	g call	202		49×10^3	–	215	202+71

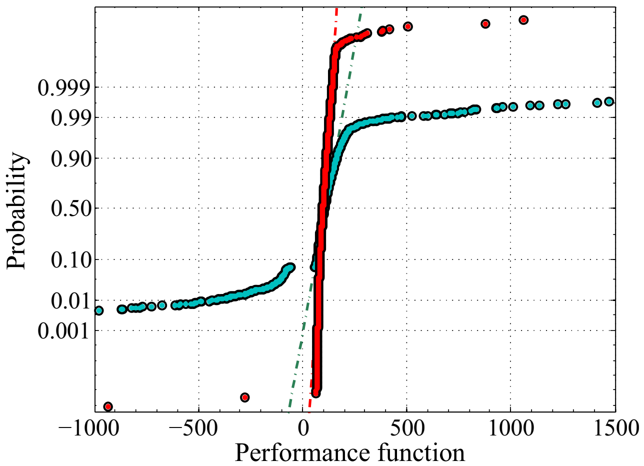

Fig. 8. Normal probability plot for MCS with 2.0×10^7 samples (red dots), and FOCM simulation with 4617 samples (green dots). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

proper design point, one may use the obtained MPP of the simulation to approximate the required first-order failure probability.

5.2. Dimensionality test problems

Case 1. A linear hyper-plane LSF with standard normal random variables is expressed as follows [56]:

$$g(\boldsymbol{U}) = \beta - \frac{1}{\sqrt{D}} \sum_{i=1}^{D} U_i, \tag{34}$$

where U_i is considered as standard normal random variables, β is the exact reliability index of the problem and D is the number of random variables. To examine the efficiency of reliability methods, the number of variables is fixed at $D = 100$, and the problem has been solved for a variety of β. Results are presented in Table 3. Due to the results, LS easily solves the

Table 4
Reliability results of the high dimension nonlinear reliability problem.

Reliability Index (g call)								
k	D	FORM[a]	MCS	CW- SS[b]	IS	LS	FOCM	B Exact
0.2	2	(6)	$3.82(10^7)$	3.83(16)	$3.84(2\times10^3)$	3.85(100)	3.83(6 + 68)	3.83
	10	(30)	$3.84(10^8)$	3.84(16)	$3.7(1.1\times10^4)$	3.78(250)	3.84(30+92)	
	50	(102)	$3.84(10^8)$	3.84(16)	$5.77(5\times10^4)$	3.80(200)	3.83(102+163)	
	100	(202)	$3.83(10^8)$	3.82(16)	Failed	3.86(550)	3.85(202+157)	
0.6	2	(6)	$3.01(10^6)$	3.00(12)	$3.03(1.8\times10^3)$	3.19(250)	2.98(46+194)	2.99
	10	(30)	$3.02(10^6)$	2.97(12)	$2.89(1.4\times10^4)$	3.16(500)	2.99(127+159)	
	50	(102)	$2.99(10^6)$	3.01(12)	$4.86(6\times10^4)$	3.07(400)	3.01(175+181)	
	100	(202)	$3.01(10^6)$	2.97(12)	Failed	2.91(650)	3.01(416+195)	
1.0	2	(6)	$2.37(10^5)$	2.37(6)	$2.34(2.1\times10^3)$	2.45(550)	2.37(30+107)	2.37
	10	(30)	$2.37(10^5)$	2.34 (6)	$2.43(2\times10^4)$	2.24(500)	2.39(261+92)	
	50	(102)	$2.37(10^5)$	2.36 (6)	$3.72(7\times10^4)$	2.45(600)	2.42(317+168)	
	100	(202)	$2.38(10^5)$	2.35 (6)	Failed	2.49(550)	2.45(487+157)	
−1.0	2	(6)	$4.22(10^8)$	4.23(27)	$4.19(3\times10^3)$	4.16(100)	4.18(6 + 1514)	4.19
	10	(22)	$4.21(10^8)$	4.19(27)	$4.27(3\times10^3)$	4.19(300)	4.22(22+1284)	
	50	(102)	$4.17(10^8)$	4.20(27)	$6.51(7\times10^4)$	4.23(250)	4.22(102+1578)	
	100	(202)	$4.19(10^8)$	4.20(27)	Failed	4.21(650)	4.19(202+1471)	
−5.0	2	(6)	$4.39(10^8)$	4.35(27)	$4.47(4\times10^3)$	4.33(350)	4.37(6 + 1574)	4.36
	10	(22)	$4.41(10^8)$	4.35 (27)	$4.06(2.2\times10^4)$	4.29(300)	4.37(22+1462)	
	50	(102)	$4.38(10^8)$	4.36(27)	$6.42(9\times10^4)$	4.30(250)	4.39(102+1247)	
	100	(202)	$4.33(10^8)$	4.35 (27)	Failed	4.38(600)	4.36(202+1277)	
−10.0	2	(6)	$4.49(10^8)$	4.35(27)	$4.52(7\times10^3)$	4.40(450)	4.45(6 + 1381)	4.45
	10	(22)	$4.46(10^8)$	4.35 (27)	$4.96(1.8\times10^4)$	4.41(200)	4.43(22+1074)	
	50	(102)	$4.46(10^8)$	4.36(27)	$6.44(5\times10^4)$	4.43(550)	4.47(102+1631)	
	100	(202)	$4.41(10^8)$	4.35 (27)	Failed	4.46(700)	4.47(202+1679)	

