# Minimal independent couplings at order $\alpha^{\prime 2}$ 

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#### Abstract

Using field redefinitions and Bianchi identities on the general form of the effective action for metric, $B$-field and dilaton, we have found that the minimum number of independent couplings at order $\alpha^{\prime 2}$ is 60 . We write these couplings in two different schemes in the string frame. In the first scheme, each coupling does not include terms with more than two derivatives and it does not include structures $R, R_{\mu \nu}, \nabla_{\mu} H^{\mu \alpha \beta}$, $\nabla_{\mu} \nabla^{\mu} \Phi$. In this scheme, 20 couplings which are the minimum number of couplings for metric and $B$-field, include dilaton trivially as the overall factor of $e^{-2 \Phi}$, and all other couplings include derivatives of dilaton. In the second scheme, the dilaton appears in all 60 couplings only as the overall factor of $e^{-2 \Phi}$. In this scheme, 20 of the couplings are the same as those in the previous scheme.


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## I. INTRODUCTION

String theory is a quantum theory of gravity with a finite number of massless fields and a tower of infinite number of massive fields reflecting the stringy nature of the gravity. An efficient way to study different phenomena in this theory is to use an effective action which includes only massless fields and their derivatives [1,2]. The effective action has double expansions: the genus expansion which includes the classical and a tower of quantum corrections, and the stringy expansion which is an expansion in powers of the Regge slope parameter $\alpha^{\prime}$. The latter expansion for the metric yields the Einstein gravity and the stringy corrections which are quadratic and higher orders in curvature. The main challenge thus is to explore different techniques to find the effective action that incorporates all such corrections, including nonperturbative effects [3]. In the bosonic and in the heterotic string theories, the higher derivative couplings begin at order $\alpha^{\prime}$, whereas, in type II superstring theory, they begin at order $\alpha^{\prime 3}$.

There are various techniques in the string theory for finding these higher derivative couplings: S-matrix element approach [4-9], sigma-model approach [10-14], supersymmetry approach [15-18], double field theory approach [19-23], and duality approach [3,24-28]. In the duality

[^0]approach, the consistency of the effective actions with T- and S-duality transformations are imposed to find the higher derivative couplings [3,27]. In particular, it has been speculated in [29] that the consistency of the effective actions at any order of $\alpha^{\prime}$ with the T-duality transformations may fix both the effective actions and the corrections to the Buscher rules [30,31]. It has been shown explicitly in [32] that the T-duality constraint fixes the effective action and the corrections to the Buscher rules at order $\alpha^{\prime}$, up to an overall factor.

In using the above techniques for finding the effective actions at the higher-derivative orders in the string theory, one needs the most general gauge invariant and minimal independent couplings at each order of $\alpha^{\prime}$. To find such couplings, one needs to impose various Bianchi identities and use field redefinitions freedom [33-35]. In the literature, the Bianchi identities are first imposed to find the minimum number of couplings at each order of $\alpha^{\prime}$, up to some total derivative terms and field redefinitions. The parameters in the resulting action are then either unambiguous, which are not changed under field redefinition, or ambiguous, which are changed under the field redefinitions. Some combinations of the latter parameters, however, remain invariant under the field redefinitions [13]. This allows one to separate the ambiguous parameters to essential parameters which are fixed by, e.g., S-matrix calculations [7,13], and some remaining arbitrary parameters. Depending on which set of parameters is chosen as essential parameters, one has different schemes. To find the minimum number of independent couplings, one sets all the arbitrary parameters to zero. This method has been used to find the 8 independent couplings for gravity, $B$-field, and dilaton at order $\alpha^{\prime}$ in [13], the 7 independent couplings for gravity and dilaton at order $\alpha^{\prime 2}$ in [36-39], and

20 independent couplings for gravity and B-field at order $\alpha^{\prime 2}$ in [40]. ${ }^{1}$

One may impose the Bianchi identities, remove the boundary terms, and use the field redefinition freedom at the same time. That is, one may first write all gauge invariant couplings at each order of $\alpha^{\prime}$ and then impose the above freedoms to reduce the couplings to the minimal couplings. The parameters in the gauge invariant action are then either unambiguous or ambiguous depending on whether or not they are changed under these freedoms. Some combinations of the ambiguous parameters, however, remain invariant. This allows one to separate the ambiguous parameters to essential parameters which may be found by S-matrix calculations, and some arbitrary parameters. Again, depending on which set of parameters is chosen as essential parameters, one has different schemes. The minimum number of independent couplings is found in the schemes where all the arbitrary parameters are set to zero. We find that this latter method is more convenient to finding the independent couplings systematically, using the Mathematica packages like "xAct" [41]. In particular, to impose the Bianchi identities we write the curvatures and the covariant derivatives in terms of the metric and its derivatives. Then we choose the local inertial frame in which the first derivative of the metric is zero. In this frame the Bianchi identities are all satisfied automatically. Using this method, we are going to find the minimal independent couplings for gravity, dilaton and B -field at order $\alpha^{\prime 2}$. We find that there are 60 parameters in the minimal couplings. We write them in two different schemes. Both schemes have the same 20 couplings between gravity and $B$-field. In one scheme the other 40 couplings include derivatives of the dilaton, and in the other scheme the 40 couplings do not include the derivative of the dilaton. The 20 common couplings in both schemes are the minimal couplings when the dilaton is constant.

The outline of the paper is as follows: In Sec. II, we write the most general gauge invariant couplings involving metric, dilaton, and B-field at order $\alpha^{\prime}$. There are 41 such terms. Then we add to them the most general boundary terms and field redefinitions with arbitrary parameters. Writing them in the local inertial frame, we then use the arbitrary parameters in the total derivative terms and in the field redefinitions to reduce the 41 couplings to 8 independent couplings that are known in the literature. We write them in two different schemes. In one scheme, there is no term in which fields have more than two derivatives and there is no term involving $R$, $R_{\mu \nu}, \nabla_{\mu} H^{\mu \alpha \beta}, \nabla_{\mu} \nabla^{\mu} \Phi$. More specifically, we write the couplings into two separate parts. One part which has 4

[^1]couplings, does not include derivatives of the dilaton, and it is the same as the set of minimal couplings when the dilaton is constant. The other part includes derivatives of the dilaton. In the second scheme, we again write the couplings into two parts: one part is the same as the minimal couplings when the dilaton is constant, and the other part includes some other couplings in which the dilaton appears trivially. In Sec. III, we extend the calculations to the order $\alpha^{\prime 2}$. We found that the most general action at this order has 705 couplings; however, adding the total derivative terms and field redefinitions to them with arbitrary parameters, and writing the result in the local frame, we find that the arbitrary parameters can be used to reduce the couplings to the minimum number of couplings, which is 60 . We write them in two different schemes as in Sec. II. Each scheme has 20 common couplings, which are the minimal couplings when the dilaton is constant, and 40 other couplings. These couplings all include derivatives of the dilaton in one scheme, whereas, in the other scheme the dilaton appears trivially.