[a] *FORM Reliability Index* = 4.
[b] *g call* $\times 10^3$.

problem. However IS fails to provide any suitable answer. It also could be found that the increase in reliability index β has no effect on the efficiency of FORM and the proposed FOCM, while it significantly decreases the efficiency of MCS and SS. It is evident that the result of FOCM always would be accurate for such D-dimension problems because of the linearity of original LSF.

Case 2. A high dimension nonlinear LSF with normal random variables is taken from Ref. [56] as follows:

$$g(\boldsymbol{u}) = \beta_0 - \frac{1}{\sqrt{D}}\sum_{i=1}^{D} U_i - \frac{k}{4}(U_1 - U_2)^2, \tag{35}$$

where k is a curvature parameter to control the nonlinearity of the function. For the proposed problem, the failure probability of the problem has an analytical solution that is independent of the dimension. The problem is solved by the proposed reliability methods for $\beta_0 = 4.0$ and a sequence of k values and dimensions D. Due to the results presented in Table 4, the IS method is unable to find any suitable answer in dimensions above 50. The results also show that for the proposed problem, FORM fails to provide proper results because of the nonlinearity of LSF. However, the proposed FOCM provided an accurate result with high efficiency compared to MCS and SS and LS. It is not presentedin in the table, but simulation has shown that δ_{PF} of SS is highly under the effect of selected initial probability and sample size. Herein, employing improper parameters would lead to improper reliability results for the problem. For the case of FORM, the common used gradient-based algorithms often fail to find the correct design point of the problem. For all presented curvatures, as shown in Fig. 9 by a red dot, they determine a point with the location of [2.8285, 2.8284] as the design point that is not a suitable design point when $k = 0.6$ and 1. For such cases, the strategy presented in Section 3.2, would lead to obtaining suitable results for the problem.

Case 3. Reliability of a problem with lognormal random variables and the following LSF is investigated [57]:

$$g(X) = \left(D + 3\sigma\sqrt{D}\right) - \sum_{i=1}^{D} X_i, \tag{36}$$

where $U_{\rm i}$ are considered as lognormal random variables with unit means and standard deviations of $\sigma = 0.2$. The problem is solved for $D = 40$, 100 and the obtained results are presented in Table 5.

Table 5
Reliability results of the high dimension non-normal reliability problem.

Method	D	g call	δ_{P_F}	P_F
FORM	40	205	–	5.36×10^{-5}
	100	505	–	2.76×10^{-4}
MCS	40	25×10^4	0.09	6.41×10^{-4}
	100	9.0×10^4	0.04	7.33×10^{-3}
CW- SS	40	7.10×10^3	0.14	6.41×10^{-4}
	100	4.32×10^3	0.11	7.25×10^{-3}
IS	40	–	–	Failed
	100	–	–	Failed
LS	40	150	0.07	6.48×10^{-3}
	100	100	0.05	7.26×10^{-3}
FOCM	40	205+1781	0.12	5.88×10^{-4}
	100	505+1319	0.09	2.87×10^{-3}

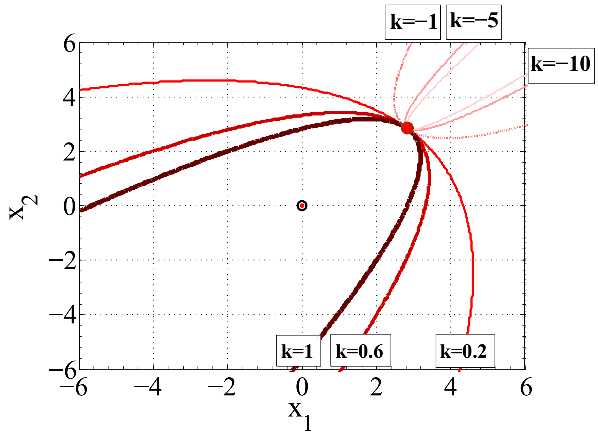

Fig. 9. The 2-D representation of the curvature governed by the parameter k. (For interpretation of the references to color in this figure, the reader is referred to the web version of this article.)

For such problems involving non-normal random variables, transferring random variable form original X-space to standard normal U-space would produce nonlinearity for LSF in U-space. As it is presented in Table 4, the obtained nonlinearity would lead to huge errors in FORM results. Also, IS is not able to converge to any proper results in high dimension problems. However, the proposed FOCM presented suitable results with suitable efficiency compared to MCS, SS, and LS.

The results obtained for the proposed three cases confirm the efficiency and accuracy of the method for solving high dimension problems with nonlinear LSFs.

6. Conclusion

A novel method presented for reliability analysis of structures with small failure probability and high dimension nonlinear LSF. The proposed approach adaptively takes the advantages of FORM and simulation methods for solving problems. By successfully solving of widely used numerical/engineering reliability problems, it has shown that the proposed approach is almost insensitive to the PDF and statistical parameters of random variables, the value of failure probability, the dimension of problem and nonlinearity of LSF. For all solved examples, FOCM has presented the same accuracy as MCS with the efficiency close to FORM. In the proposed approach, it is possible to approximate and also control the variance of estimation during the simulation process. However, for the case of SS, the variance may be calculated after the procedure, and the researcher has no control over the accuracy of the results during the process. These advantages confirm the high potential of the method for application in reliability analysis of structures.

Appendix

	LSF significance	Case No.	Limit state function (s)	Stochastic variables	β	Ref.
Benchmark problems with different probability density functions	Iceland shape	(1)	$g(X) = 7 - (8.\exp(-(X_1+1)^2 + (X_2+1)^2) + 2.\exp(-((X_1-5)^2 + (X_2-4)^2)) + 1 + \frac{X_1X_2}{10})$	X_1: N(2.0,1.0) X_2: N(2.0,1.0)	4.0	[11]
	-Non-Normal - Highly Nonlinear	(2)	$g(\boldsymbol{X}) = X_1X_2X_3X_4 - \frac{X_5X_6^2}{8}$	X_1: W(4.0,0.1) X_2: LN(2.5×10^4,2.0×10^3) X_3: G(8.75×10^{-2},0.1) X_4: Un(20.0,0.1) X_5: Ex(100.0,100.0) X_6: N(150.0,10.0)	2.70	[35]
	With high-frequency, artificial noise term	(3)	$g(\boldsymbol{X}) = X_1 + 2X_2 + 2X_3 + X_4 - 5X_5 - 5X_6 + 0.001\sum_{i=1}^{6}\sin(100X_i)$	$X_{1\ldots4}$: LN(120.0,12.0) X_5: LN(50.0,15.0) X_6: LN(40.0,12.0)	2.247	[23,34,37,43,46,49]
	-Non-Normal -Nonlinear	(4)	$g(\boldsymbol{X}) = 1.1 - 0.00115X_1X_2 + 0.00157X_2^2 + 0.00117X_1^2 + 0.0135X_3X_2 - 0.0705X_2 - 0.00534X_1 - 0.0149X_1X_3 - 0.0611X_4X_2 + 0.0717X_1X_4 - 0.226X_3 + 0.0333X_3^2 - 0.558X_3X_4 + 0.998X_4 - 1.339X_4^2$	X_1: EX-II(10.0,5.0) X_2: N(25.0,5.0) X_3: N(0.8,0.2) X_4: LN(6.25×10^{-2},6.25×10^{-2})	1.39	[22,33,34,40]
	Highly Nonlinear	(5)	$g(\boldsymbol{X}) = \frac{120}{X_1} - \frac{X_2}{X_1} - 1$	X_1: N(4.0,1.0) X_2: N(4.0,1.0)	4.0	[36]
	Cubic performance function	(6)	$g(\boldsymbol{X}) = 2.2257 - \frac{0.025\sqrt{2}}{27}(X_1 + X_2 - 20)^3 + \frac{33}{140}(X_1 - X_2)$	X_1: N(10.0,3.0) X_2: N(10.0,3.0)	2.07	[30,49]
	Quartic performance function	(7)	$g(\boldsymbol{X}) = \frac{5}{2} + \frac{1}{216}(X_1 + X_2 - 20)^4 - \frac{33}{140}(X_1 + X_2)$	X_1: N(10.0,3.0) X_2: N(10.0,3.0)	2.76	[30,49]
	Highly Nonlinear	(8)	$g(\boldsymbol{X}) = X_1^3 + X_2^3 - 18$	X_1: N(10.0,5.0) X_2: N(9.9,5.0)	2.53	[32,38,40,49,51]
	-Non-Normal -with positive curvature failure surface	(9)	$g(\boldsymbol{X}) = \pm\sum_{i=1}^{2}X_i \mp 6.6384$	X_1: Ex (0.0,1.0) X_2: Ex (0.0,1.0)	2.52	[39]
	-Non-Normal -with positive curvature failure surface	(10)	$g(\boldsymbol{X}) = \pm\sum_{i=1}^{10}X_i \mp 26.193$	$X_{1\ldots10}$: Ex (0.0,1.0)	3.72	[39]
	Highly Nonlinear	(11)	$g(\boldsymbol{X}) = X_1^3 + X_2^3 - 67.5$	X_1: N(10.0,5.0) X_2: N(9.9,5.0)	2.23	[40]
	Highly Nonlinear	(12)	$g(\boldsymbol{X}) = X_1^3 + X_2X_1^2 + X_2^3 - 18$	X_1: N(10.0,5.0) X_2: N(9.9,5.0)	2.52	[22,41,49]
	Highly Nonlinear	(13)	$g(\boldsymbol{X}) = X_1^4 + 2X_2^4 - 20$	X_1: N(10.0,5.0) X_2: N(10.0,5.0)	2.90	[8,12,41,49]
	With Multiple failure points	(14)	$g(\boldsymbol{X}) = X_1X_2 - 146.14$	X_1: N(7.80644×10^4,1.17097×10^4) X_2: N(0.0104,0.00156)	5.11	[36]
	-Non-Normal-Nonlinear	(15)	$g(\boldsymbol{X}) = X_1X_2 - 1140$	X_1: LN(38.0,3.8) X_2: LN(54.0,2.7)	5.21	[42]
	-Non-Normal -Nonlinear	(16)	$g(\boldsymbol{X}) = X_1X_2 - 2000X_3$	X_1: N(0.32,0.032) X_2: N(1.4×10^6,7.0×10^4) X_3: LN(100.0,40.0)	2.184	[40]
Benchmark problems with standard normal variables	Circle Shape	(17)	$g(\boldsymbol{X}) = 9 - X_1^2 - X_2^2$	X_1: N(0.0,1.0) X_2: N(0.0,1.0)	2.28	[36]