## II. MINIMAL COUPLINGS AT ORDER $\boldsymbol{\alpha}^{\prime}$

The effective action of string theory has double expansions. One expansion is the genus expansion which includes the classical sphere-level and a tower of quantum effects. The other one is a stringy expansion which is an expansion in terms of higher-derivative couplings. The number of derivatives in each coupling can be accounted by the order of $\alpha^{\prime}$. The sphere-level effective action has the following power series of $\alpha^{\prime}$ in the string frame:

$$
\begin{align*}
S_{\mathrm{eff}} & =\sum_{n=0}^{\infty} \alpha^{\prime n} S_{n}=S_{0}+\alpha^{\prime} S_{1}+\alpha^{\prime 2} S_{2}+\cdots \\
S_{n} & =\int d^{D} x \sqrt{-g} e^{-2 \Phi} \mathcal{L}_{n} \tag{1}
\end{align*}
$$

The effective action must be invariant under the coordinate transformations and under the $B$-field gauge transformations, so the metric $g$, the antisymmetric $B$-field, and the dilaton must appear in the Lagrangian $\mathcal{L}_{n}$ through their field strengths and their covariant derivatives, e.g., the Lagrangian at the leading order of $\alpha^{\prime}$ is: ${ }^{2}$

$$
\begin{equation*}
\mathcal{L}_{0}=R-\frac{1}{12} H_{\alpha \beta \gamma} H^{\alpha \beta \gamma}+4 \nabla_{\alpha} \Phi \nabla^{\alpha} \Phi \tag{3}
\end{equation*}
$$

The higher-derivative field redefinitions and Bianchi identity can not change the form of this action.

$$
\begin{align*}
& { }^{2} \text { Throughout this paper, we use the conventions } \\
& \qquad \begin{aligned}
R^{\lambda}{ }_{\mu \nu \rho} & =\partial_{\nu} \Gamma_{\mu \rho}^{\lambda}-\cdots, \quad R_{\mu \nu}=R^{\lambda}{ }_{\mu \lambda \rho} \\
H_{\lambda \mu \nu} & =3 \partial_{[\lambda} B_{\mu \nu]}, \quad \nabla_{\mu} A^{\lambda}=\partial_{\mu} A^{\lambda}+\Gamma_{\mu \nu}^{\lambda} A^{\nu}, \\
T_{\left[\mu_{1} \ldots \mu_{n}\right]} & =\frac{1}{n!}\left(T_{\mu_{1} \ldots \mu_{n}}+\cdots\right) .
\end{aligned}
\end{align*}
$$

Using the Bianchi identities, it has been shown in [13] that, up to some boundary terms, the Lagrangian $\mathcal{L}_{1}$ has 20 couplings, each with an arbitrary parameter. Three of these parameters are unambiguous because they are not changed under field redefinitions, and all others are ambiguous. The field redefinition freedom then has been used to show that only 5 couplings among the ambiguous couplings are essential and all others are arbitrary. To find the minimal
independent couplings, one sets the arbitrary parameters to zero [13]. In this section, we are going to rederive the 8 independent couplings by using a systematic method for using total derivative terms, applying the field redefinitions and the Bianchi identities, that can easily be extended to the higher order couplings.

Using the package "xAct", one finds the most general gauge invariant Lagrangian at order $\alpha^{\prime}$ has the following couplings:

$$
\begin{align*}
\mathcal{L}_{1}= & B_{1} R_{\alpha \beta \gamma \delta} R^{\alpha \beta \gamma \delta}+B_{2} R_{\alpha \gamma \beta \delta} R^{\alpha \beta \gamma \delta}+B_{3} R_{\alpha \beta} R^{\alpha \beta}+B_{4} R^{2}+B_{5} \nabla_{\beta} \nabla_{\alpha} R^{\alpha \beta}+B_{6} \nabla_{\alpha} \nabla^{\alpha} R \\
& +B_{7} R^{\alpha \beta} \nabla_{\beta} \nabla_{\alpha} \Phi+B_{8} R_{\alpha \beta} \nabla^{\alpha} \Phi \nabla^{\beta} \Phi+B_{9} R \nabla_{\alpha} \nabla^{\alpha} \Phi+B_{10} R \nabla_{\alpha} \Phi \nabla^{\alpha} \Phi+B_{11} \nabla^{\alpha} \Phi \nabla_{\beta} R_{\alpha}{ }^{\beta} \\
& +B_{12} \nabla_{\alpha} \Phi \nabla^{\alpha} R+B_{13} \nabla_{\beta} \nabla^{\beta} \nabla_{\alpha} \nabla^{\alpha} \Phi+B_{14} \nabla_{\beta} \nabla_{\alpha} \nabla^{\beta} \nabla^{\alpha} \Phi+B_{15} \nabla_{\alpha} \nabla_{\beta} \nabla^{\beta} \Phi \nabla^{\alpha} \Phi \\
& +B_{16} \nabla^{\alpha} \Phi \nabla_{\beta} \nabla^{\beta} \nabla_{\alpha} \Phi+B_{17} \nabla_{\beta} \nabla_{\alpha} \Phi \nabla^{\beta} \nabla^{\alpha} \Phi+B_{18} \nabla_{\alpha} \nabla^{\alpha} \Phi \nabla_{\beta} \nabla^{\beta} \Phi \\
& +B_{19} \nabla^{\alpha} \Phi \nabla_{\beta} \nabla_{\alpha} \Phi \nabla^{\beta} \Phi+B_{20} \nabla_{\alpha} \Phi \nabla^{\alpha} \Phi \nabla_{\beta} \nabla^{\beta} \Phi+B_{21} \nabla_{\alpha} \Phi \nabla^{\alpha} \Phi \nabla_{\beta} \Phi \nabla^{\beta} \Phi \\
& +B_{22} H_{\alpha}^{\delta \epsilon} H^{\alpha \beta \gamma} H_{\beta \delta}^{\zeta} H_{\gamma \epsilon \zeta}+B_{23} H_{\alpha \beta}^{\delta} H^{\alpha \beta \gamma} H_{\gamma}^{\epsilon \zeta} H_{\delta \epsilon \zeta}+B_{24} H_{\alpha \beta \gamma} H^{\alpha \beta \gamma} H_{\delta \epsilon \zeta} H^{\delta \epsilon \zeta} \\
& +B_{25} H^{\alpha \beta \gamma} \nabla_{\delta} \nabla^{\delta} H_{\alpha \beta \gamma}+B_{26} H^{\alpha \beta \gamma} \nabla_{\delta} \nabla_{\gamma} H_{\alpha \beta}^{\delta}+B_{27} H^{\alpha \beta \gamma} \nabla_{\gamma} \nabla_{\delta} H_{\alpha \beta}^{\delta}+B_{28} \nabla_{\delta} H_{\alpha \beta \gamma} \nabla^{\delta} H^{\alpha \beta \gamma} \\
& +B_{29} \nabla_{\alpha} H^{\alpha \beta \gamma} \nabla_{\delta} H_{\beta \gamma}^{\delta}+B_{30} \nabla_{\gamma} H_{\alpha \beta \delta} \nabla^{\delta} H^{\alpha \beta \gamma}+B_{31} H_{\alpha}^{\delta \epsilon} H^{\alpha \beta \gamma} R_{\beta \gamma \delta \epsilon}+B_{32} H_{\alpha}^{\delta \epsilon} H^{\alpha \beta \gamma} R_{\beta \delta \gamma \epsilon} \\
& +B_{33} H_{\alpha}^{\gamma \delta} H_{\beta \gamma \delta} R^{\alpha \beta}+B_{34} H_{\alpha \beta \gamma} H^{\alpha \beta \gamma} R+B_{35} H^{\beta \gamma \delta} \nabla_{\alpha} H_{\beta \gamma \delta} \nabla^{\alpha} \Phi+B_{36} H^{\beta \gamma \delta} \nabla^{\alpha} \Phi \nabla_{\delta} H_{\alpha \beta \gamma} \\
& +B_{37} H_{\alpha}^{\beta \gamma} \nabla^{\alpha} \Phi \nabla_{\delta} H_{\beta \gamma}^{\delta}+B_{38} H_{\beta \gamma \delta} H^{\beta \gamma \delta} \nabla_{\alpha} \nabla^{\alpha} \Phi+B_{39} H_{\alpha}^{\gamma \delta} H_{\beta \gamma \delta} \nabla^{\beta} \nabla^{\alpha} \Phi \\
& +B_{40} H_{\beta \gamma \delta} H^{\beta \gamma \delta} \nabla_{\alpha} \Phi \nabla^{\alpha} \Phi+B_{41} H_{\alpha}^{\gamma \delta} H_{\beta \gamma \delta} \nabla^{\alpha} \Phi \nabla^{\beta} \Phi, \tag{4}
\end{align*}
$$