(*continued on next page*)

(*continued*)

	LSF significance	Case No.	Limit state function (s)	Stochastic variables	β	Ref.
	Highly Nonlinear	(18)	$g(\boldsymbol{X}) = 3 - X_2 + (4X_1)^4$	X_1: N(0.0,1.0) X_2: N(0.0,1.0)	3.57	[54]
	Nonlinear	(19)	$g(\boldsymbol{X}) = 2 + 0.015\sum_{i=1}^{9} X_i^2 - X_{10}$	$X_{1\ldots10}$: N(0.0,1.0)	2.13	[16]
	Highly nonlinear with non-convex domains	(20)	$g(\boldsymbol{X}) = 10 - \sum_{i=1}^{2}(X_i^2 - 5\cos(2\pi X_i))$	X_1: N(0.0,1.0) X_2: N(0.0,1.0)	1.45	[44]
	Nonlinear LS with saddle point	(21)	$g(\boldsymbol{X}) = 2 - X_2 - 0.1X_1^2 + 0.06X_1^3$	X_1: N(0.0,1.0) X_2: N(0.0,1.0)	1.819	[37]
	Nonlinear Concave LS	(22)	$g(\boldsymbol{X}) = -0.5(X_1 - X_2)^2 - \frac{(X_1+X_2)}{\sqrt{2}} + 3$	X_1: N(0.0,1.0) X_2: N(0.0,1.0)	1.255	[45]
	Nonlinear	(23)	$g(\boldsymbol{X}) = 25 - 2(X_1 - X_2)^2 - 2(X_1^2 - X_2^2)$	X_1: N(0.0,1.0), X2: N(0.0,1.0)	1.705	[30]
	Nonlinear	(24)	$g(\boldsymbol{X}) = -\frac{4}{25}(X_1 - 1)^2 - X_2 + 4$	X_1: N(0.0,1.0), X_2: N(0.0,1.0)	3.156	[19]
	Highly Nonlinear	(25)	$g(\boldsymbol{X}) = -0.16(X_1 - 1)^3 - X_2 + 4 - 0.04\cos(X_1X_2)$	X_1: N(0.0,1.0), X_2: N(0.0,1.0)	4.16	[49,47]
	Highly Nonlinear	(26)	$g(\boldsymbol{X}) = \frac{1}{40}X_1^4 + 2X_3^3 + X_3 + 3$	X_1: N(0.0,1.0), X_2: N(0.0,1.0), X_2: N(0.0,1.0)	3.43	[19]
	Highly Nonlinear	(27)	$g(\boldsymbol{X}) = exp[0.4(X_1 + 2) + 6.2] - exp[0.3X_2 + 5] - 200$	X_1: N(0.0,1.0), X_2: N(0.0,1.0)	2.709	[49,48]
	Highly Nonlinear	(28)	$g(\boldsymbol{X}) = exp[0.2X_1 + 1.4] - X_2$	X_1: N(0.0,1.0), X_2: N(0.0,1.0)	3.385	[12,18,48]
	Highly Nonlinear	(29)	$g(\boldsymbol{X}) = X_3 + (\frac{X_1-1.1}{1.5})^2 - (\frac{X_2-.2}{3.0})^2 + 3.6$	X_1: N(0.0,1.0), X_2: N(0.0,1.0), X_3: N(0.0,1.0)	3.7050	[49]
	Highly Nonlinear	(30)	$g(\boldsymbol{X}) = exp[.2(1 + X_1 - X_2)] - exp[.2(5 - 5X_1 - X_2)] - 1$	X_1: N(0.0,1.0) X_2: N(0.0,1.0)	4.44	[49]
	Nonlinear	(31)	$g(\boldsymbol{X}) =$ $-0.5(X_1^2 + X_2^2 + X_3^2 - 2X_1X_2 - 2X_3X_2 - 2X_3X_1) - \frac{(X_1+X_2+X_3)}{\sqrt{3}} + 3$	X_1: N(0.0,1.0) X_2: N(0.0,1.0) X_3: N(0.0,1.0)	0.849	[30]
	Nonlinear	(32)	$g(\boldsymbol{X}) = exp[0.2X_1 + 6.2] - exp[0.47X_2 + 5.0]$	X_1: N(0.0,1.0), X_2: N(0.0,1.0)	2.35	[30]
Engineering problems	SDOF	(33)	$g(\boldsymbol{X}) = 3X_4 - \frac{2X_5}{X_2+X_3} \times \sin(\frac{X_6}{2}.\sqrt{\frac{X_2+X_3}{X_1}})$	X_1: LN(1.0,0.05) X_2: LN(1.0,0.1) X_3: LN(0.1,0.01) X_4: LN(0.5,0.05) X_5: LN(1.0,0.2) X_6: LN(1.0,0.2)	1.85	[49,44]
	Fatigue	(34)	$g(\boldsymbol{X}) = 2 - e^{(\frac{X_5X_3}{X_1})} + \frac{e^{X_5}-2}{e^{-X_6}-1}(e^{-(\frac{X_6X_3}{X_1})} - 1) - \frac{X_4}{X_2}$	X_1: LN(5490.0,1098.0) X_2: LN(17,100.0,3420.0) X_3: LN(549.0,109.8) X_4: LN(4.0×10^3,8.0×10^2) X_5: N(0.42,0.084) X_6: N(6.0,1.2)	3.633	[30]

(*continued on next page*)

(*continued*)

LSF significance	Case No.	Limit state function (s)	Stochastic variables	β	Ref.
Burst margin of a disk	(35)	$g(\boldsymbol{X}) = \left[\frac{X_1X_2}{\frac{X_3(X_4\frac{2\pi}{60})^2(X_5^3-X_6^3)}{3(385.82)(X_5-X_6)}}\right]^{1/2} - 0.37473$	X_1: W(0.9377,4.59 $\times 10^{-2}$) X_2: N(2.2 $\times 10^5$,5.0 $\times 10^3$) X_3: Un(0.29,5.8 $\times 10^{-3}$) X_4: N(2.1 $\times 10^4$,1.0 $\times 10^3$) X_5: N(24.0,0.5) X_6: N(8,0.3)	3.045	[30]
Steel joint	(36)	$g(\boldsymbol{X}) = X_1 - 10^4 \times \left[\frac{X_2(X_4X_5)^{1.71}}{X_3} + \frac{(1-X_2)(X_4X_5)^{1.188}}{X_6}\right]$	X_1: LN(1.044,0.31320) X_2: N(0.7,0.07) X_3: LN(0.2391,0.09564) X_4: LN(1.0110,0.15165) X_5: Ex(5.0 $\times 10^{-4}$,8.0 $\times 10^{-5}$) X_6: LN(1.8020,0.72080)	2.323	[36]
Beam/ Bending	(37)	$g(\boldsymbol{X}) = X_1X_2 - 10^6X_3$	X_1: LN(275.52,34.44) X_2: LN(8.19 $\times 10^{-4}$, 4.1 $\times 10^{-5}$) X_3: EX-I (1.13 $\times 10^5$, 2.26 $\times 10^4$)	2.721	[30]
Retaining wall/ Overturning	(38)	$g(\boldsymbol{X}) = 27.668X_1 + 18.595X_3 - 121.5X_1\tan^2(45 - \frac{X_2}{2})$	X_1: N(16.0,1.12) X_2: N(30.0,3.0) X_3: N(25.0,1.0)	2.74	[50]
Conical structure	(39)	$g(\boldsymbol{X}) = 1 - \frac{\sqrt{3(1-0.3^2)}}{\pi X_1X_2^2\cos^2X_3} \times (\frac{X_6}{0.66} + \frac{X_5}{0.41X_4})$	X_1: N(7.0 $\times 10^{10}$,3.50 $\times 10^9$) X_2: N(2.50 $\times 10^{-3}$,1.25 $\times 10^{-4}$) X_3: N(0.524,0.010480) X_4: N(0.90,0.0225) X_5: N(8.0 $\times 10^4$,6.4 $\times 10^3$) X_6: N(7.0 $\times 10^4$,5.6.0 $\times 10^3$)	4.78	[23]
Cantilever beam	(40)	$g(\boldsymbol{X}) = 18.461 - 7.477 \times 10^{10}\frac{X_1}{X_2^3}$	X_1: N(0.001,0.00002) X_2: N(250.0,37.5)	2.41	[48]
Roof truss	(41)	$g(\boldsymbol{X}) = 0.03 - (\frac{X_1X_2^2}{2})(\frac{3.81}{X_4X_6} + \frac{1.13}{X_3X_5})$	X_1: N(2.0 $\times 10^4$,1.4.0 $\times 10^3$) X_2: N(12,0.12) X_3: N(9.82 $\times 10^{-4}$,5.9852 $\times 10^{-5}$) X_4: N(0.04, 4.8 $\times 10^{-3}$) X_5: N(1.0 $\times 10^{11}$, 1.0 $\times 10^9$) X_6: N(2.0 $\times 10^{10}$, 1.2 $\times 10^9$)	2.59	[32,55]
Tuned vibration absorber	(42)	$g(\boldsymbol{X}) = 27 - \frac{\lvert 1-(\frac{1}{X_2})^2\rvert}{\sqrt{[a]^2 + 4(0.01)^2[(\frac{1}{X_1})^2 - \frac{1}{X_1X_2}]^2}}$, $a = 1 - 0.01(\frac{1}{X_1})^2 - (\frac{1}{X_1})^2 - (\frac{1}{X_2})^2 + (\frac{1}{X_1X_2})^2$	X_1: N(1.0,0.025) X_2: N(1.0,0.025)	2.29	[8]