where $B_{1}, \ldots, B_{41}$ are some parameters. The above couplings are not all independent. Some of them are related by total derivative terms, some of them are related by field redefinitions, and some others are related by various Bianchi identities.

To remove the total derivative terms from the above couplings, we consider the most general total derivative terms at order $\alpha^{\prime}$ which has the following structure:

$$
\begin{equation*}
\alpha^{\prime} \int d^{D} x \sqrt{-g} e^{-2 \Phi} \mathcal{J}_{1}=\alpha^{\prime} \int d^{D} x \sqrt{-g} \nabla_{\alpha}\left(e^{-2 \Phi} J_{1}^{\alpha}\right) \tag{5}
\end{equation*}
$$

where the vector $J^{\alpha}$ is all possible covariant and gauge invariant terms at three-derivative level, i.e.,

$$
\begin{align*}
J_{1}^{\alpha}= & J_{1} \nabla_{\beta} R^{\alpha \beta}+J_{2} \nabla^{\alpha} R+J_{3} R^{\alpha}{ }_{\beta} \nabla^{\beta} \Phi+J_{4} R \nabla^{\alpha} \Phi+J_{5} \nabla_{\beta} \nabla^{\beta} \nabla^{\alpha} \Phi+J_{6} \nabla^{\alpha} \nabla_{\beta} \nabla^{\beta} \Phi \\
& +J_{7} \nabla_{\beta} \nabla^{\alpha} \Phi \nabla^{\beta} \Phi+J_{8} \nabla^{\alpha} \Phi \nabla_{\beta} \nabla^{\beta} \Phi+J_{9} \nabla^{\alpha} \Phi \nabla_{\beta} \Phi \nabla^{\beta} \Phi+J_{10} H^{\beta \gamma \delta} \nabla^{\alpha} H_{\beta \gamma \delta} \\
& +J_{11} H^{\beta \gamma \delta} \nabla_{\delta} H^{\alpha}{ }_{\beta \gamma}+J_{12} H^{\alpha \beta \gamma} \nabla_{\delta} H_{\beta \gamma}^{\delta}+J_{13} H_{\beta \gamma \delta} H^{\beta \gamma \delta} \nabla^{\alpha} \Phi+J_{14} H^{\alpha \gamma \delta} H_{\beta \gamma \delta} \nabla^{\beta} \Phi \tag{6}
\end{align*}
$$

where the coefficients $J_{1}, \ldots, J_{14}$ are arbitrary parameters. Inserting this into (5), one finds

$$
\begin{align*}
\mathcal{J}_{1}= & J_{1} \nabla_{\beta} \nabla_{\alpha} R^{\alpha \beta}+J_{2} \nabla_{\alpha} \nabla^{\alpha} R+J_{4} R \nabla_{\alpha} \nabla^{\alpha} \Phi+\left(-2 J_{2}+J_{4}\right) \nabla_{\alpha} \Phi \nabla^{\alpha} R \\
& +\left(-2 J_{10}+2 J_{13}\right) H^{\beta \gamma \delta} \nabla_{\alpha} H_{\beta \gamma \delta} \nabla^{\alpha} \Phi-2 J_{13} H_{\beta \gamma \delta} H^{\beta \gamma \delta} \nabla_{\alpha} \Phi \nabla^{\alpha} \Phi-2 J_{4} R \nabla_{\alpha} \Phi \nabla^{\alpha} \Phi \\
& +\left(-2 J_{6}+J_{8}\right) \nabla_{\alpha} \nabla_{\beta} \nabla^{\beta} \Phi \nabla^{\alpha} \Phi+\left(-2 J_{1}+J_{3}\right) \nabla^{\alpha} \Phi \nabla_{\beta} R_{\alpha}^{\beta}+J_{3} R^{\alpha \beta} \nabla_{\beta} \nabla_{\alpha} \Phi \\
& +J_{8} \nabla_{\alpha} \nabla^{\alpha} \Phi \nabla_{\beta} \nabla^{\beta} \Phi+\left(-2 J_{8}+J_{9}\right) \nabla_{\alpha} \Phi \nabla^{\alpha} \Phi \nabla_{\beta} \nabla^{\beta} \Phi+\left(-2 J_{5}+J_{7}\right) \nabla^{\alpha} \Phi \nabla_{\beta} \nabla^{\beta} \nabla_{\alpha} \Phi \\
& +J_{6} \nabla_{\beta} \nabla^{\beta} \nabla_{\alpha} \nabla^{\alpha} \Phi-2 J_{14} H_{\alpha}^{\gamma \delta} H_{\beta \gamma \delta} \nabla^{\alpha} \Phi \nabla^{\beta} \Phi-2 J_{3} R_{\alpha \beta} \nabla^{\alpha} \Phi \nabla^{\beta} \Phi-2 J_{9} \nabla_{\alpha} \Phi \nabla^{\alpha} \Phi \nabla_{\beta} \Phi \nabla^{\beta} \Phi \\
& +\left(-2 J_{7}+2 J_{9}\right) \nabla^{\alpha} \Phi \nabla_{\beta} \nabla_{\alpha} \Phi \nabla^{\beta} \Phi+J_{14} H_{\alpha}^{\gamma \delta} H_{\beta \gamma \delta} \nabla^{\beta} \nabla^{\alpha} \Phi+J_{7} \nabla_{\beta} \nabla_{\alpha} \Phi \nabla^{\beta} \nabla^{\alpha} \Phi \\
& +J_{12} H^{\alpha \beta \gamma} \nabla_{\gamma} \nabla_{\delta} H_{\alpha \beta}^{\delta}+\left(-2 J_{11}+J_{14}\right) H^{\beta \gamma \delta} \nabla^{\alpha} \Phi \nabla_{\delta} H_{\alpha \beta \gamma}+J_{12} \nabla_{\alpha} H^{\alpha \beta \gamma} \nabla_{\delta} H_{\beta \gamma}^{\delta} \\
& +\left(-2 J_{12}+J_{14}\right) H_{\alpha}^{\beta \gamma} \nabla^{\alpha} \Phi \nabla_{\delta} H_{\beta \gamma}^{\delta}+J_{11} H^{\alpha \beta \gamma} \nabla_{\delta} \nabla_{\gamma} H_{\alpha \beta}^{\delta}+J_{10} H^{\alpha \beta \gamma} \nabla_{\delta} \nabla^{\delta} H_{\alpha \beta \gamma} \\
& +J_{11} \nabla_{\gamma} H_{\alpha \beta \delta} \nabla^{\delta} H^{\alpha \beta \gamma}+J_{10} \nabla_{\delta} H_{\alpha \beta \gamma} \nabla^{\delta} H^{\alpha \beta \gamma}+J_{13} H_{\beta \gamma \delta} H^{\beta \gamma \delta} \nabla_{\alpha} \nabla^{\alpha} \Phi+J_{5} \nabla_{\beta} \nabla_{\alpha} \nabla^{\beta} \nabla^{\alpha} \Phi . \tag{7}
\end{align*}
$$

One is free to add $\mathcal{J}_{1}$ to $\mathcal{L}_{1}$ and choose the parameters $J_{1}, \ldots, J_{14}$ to reduce the couplings in (4).

The couplings in $\mathcal{J}_{1}+\mathcal{L}_{1}$, however, are in fixed field variables. One is free to change the field variables as

$$
\begin{align*}
g_{\mu \nu} & \rightarrow g_{\mu \nu}+\alpha^{\prime} \delta g_{\mu \nu}^{(1)} \\
B_{\mu \nu} & \rightarrow B_{\mu \nu}+\alpha^{\prime} \delta B_{\mu \nu}^{(1)} \\
\Phi & \rightarrow \Phi+\alpha^{\prime} \delta \Phi^{(1)}, \tag{8}
\end{align*}
$$

where the tensors $\delta g_{\mu \nu}^{(1)}, \delta B_{\mu \nu}^{(1)}$, and $\delta \Phi^{(1)}$ are all possible covariant and gauge invariant terms at two-derivative level, i.e.,

$$
\begin{align*}
\delta g_{\mu \nu}^{(1)}= & a_{1} R_{\mu \nu}+a_{2} H_{\mu}^{\alpha \beta} H_{\nu \alpha \beta}+a_{3} \nabla_{\nu} \nabla_{\mu} \Phi+a_{4} \nabla_{\mu} \Phi \nabla_{\nu} \Phi \\
& +g_{\mu \nu}\left(a_{5} R+a_{6} H_{\alpha \beta \gamma} H^{\alpha \beta \gamma}\right. \\
& \left.+a_{7} \nabla_{\alpha} \nabla^{\alpha} \Phi+a_{8} \nabla_{\alpha} \Phi \nabla^{\alpha} \Phi\right) \\
\delta B_{\mu \nu}^{(1)}= & a_{9} \nabla_{\alpha} H_{\mu \nu}^{\alpha}+a_{10} H_{\mu \nu \alpha} \nabla^{\alpha} \Phi \\
\delta \Phi^{(1)}= & a_{11} R+a_{12} H_{\alpha \beta \gamma} H^{\alpha \beta \gamma}+a_{13} \nabla_{\alpha} \nabla^{\alpha} \Phi \\
& +a_{14} \nabla_{\alpha} \Phi \nabla^{\alpha} \Phi . \tag{9}
\end{align*}
$$

The coefficients $a_{1}, \ldots, a_{14}$ are arbitrary parameters. When the field variables in $\sqrt{-g} e^{-2 \Phi}\left(\mathcal{J}_{1}+\mathcal{L}_{1}\right)$ are changed according to the above field redefinitions, they produce some couplings at order $\alpha^{12}$ in which we are not interested in this section. However, when the field variables in $S_{0}$ are changed, up to some total derivative terms, the following couplings at order $\alpha^{\prime}$ are produced:

$$
\begin{align*}
\delta S_{0}= & \frac{\delta S_{0}}{\delta g_{\alpha \beta}} \delta g_{\alpha \beta}^{(1)}+\frac{\delta S_{0}}{\delta B_{\alpha \beta}} \delta B_{\alpha \beta}^{(1)}+\frac{\delta S_{0}}{\delta \Phi} \delta \Phi^{(1)} \\
\equiv & \int d^{D} x \sqrt{-g} e^{-2 \Phi} \mathcal{K}_{1} \\
= & \int d^{D} x \sqrt{-g} e^{-2 \Phi}\left[\left(\frac{1}{2} \nabla_{\gamma} H^{\alpha \beta \gamma}-H^{\alpha \beta}{ }_{\gamma} \nabla^{\gamma} \Phi\right) \delta B_{\alpha \beta}^{(1)}\right. \\
& -\left(R^{\alpha \beta}-\frac{1}{4} H^{\alpha \gamma \delta} H^{\beta}{ }_{\gamma \delta}+2 \nabla^{\beta} \nabla^{\alpha} \Phi\right) \delta g_{\alpha \beta}^{(1)} \\
& -2\left(R-\frac{1}{12} H_{\alpha \beta \gamma} H^{\alpha \beta \gamma}+4 \nabla_{\alpha} \nabla^{\alpha} \Phi-4 \nabla_{\alpha} \Phi \nabla^{\alpha} \Phi\right) \\
& \left.\times\left(\delta \Phi^{(1)}-\frac{1}{4} \delta g^{(1) \mu}{ }_{\mu}\right)\right] . \tag{10}
\end{align*}
$$