(*continued on next page*)

(*continued*)

LSF significance	Case No.	Limit state function (s)	Stochastic variables	β	Ref.
Reinforced concrete beam	(43)	$g(\boldsymbol{X}) = X_2X_3X_4 - \frac{X_5X_3^2X_4^2}{X_6X_7} - X_1$	X_1: EX-1(0.01,3 × 10^{-3}) X_2: N(0.3,1.5 × 10^{-2}) X_3: N(360.0,36.0) X_4: LN(226 × 10^{-6},11.3 × 10^{-6}) X_5: N(0.5,0.05) X_6: N(0.12,6 × 10^{-3}) X_7: LN(40,6.0)	3.50	[12,33,49]
Front Axle	(44)	$g = 460 - \sqrt{(Sb)^2 + 3(Ts)^2}$, $Sb = \frac{X_5}{\frac{X_1(X_4 - 2X_3)^3}{6X_4} + \frac{X_2}{6X_4}({X_4}^3 - (X_4 - 2X_3)^3)}$, $Ts = \frac{X_6}{0.8X_2{X_3}^2 + 0.4(\frac{{X_1}^3(X_4 - 2X_3)}{X_3})}$	X_1: N(12.0,0.06) X_2: N(65.0,0.325) X_3: N(14.0,0.07) X_4: N(85.0,0.425) X_5: N(3.5 × 10^6,1.75 × 10^5) X_6: N(3.1 × 10^6,1.55 × 10^5)	2.05	[52]
Tower of a cable-stayed bridge	(45)	$g = X_5 - \frac{5X_3X_4(47.4)^3}{3(5X_1X_2 - 2X_3(47.4)^2)}$ $- \frac{5(1170 \times 47.4)X_4(0.4)^2}{3(5X_1X_2 - 2X_3(47.4)^2)} - 47.4X_4$	X_1: N(17.6,1.76) X_2: N(4.0 × 10^7,3.2 × 10^6) X_3: N(9.01 × 10^4,9.01 × 10^3) X_4: N(2120,318) X_5: N(2.6 × 10^5,3.90 × 10^4)	3.50	[53]

Note: *N*=normal; LN=log-normal; Un= Uniform; *G*= Gumbel; EX-II=Extreme II; EX-*I*=Extreme I; Ex= exponential; *W*= weibull.