Replacing (9) into (10), one finds

$$
\begin{align*}
\mathcal{K}_{1}= & -a_{1} R_{\alpha \beta} R^{\alpha \beta}+\frac{1}{2}\left(a_{1}-4 a_{11}+a_{5}(-2+D)\right) R^{2}+\frac{1}{4} a_{2} H_{\alpha \beta}^{\delta} H^{\alpha \beta \gamma} H_{\gamma} \epsilon^{\epsilon \zeta} H_{\delta \epsilon \zeta} \\
& +\frac{1}{24}\left(4 a_{12}-a_{2}-a_{6}(-6+D)\right) H_{\alpha \beta \gamma} H^{\alpha \beta \gamma} H_{\delta \epsilon \zeta} H^{\delta \epsilon \zeta}+\frac{1}{4}\left(a_{1}-4 a_{2}\right) H_{\alpha}^{\gamma \delta} H_{\beta \gamma \delta} R^{\alpha \beta} \\
& +\frac{1}{24}\left(-a_{1}+4 a_{11}-48 a_{12}+12 a_{2}+6 a_{5}-24 a_{6}-a_{5} D+12 a_{6} D\right) H_{\alpha \beta \gamma} H^{\alpha \beta \gamma} R \\
& +\frac{1}{24}\left(-192 a_{12}+4 a_{13}+48 a_{2}-a_{3}-48 a_{6}+6 a_{7}+48 a_{6} D-a_{7} D\right) H_{\beta \gamma \delta} H^{\beta \gamma \delta} \nabla_{\alpha} \nabla^{\alpha} \Phi \\
& +\frac{1}{2}\left(4 a_{1}-16 a_{11}-4 a_{13}+a_{3}-4 a_{5}-2 a_{7}+4 a_{5} D+a_{7} D\right) R \nabla_{\alpha} \nabla^{\alpha} \Phi \\
& +\frac{1}{24}\left(192 a_{12}+4 a_{14}-48 a_{2}-a_{4}+6 a_{8}-48 a_{6} D-a_{8} D\right) H_{\beta \gamma \delta} H^{\beta \gamma \delta} \nabla_{\alpha} \Phi \nabla^{\alpha} \Phi \\
& +\frac{1}{2}\left(-4 a_{1}+16 a_{11}-4 a_{14}+a_{4}-2 a_{8}-4 a_{5} D+a_{8} D\right) R \nabla_{\alpha} \Phi \nabla^{\alpha} \Phi+\left(-2 a_{1}-a_{3}\right) R^{\alpha \beta} \nabla_{\beta} \nabla_{\alpha} \Phi \\
& +2\left(-4 a_{13}+a_{3}+a_{7}(-1+D)\right) \nabla_{\alpha} \nabla^{\alpha} \Phi \nabla_{\beta} \nabla^{\beta} \Phi+\frac{1}{4}\left(-4 a_{10}+a_{4}\right) H_{\alpha}^{\gamma \delta} H_{\beta \gamma \delta} \nabla^{\alpha} \Phi \nabla^{\beta} \Phi \\
& +2\left(4 a_{13}-4 a_{14}-a_{3}+a_{4}-a_{8}-a_{7} D+a_{8} D\right) \nabla_{\alpha} \Phi \nabla^{\alpha} \Phi \nabla_{\beta} \nabla^{\beta} \Phi-a_{4} R_{\alpha \beta} \nabla^{\alpha} \Phi \nabla^{\beta} \Phi \\
& +\left(8 a_{14}-2\left(a_{4}+a_{8} D\right)\right) \nabla_{\alpha} \Phi \nabla^{\alpha} \Phi \nabla_{\beta} \Phi \nabla^{\beta} \Phi-2 a_{4} \nabla^{\alpha} \Phi \nabla_{\beta} \nabla_{\alpha} \Phi \nabla^{\beta} \Phi \\
& +\frac{1}{4}\left(-8 a_{2}+a_{3}\right) H_{\alpha}^{\gamma \delta} H_{\beta \gamma \delta} \nabla^{\beta} \nabla^{\alpha} \Phi-2 a_{3} \nabla_{\beta} \nabla_{\alpha} \Phi \nabla^{\beta} \nabla^{\alpha} \Phi \\
& +\frac{1}{2} a_{9} \nabla_{\alpha} H^{\alpha \beta \gamma} \nabla_{\delta} H_{\beta \gamma}^{\delta}+\frac{1}{2}\left(a_{10}-2 a_{9}\right) H_{\alpha}^{\beta \gamma} \nabla^{\alpha} \Phi \nabla_{\delta} H_{\beta \gamma}^{\delta} \tag{11}
\end{align*}
$$

where $D$ is the number of spacetime dimensions.

Not all arbitrary parameters $a_{1}, \ldots, a_{14}$ produce nonzero $\mathcal{K}_{1}$. For some relations between the parameters, one finds the field redefinition (8) is the general coordinate transformation, which obviously leaves $\mathcal{K}_{1}$ invariant up to some boundary term. In fact under the coordinate transformation $x^{\mu} \rightarrow x^{\mu}+\varepsilon^{\mu}=x^{\mu}+a \nabla^{\mu} \Phi$, one has the following transformations for fields:

$$
\begin{align*}
\delta g_{\mu \nu} & =\nabla_{\mu} \varepsilon_{\nu}+\nabla_{\nu} \varepsilon_{\mu}=2 a \nabla_{\mu} \nabla_{\nu} \Phi \\
\delta \Phi & =\varepsilon^{\mu} \nabla_{\mu} \Phi=a \nabla^{\mu} \Phi \nabla_{\mu} \Phi \\
\delta B_{\mu \nu} & =\varepsilon^{\gamma} H_{\gamma \mu \nu}=a H_{\alpha \mu \nu} \nabla^{\alpha} \Phi . \tag{12}
\end{align*}
$$

Hence, if $a_{3}=2 a_{10}$ and $a_{14}=a_{10}$, the corresponding field redefinitions in (9) are a coordinate transformation which leaves $\mathcal{K}_{1}$ invariant, up to some boundary term. For some other relations between the parameters, $\mathcal{K}_{1}$ may still be invariant; however, the corresponding transformation is not the coordinate transformation. If one removes the parameters that leave $\mathcal{K}_{1}$ invariant, then the remaining parameters all would be fixed after using the field redefinitions. On the other hand, if one keeps all parameters $a_{1}, \ldots, a_{14}$, then some of the parameters remain arbitrary after using the field redefinitions. We use this latter method and work with all parameters $a_{1}, \ldots, a_{14}$.

One is free to add $\mathcal{K}_{1}$ to $\mathcal{L}_{1}+\mathcal{J}_{1}$ and choose the parameters $J_{1}, \ldots, J_{14}$ and $a_{1}, \ldots, a_{14}$ to reduce the couplings in (4). The Bianchi identities and the commutation relation of the covariant derivatives, however, are not yet used in $\mathcal{L}_{1}+\mathcal{J}_{1}+\mathcal{K}_{1}$. They are

$$
\begin{align*}
R_{\alpha[\beta \gamma \delta]} & =0, & & \nabla_{[\mu} R_{\alpha \beta] \gamma \delta}=0, \\
\nabla_{[\mu} H_{\alpha \beta \gamma]} & =0, & & {[\nabla, \nabla] \mathcal{O}=R \mathcal{O} . } \tag{13}
\end{align*}
$$

They can further reduce the independent couplings in (4). The above identities may be contracted with tensors $R, H$, $\nabla \Phi$, and their derivatives with arbitrary parameters, and then
add them to $\mathcal{L}_{1}+\mathcal{J}_{1}+\mathcal{K}_{1}$. By manipulating the arbitrary parameters, one may find the independent couplings in (4). Instead of imposing the identities (13) with arbitrary parameters, we use the locally inertial frame in which the above identities are almost automatically satisfied.

In the locally inertial frame, the metric $g_{\alpha \beta}$ takes its canonical form and its first derivatives are all vanished, i.e.,

$$
g_{\alpha \beta}=\eta_{\alpha \beta}, \quad \partial_{\mu} g_{\alpha \beta}=0 .
$$

The second and higher derivatives of the metric, however, are nonzero. In this coordinate, by rewriting the covariant derivative in terms of partial derivatives, one finds, except the Bianchi identity $d H=0$, all other identities in (13) are satisfied. To satisfy the Bianchi identity $d H=0$ as well, in the couplings which involve derivatives of $H$, we rewrite $H$ in terms of $B$-field, i.e., $H_{\mu \nu \alpha}=\partial_{\mu} B_{\nu \alpha}+\partial_{\alpha} B_{\mu \nu}+\partial_{\nu} B_{\alpha \mu}$. When writing the couplings $\mathcal{L}_{1}+\mathcal{J}_{1}+\mathcal{K}_{1}$ in this local frame, then all resulting terms in $\mathcal{L}_{1}+\mathcal{J}_{1}+\mathcal{K}_{1}$ become independent. Using the arbitrary parameters $J_{1}, \ldots, J_{14}$ and $a_{1}, \ldots, a_{14}$, one may find the couplings in many different schemes.

To clarify this point, one may write the final form of the couplings as $\mathcal{L}_{1}^{\prime}$ which is the same as the couplings (4) with different parameters $B_{1}^{\prime}, \ldots, B_{41}^{\prime}$. Then writing

$$
\begin{equation*}
\mathcal{L}_{1}^{\prime}-\mathcal{L}_{1}=\mathcal{J}_{1}+\mathcal{K}_{1} \tag{14}
\end{equation*}
$$

in the local frame, one finds some relations between the arbitrary parameters $J_{1}, \ldots, J_{14}, a_{1}, \ldots, a_{14}$, and $\delta B_{1}, \ldots, \delta B_{41}$, where $\delta B_{i}=B_{i}^{\prime}-B_{i}$. These are very lengthy expressions that we do not write here explicitly. There are two parameters $a_{10}, a_{8}$ in these relations which remain unfixed.

The Eq. (14) produces the following 8 relations between only $\delta B_{1}, \ldots, \delta B_{41}$ as well:

$$
\begin{align*}
& \delta B_{22}=0 \\
& 2 \delta B_{1}+\delta B_{2}=0 \\
& 16 \delta B_{4}-8 \delta B_{9}-4 \delta B_{10}+4 \delta B_{18}+2 \delta B_{20}+\delta B_{21}=0 \\
& 6 \delta B_{25}+2 \delta B_{26}-6 \delta B_{28}-2 \delta B_{30}-2 \delta B_{31}-\delta B_{32}=0 \\
& \delta B_{3}+16 \delta B_{23}+12 \delta B_{25}+4 \delta B_{26}-12 \delta B_{28}-4 \delta B_{30}+4 \delta B_{33}=0 \\
& 2 \delta B_{14}+\delta B_{16}-\delta B_{17}+16 \delta B_{29}-4 \delta B_{3}+8 \delta B_{37}+4 \delta B_{41}+2 \delta B_{7}+\delta B_{8}=0 \\
& 10 \delta B_{10}-4 \delta B_{11}-8 \delta B_{12}+8 \delta B_{13}+4 \delta B_{14}+4 \delta B_{15}+2 \delta B_{16}-8 \delta B_{18}+\delta B_{19}-2 \delta B_{20} \\
& \quad-96 \delta B_{34}+48 \delta B_{38}-80 \delta B_{4}+24 \delta B_{40}-8 \delta B_{5}-16 \delta B_{6}-2 \delta B_{8}+28 \delta B_{9}=0 \\
& 5 \delta B_{11}+10 \delta B_{12}-4 \delta B_{13}-4 \delta B_{14}-2 \delta B_{15}-2 \delta B_{16}+2 \delta B_{17}+2 \delta B_{18}+288 \delta B_{24}+24 \delta B_{25} \\
& \quad+8 \delta B_{26}+12 \delta B_{3}+8 \delta B_{33}+120 \delta B_{34}+12 \delta B_{35}+4 \delta B_{36}-24 \delta B_{38}-4 \delta B_{39}+50 \delta B_{4}+10 \delta B_{5} \\
& \quad+20 \delta B_{6}-5 \delta B_{7}-10 \delta B_{9}=0 . \tag{15}
\end{align*}
$$

The first relation, $\delta B_{22}=0$, indicates that the parameter $B_{22}^{\prime}=B_{22}$ is not changed by the field redefinition, by adding the total derivative term or by using the Bianchi identities. It is an unambiguous parameter. All other relations indicate that the other parameters are ambiguous parameters because they are changed by the field redefinition, by the total derivative term or by the Bianchi identities.

To find the minimum number of couplings in $\mathcal{L}_{1}^{\prime}$, one may choose 33 parameters in $\mathcal{L}_{1}^{\prime}$ to be zero. These parameters, however, should change the 8 equations in (15) to 8 equations $\delta B_{i}=f_{i}\left(B_{1}, \ldots, B_{41}\right)$, where $i$ is 8 numbers among $1, \ldots, 41$ depending on the scheme that one uses for the terms in $\mathcal{L}_{1}^{\prime}$ to be zero. It is obvious that one of them is $i=22$ for which $f_{22}=0$. If one chooses $B_{2}^{\prime}=0$, then the second equation above indicates that $\delta B_{1}=B_{2} / 2$. Alternatively, if one chooses $B_{1}^{\prime}=0$, then the second equation becomes $\delta B_{2}=2 B_{1}$. Similarly for all other equations which have many different schemes. In any scheme, the nonzero parameters in $\mathcal{L}_{1}^{\prime}$ are $B_{i}^{\prime}=B_{i}+f_{i}\left(B_{1}, \ldots, B_{41}\right)$.

We choose a set of zero parameters in $\mathcal{L}_{1}^{\prime}$ to be those that their corresponding couplings have terms with more than two derivatives or have $R, R_{\mu \nu}, \nabla_{\mu} H^{\mu \alpha \beta}, \nabla_{\mu} \nabla^{\mu} \Phi$. There are, however, 24 such parameters. One may set to zero the other 9 parameters in $\mathcal{L}_{1}^{\prime}$ such that the remaining nonzero parameters become the ones considered in [13], i.e., $B_{1}^{\prime}, B_{21}^{\prime}, B_{22}^{\prime}, B_{23}^{\prime}, B_{24}^{\prime}, B_{31}^{\prime}, B_{40}^{\prime}, B_{41}^{\prime}$.

However, we choose two other schemes. In one scheme, we write the couplings as the following:

$$
\begin{equation*}
\mathcal{L}_{1}^{\prime}=\mathcal{L}_{1}^{1}+\mathcal{L}_{1}^{2}, \tag{16}
\end{equation*}
$$

where $\mathcal{L}_{1}^{1}$ includes the minimum number of couplings which do not include the dilaton, i.e.,

$$
\begin{align*}
\mathcal{L}_{1}^{1}= & B_{1} R_{\alpha \gamma \beta \delta} R^{\alpha \beta \gamma \delta}+B_{2} H_{\alpha}^{\delta \epsilon} H^{\alpha \beta \gamma} H_{\beta \delta}{ }^{\zeta} H_{\gamma \epsilon \zeta} \\
& +B_{3} H_{\alpha \beta}{ }^{\delta} H^{\alpha \beta \gamma} H_{\gamma}{ }^{\epsilon \zeta} H_{\delta \epsilon \zeta}+B_{4} H_{\alpha}^{\delta \epsilon} H^{\alpha \beta \gamma} R_{\beta \delta \gamma \epsilon}, \tag{17}
\end{align*}
$$

and $\mathcal{L}_{1}^{2}$ includes the other couplings which all include nontrivially the dilaton, i.e.,

$$
\begin{align*}
\mathcal{L}_{1}^{2}= & B_{5} H_{\beta \gamma \delta} H^{\beta \gamma \delta} \nabla_{\alpha} \Phi \nabla^{\alpha} \Phi+B_{6} H_{\alpha}{ }^{\gamma \delta} H_{\beta \gamma \delta} \nabla^{\alpha} \Phi \nabla^{\beta} \Phi \\
& +B_{7} H_{\alpha}{ }^{\gamma \delta} H_{\beta \gamma \delta} \nabla^{\beta} \nabla^{\alpha} \Phi+B_{8} \nabla_{\alpha} \Phi \nabla^{\alpha} \Phi \nabla_{\beta} \Phi \nabla^{\beta} \Phi, \tag{18}
\end{align*}
$$

where we have also dropped the prime on the coefficients and relabeled then from 1 to 8 . The reason for the couplings in (17) to be the minimum number of couplings which do not include the dilaton, is that when one sets the dilaton to be constant, there are only 4 relations between $\delta B \mathrm{~s}$.