References

[1] P. Castaldo, M. Calvello, B. Palazzo, Probabilistic analysis of excavation-induced damages to existing structures, Comput. Geotech. 53 (2013) 17–30.
[2] P. Castaldo, F. Jalayer, B. Palazzo, Probabilistic assessment of groundwater leakage in diaphragm wall joints for deep excavations, Tunn. Undergr. Space Technol. 71 (2018) 531–543.
[3] P. Castaldo, D. Gino, G. Bertagnoli, G. Mancini, Partial safety factor for resistance model uncertainties in 2D non-linear finite element analysis of reinforced concrete structures, Eng. Struct. 176 (2018) 746–762.
[4] C. Wang, H. Zhang, Q. Li, Moment-based evaluation of structural reliability, Reliab. Eng. Syst. Saf. 181 (2019) 38–45.
[5] Z. Meng, H. Zhou, H. Hu, B. Keshtegar, Enhanced sequential approximate programming using second order reliability method for accurate and efficient structural reliability-based design optimization, Appl. Math. Model. 62 (2018) 562–579.
[6] A. Alban, H.A. Darji, A. Imamura, M.K. Nakayama, Efficient Monte Carlo methods for estimating failure probabilities, Reliab. Eng. Syst. Saf. 165 (2017) 376–394.
[7] C.H. Juang, W. Gong, J.R. Martin, Subdomain sampling methods – efficient algorithm for estimating failure probability, Struct. Saf. 66 (2017) 62–73.
[8] H. Ghohani-Arab, M.R. Ghasemi, A fast and robust method for estimating the failure probability of structures, Pron. Inst. Civil Eng. Struct. B 168 (4) (2015) 298–309.
[9] D.K.E. Green, Efficient Markov chain Monte Carlo for combined subset simulation and nonlinear finite element analysis, Comput. Methods Appl. Mech. Eng. 313 (2017) 337–361.
[10] H.J. Pradlwarter, G.I. Schueller, P.S. Koutsourelakis, D.C. Charmpis, Application of line sampling simulation method to reliability benchmark problems, Struct. Saf. 29 (2007) 208–221.
[11] M. Rashki, Hybrid control variates-based simulation method for structural reliability analysis of some problems with low failure probability, Appl. Math. Model. 60 (2018) 220–234.
[12] M. Rashki, M. Miri, M. Azhdary-Moghaddam, A new efficient simulation method to approximate the probability of failure and most probable point, Struct. Saf. 39 (2012) 22–29.
[13] N.M. Okasha, An improved weighted average simulation approach for solving reliability-based analysis and design optimization problems, Struct. Saf. 60 (2016) 47–55.
[14] G. Xue, H. Dai, H. Zhang, W. Wang, A new unbiased metamodel method for efficient reliability analysis, Struct. Saf. 67 (2017) 1–10.
[15] Q. Pan, D. Dias, An efficient reliability method combining adaptive support vector machine and Monte Carlo simulation, Struct. Saf. 67 (2017) 85–95.
[16] M.A. Shayanfar, M.A. Barkhordari, M.A. Roudak, An efficient reliability algorithm for locating design point using the combination of importance sampling concepts and response surface, Commun. Nonlinear Sci. 47 (2017) 223–237.
[17] M.R. Rajashekhar, B.R. Ellingwood, A new look at the response approach for reliability analysis, Struct. Saf. 12 (1993) 205–220.
[18] N. Gayton, J.M. Bourinet, M. Lemaire, CQ2RS: a new statistical approach to the response surface method for reliability analysis, Struct. Saf. 25 (1) (2003) 99–121.
[19] X.L. Guan, R.E. Melchers, Effect of response surface parameter variation on structural reliability estimates, Struct. Saf. 23 (2001) 429–444.
[20] C.A. Cornell, A probability based structural code, ACI 12 (1969) 974–985.
[21] A.M. Hasofer, N.C. Lind, Exact and invariant second-moment code format, Eng. Mech. Div. ASCE 100 (1974) 111–121.
[22] B. Keshtegar, S. Chakraborty, A hybrid self-adaptive conjugate first order reliability method for robust structural reliability analysis, Appl. Math. Model. 53 (2018) 319–332.
[23] N. Shabakhty, N. Kazemi, M.A. Kia, Structural reliability assessment using improved harmony search evolutionary algorithm. Proceedings of the 2nd Iranian Conference on Reliability Engineering, Aerospace Research Institute, Tehran, Iran, 1–9.
[24] Z. Meng, G. Li, B.P. Wang, P. Hao, A hybrid chaos control approach of the performance measure functions for reliability-based design optimization, Comput. Struct. 146 (1) (2014) 32–43.
[25] Z. Meng, G. Li, D. Yang, L. Zhan, A new directional stability transformation method of chaos control for first order reliability analysis, Struct. Multidiscip. Optim. 55 (2) (2016) 601–612.
[26] Z. Meng, H. Zhou, New target performance approach for a super parametric convex model of non-probabilistic reliability-based design optimization, Comput. Methods Appl. Mech. Eng. 339 (9) (2018) 644–662.
[27] K. Breitung, Asymptotic approximations for multinormal integrals, J. Eng. Mech. ASCE 10 (1984) 357–366.