In the other scheme, we write the couplings as the following:

$$
\begin{equation*}
\mathcal{L}_{1}^{\prime}=\mathcal{L}_{1}^{1}+\mathcal{L}_{1}^{3} \tag{19}
\end{equation*}
$$

where $\mathcal{L}_{1}^{1}$ is the same as in (17), and $\mathcal{L}_{1}^{3}$ includes 4 couplings other than those that appear in $\mathcal{L}_{1}^{1}$ in which the dilaton does not appear, i.e.,

$$
\begin{align*}
\mathcal{L}_{1}^{3}= & B_{5} R^{2}+B_{6} H_{\alpha \beta \gamma} H^{\alpha \beta \gamma} R+B_{7} \nabla_{\alpha} H^{\alpha \beta \gamma} \nabla_{\delta} H_{\beta \gamma}{ }^{\delta} \\
& +B_{8} H_{\alpha \beta \gamma} H^{\alpha \beta \gamma} H_{\delta \epsilon \zeta} H^{\delta \epsilon \zeta} . \tag{20}
\end{align*}
$$

The 8 parameters in (16) or (19) have been fixed by comparison with the three- and four-point string amplitudes [13]. They have been also fixed, up to an overall factor, by the T-duality constraint [32]. Only the parameters in $\mathcal{L}_{1}^{1}$ are nonzero!

The parameters in the field redefinitions $\delta g_{\alpha \beta}^{(1)}, \delta B_{\alpha \beta}^{(1)}$, and $\delta \Phi^{(1)}$ that change the action (4) to (16) or (19), are functions of $\delta B_{1}, \ldots, \delta B_{41}$ and $a_{10}, a_{8}$. In the above schemes, $\delta B_{i}=f_{i}\left(B_{1}, \ldots, B_{41}\right)$, where $i$ is 8 numbers among $1, \ldots, 41$, and all others are $\delta B_{j}=-B_{j}$. The parameter $a_{10}$ produces coordinate transformations in which we are not interested, and parameter $a_{8}$ produces the transformation that leaves $\mathcal{K}_{1}$ invariant. We will discuss more about this term in the next section.

## III. MINIMAL COUPLINGS AT ORDER $\alpha^{\prime 2}$

In this section we extend the calculations in the previous section to the order $\alpha^{\prime 2}$. We begin by writing all possible covariant and gauge invariant couplings at six-derivative order, i.e.,

$$
\begin{align*}
\mathcal{L}_{2}= & C_{1} R_{\alpha}{ }^{\epsilon}{ }_{\gamma}^{\zeta} R^{\alpha \beta \gamma \delta} R_{\beta \zeta \delta \epsilon}+C_{2} R_{\alpha \beta}^{\epsilon \zeta} R^{\alpha \beta \gamma \delta} R_{\gamma \epsilon \delta \zeta} \\
& +C_{3} H_{\alpha}{ }^{\gamma} \delta H_{\gamma}{ }^{\epsilon \zeta} R_{\beta \epsilon \delta \zeta} \nabla^{\beta} \nabla^{\alpha} \Phi+\cdots . \tag{21}
\end{align*}
$$

There are 705 such couplings; however, they are not independent. To remove the total derivative terms from the above couplings, we consider the most general total derivative terms at order $\alpha^{\prime 2}$ which has the following structure:

$$
\begin{equation*}
\alpha^{\prime 2} \int d^{D} x \sqrt{-g} e^{-2 \Phi} \mathcal{J}_{2}=\alpha^{\prime 2} \int d^{D} x \sqrt{-g} \nabla_{\alpha}\left(e^{-2 \Phi} J_{2}^{\alpha}\right) \tag{22}
\end{equation*}
$$

where the vector $J_{2}^{\alpha}$ is all possible covariant and gauge invariant terms at five-derivative level, i.e.,

$$
\begin{equation*}
J_{2}^{\alpha}=J_{1} \nabla^{\alpha} \Phi R^{2}+J_{2} H^{\alpha \mu \nu} H_{\beta \mu \nu} \nabla^{\beta} \Phi R+\cdots \tag{23}
\end{equation*}
$$

There are 315 such terms with arbitrary parameters. The corresponding $\mathcal{J}_{2}$ has 641 terms.

Now we consider the field redefinitions at order $\alpha^{\prime 2}$. Under the field redefinitions

$$
\begin{align*}
g_{\mu \nu} & \rightarrow g_{\mu \nu}+\alpha^{\prime} \delta g_{\mu \nu}^{(1)}+\alpha^{\prime 2} \delta g_{\mu \nu}^{(2)} \\
B_{\mu \nu} & \rightarrow B_{\mu \nu}+\alpha^{\prime} \delta B_{\mu \nu}^{(1)}+\alpha^{\prime 2} \delta B_{\mu \nu}^{(2)} \\
\Phi & \rightarrow \Phi+\alpha^{\prime} \delta \Phi^{(1)}+\alpha^{\prime 2} \delta \Phi^{(2)} \tag{24}
\end{align*}
$$

where the deformations at orders $\alpha^{\prime}$ and $\alpha^{\prime 2}$ are arbitrary, the actions $S_{0}$ and $S_{1}$ produce the following contributions at order $\alpha^{\prime 2}$, up to some total derivative terms:

$$
\begin{align*}
\delta S_{0}+\delta S_{1}= & \frac{\delta S_{0}}{\delta g_{\alpha \beta}} \delta g_{\alpha \beta}^{(2)}+\frac{\delta S_{0}}{\delta B_{\alpha \beta}} \delta B_{\alpha \beta}^{(2)}+\frac{\delta S_{0}}{\delta \Phi} \delta \Phi^{(2)} \\
& +\frac{\delta S_{1}}{\delta g_{\alpha \beta}} \delta g_{\alpha \beta}^{(1)}+\frac{\delta S_{1}}{\delta B_{\alpha \beta}} \delta B_{\alpha \beta}^{(1)}+\frac{\delta S_{1}}{\delta \Phi} \delta \Phi^{(1)} \\
& +S_{0}\left(\delta g^{(1)}, \delta g^{(1)}\right)+S_{0}\left(\delta g^{(1)}, \delta B^{(1)}\right) \\
& +S_{0}\left(\delta g^{(1)}, \delta \Phi^{(1)}\right)+S_{0}\left(\delta B^{(1)}, \delta B^{(1)}\right) \\
& +S_{0}\left(\delta B^{(1)}, \delta \Phi^{(1)}\right)+S_{0}\left(\delta \Phi^{(1)}, \delta \Phi^{(1)}\right) \tag{25}
\end{align*}
$$