[28] X. Du, A. Sudjianto, The first order saddlepoint approximation for reliability analysis, AIAA J. 42 (6) (2004) 1199–1207.
[29] K. Piric, Reliability analysis method based on determination of the performance function's PDF using the univariate dimension reduction method, Struct. Saf. 57 (2015) 18–25.
[30] D. Wei, S. Rahman, Structural reliability analysis by univariate decomposition and numerical integration, Probab. Eng. Mech 22 (1) (2007) 27–38.
[31] B. Tracey, D. Wolper, Reducing the error of Monte Carlo algorithms by learning control variates, in: Proceedings of the 29th Conference on Neural Information Processing Systems, NIPS, Barcelona, Spain, 2016.
[32] B. Keshtegar, Z. Meng, A hybrid relaxed first-order reliability method for efficient structural reliability analysis, Struct. Saf. 66 (2017) 84–93.
[33] J.X. Gong, P. Yi, A robust iterative algorithm for structural reliability analysis, Struct. Multidiscip. Optim. 43 (4) (2011) 519–527.
[34] P.L. Liu, A. Der Kiureghian, Optimization algorithms for structural reliability, Struct. Saf. 9 (3) (1991) 161–177.
[35] R.E. Melchers, M. Ahammed, A fast approximate method for parameter sensitivity estimation in Monte Carlo structural reliability, Comput. Struct. 82 (2004) 55–61.
[36] S.N. Xuan, Algorithmes Probabilistes Appliqués à la Durabilité et à la Mecanique des Ouvrages de gé de Génie Civil Ph.D. thesis, University of Toulouse, Toulouse, France, 2007.
[37] S. Mahadevan, S. Pan, Multiple linearization methods for linear reliability analysis, J. Eng. Mech. ASCE 127 (11) (2001) 1165–1172.
[38] L. Wang, R.V. Grandhi, Efficient safety index calculation for structural reliability analysis, Comput. Struct. 52 (1) (1994) 103–111.
[39] D. Hongzhe, W. Wei, Application of low-discrepancy sampling method in structural reliability analysis, Struct. Saf. 31 (2009) 55–64.
[40] T.V. Santosh, R.K. Saraf, A.K. Ghosh, H.S. Kushwaha, Optimum step length selection rule in modified HLRF method for structural reliability, Int. J. Press. Vessels pip. 83 (2006) 742–748.
[41] L. Wang, R.V. Grandhi, Safety index calculation using intervening variables for structural reliability analysis, Comput. Struct. 59 (6) (1996) 1139–1148.
[42] A. Haldar, S. Mahadevan, Probability, Reliability and Statistical Methods in Engineering Design, John Wiley & Sons, 2000.
[43] A. Der Kiureghian, H.Z. Lin, S.J. Hwang, Second-order reliability approximations, J. Eng. Mech. Div. ASCE 113 (8) (1987) 1208–1225.
[44] B. Echard B, N. Gayton, M. Lemaire, AK-MCS: an active learning reliability method combining Kriging and Monte Carlo simulation, Struct. Saf. 33 (2011) 145–154.
[45] A. Borri, E. Speranzini, Structural reliability analysis using a standard deterministic finite element code, Struct. Saf. 19 (4) (1997) 361–382.
[46] L. Wang, R.V. Grandhi, Higher-order failure probability calculation using nonlinear approximation, Comput. Methods Appl. Mech. Eng. 168 (1999) 185–206.
[47] H.P. Gavin, S.C. Yau, High-order limit state functions in the response surface method for structural reliability analysis, Struct. Saf. 30 (2) (2008) 162–179.
[48] C. Elegbede, Structural reliability assessment based on particles swarm optimization, Struct. Saf. 27 (2005) 171–186.
[49] J.X. Gong, P. Yi, N. Zhao, Non-gradient-based algorithm for structural reliability analysis, J. Eng. Mech. ASCE 140 (6) (2014) 04014029.
[50] A.K. Mandali, M.S. Sujith, B.N. Rao, J. Maganti, Reliability analysis of counterfort retaining walls, Electron. J. Struct. Eng. 11 (1) (2011) 42–56.
[51] R.C. Penmetsa, R.V. Grandhi, Adaptation of fast fourier transformations to estimate structural failure probability, Finite Elem. Anal. Des. 39 (2003) 473–485.

[52] S. Xiao, Z. Lu, Structural reliability analysis using combined space partition technique and unscented transformation, ASCE J. Struct. Eng. 142 (2016) 04001689.
[53] C.M. Lan, H. Li, J.Y. Peng, D.B. Sun, A structural reliability-based sensitivity analysis method using particles swarm optimization: relative convergence rate, J. Zhejiang Univ. – Sci. A (Appl. Phys. Eng.) 17 (12) (2016) 961–973.
[54] S. Shan, G.G. Wang, Failure surface frontier for reliability assessment on expensive performance function, J. Mech. Des. ASCE 128 (2006) 1227–1235.
[55] M. Rashki, M. Miri, M. Azhdary-Moghaddam, A simulation-based method for reliability based design optimization problems with highly nonlinear constraints, Automat. Constr. 47 (2014) 24–36.
[56] O. Ditlevsen, H.O. Madsen, Structural Reliability Methods, 178, Wiley New York, 1996.
[57] R. Rackwitz, Reliability analysis – a review and some perspectives, Struct. Saf. 23 (4) (2001) 365–395.