where $S_{0}\left(\delta g^{(1)}, \delta g^{(1)}\right)$, which includes $\delta g^{(1)} \delta g^{(1)}$-terms, is resulted from replacing the transformation $g \rightarrow g+\alpha^{\prime} \delta g^{(1)}$, $B \rightarrow B+\alpha^{\prime} \delta B^{(1)}$, and $\Phi \rightarrow \Phi+\alpha^{\prime} \delta \Phi^{(1)}$ into $S_{0}$. Similarly for all other terms in the second and the third line above. Up to some total derivative terms, one may write $S_{0}\left(\delta g^{(1)}, \delta g^{(1)}\right)=(\cdots) \delta g^{(1)}$. Similarly for other terms in (25). As a result, one may rewrite the above equation as

$$
\begin{align*}
\delta S_{0}+\delta S_{1}= & \frac{\delta S_{0}}{\delta g_{\alpha \beta}} \delta g_{\alpha \beta}^{(2)}+\frac{\delta S_{0}}{\delta B_{\alpha \beta}} \delta B_{\alpha \beta}^{(2)}+\frac{\delta S_{0}}{\delta \Phi} \delta \Phi^{(2)} \\
& +\left(\frac{\delta S_{1}}{\delta g_{\alpha \beta}}+\cdots\right) \delta g_{\alpha \beta}^{(1)}+\left(\frac{\delta S_{1}}{\delta B_{\alpha \beta}}+\cdots\right) \delta B_{\alpha \beta}^{(1)} \\
& +\left(\frac{\delta S_{1}}{\delta \Phi}+\cdots\right) \delta \Phi^{(1)} \tag{26}
\end{align*}
$$

However, in the previous section, we have adjusted $\delta g_{\alpha \beta}^{(1)}$, $\delta B_{\alpha \beta}^{(1)}$, and $\delta \Phi^{(1)}$ so as to obtain the action $S_{1}$ with fixed parameters. All parameters in (9) are fixed except the two parameters $a_{10}, a_{8}$. The parameter $a_{10}$ which produces the
coordinate transformation should not be included in the field redefinitions, and the parameter $a_{8}$ which leaves $\mathcal{K}_{1}$ invariant but is not corresponding to the coordinate transformation may be included in the field redefinition. We call the corresponding field deformations $\delta \hat{g}_{\alpha \beta}^{(1)}, \delta \hat{B}_{\alpha \beta}^{(1)}$, and $\delta \hat{\Phi}^{(1)}$. In fact the equation $\mathcal{K}_{1}=0$ has the following solution:

$$
\begin{aligned}
a_{1} & =a_{2}=a_{4}=a_{9}=a_{10}=0 \\
-4 a_{5} & =48 a_{6}=-a_{7}=-\frac{16}{D-2} a_{11}=\frac{192}{D-6} a_{12} \\
& =-\frac{4}{D-1} a_{13}=\frac{4}{D} a_{14}=a_{8}
\end{aligned}
$$

The corresponding field deformations are

$$
\begin{align*}
\delta \hat{g}_{\mu \nu}^{(1)} & =-a_{8} g_{\mu \nu}\left(\frac{1}{4} R-\frac{1}{48} H_{\alpha \beta \gamma} H^{\alpha \beta \gamma}+\nabla_{\alpha} \nabla^{\alpha} \Phi-\nabla_{\alpha} \Phi \nabla^{\alpha} \Phi\right) \\
\delta \hat{B}_{\mu \nu}^{(1)} & =0 \\
\delta \hat{\Phi}^{(1)} & =-\frac{a_{8}}{4}\left(\frac{D-2}{4} R-\frac{D-6}{48} H_{\alpha \beta \gamma} H^{\alpha \beta \gamma}+(D-1) \nabla_{\alpha} \nabla^{\alpha} \Phi\right. \\
& \left.-D \nabla_{\alpha} \Phi \nabla^{\alpha} \Phi\right) . \tag{27}
\end{align*}
$$

Since we have already fixed the field redefinitions at order $\alpha^{\prime}$ to choose the schemes (16) or (19), one should consider only the above residual deformations in (26).

Up to some total derivative terms, (27) can be written as

$$
\begin{aligned}
\int d^{D} x \sqrt{-g} e^{-2 \Phi} \delta \hat{g}_{\mu \nu}^{(1)} & =\frac{1}{8} a_{8} g_{\mu \nu} \frac{\delta S_{0}}{\delta \Phi} \\
\delta \hat{B}_{\mu \nu}^{(1)} & =0
\end{aligned}
$$

$$
\begin{equation*}
\int d^{D} x \sqrt{-g} e^{-2 \Phi} \delta \hat{\Phi}^{(1)}=\frac{D}{8} a_{8} \frac{\delta S_{0}}{\delta \Phi}-\frac{1}{8} a_{8} g_{\mu \nu} \frac{\delta S_{0}}{\delta g_{\mu \nu}} \tag{28}
\end{equation*}
$$

Replacing it into (26), one can rewrite (26) as

$$
\begin{align*}
\delta S_{0}+\delta S_{1}= & \frac{\delta S_{0}}{\delta g_{\alpha \beta}} \delta g_{\alpha \beta}^{(2)}+\frac{\delta S_{0}}{\delta B_{\alpha \beta}} \delta B_{\alpha \beta}^{(2)}+\frac{\delta S_{0}}{\delta \Phi} \delta \Phi^{\prime(2)} \\
\equiv & \int d^{D} x \sqrt{-g} e^{-2 \Phi} \mathcal{K}_{2} \\
= & \int d^{D} x \sqrt{-g} e^{-2 \Phi}\left[\left(\frac{1}{2} \nabla_{\gamma} H^{\alpha \beta \gamma}-H^{\alpha \beta}{ }_{\gamma} \nabla^{\gamma} \Phi\right) \delta B_{\alpha \beta}^{(2)}-\left(R^{\alpha \beta}-\frac{1}{4} H^{\alpha \gamma \delta} H^{\beta}{ }_{\gamma \delta}+2 \nabla^{\beta} \nabla^{\alpha} \Phi\right) \delta g^{\prime}(2)\right. \\
& \left.-2\left(R-\frac{1}{12} H_{\alpha \beta \gamma} H^{\alpha \beta \gamma}+4 \nabla_{\alpha} \nabla^{\alpha} \Phi-4 \nabla_{\alpha} \Phi \nabla^{\alpha} \Phi\right)\left(\delta \Phi^{\prime(2)}-\frac{1}{4} \delta g^{\prime(2) \mu}{ }_{\mu}\right)\right] \tag{29}
\end{align*}
$$

where the deformations $\delta g^{\prime(2)}, \delta \Phi^{\prime(2)}$ and $\delta g_{\alpha \beta}^{(2)}, \delta \Phi^{(2)}$ have different parameters; however, since we have not yet fixed the parameters in $\delta g_{\alpha \beta}^{(2)}, \delta \Phi^{(2)}$, we consider the field redefinitions

$$
\begin{align*}
g_{\mu \nu} & \rightarrow g_{\mu \nu}+\alpha^{\prime} \delta g_{\mu \nu}^{(1)}+\alpha^{\prime 2} \delta g_{\mu \nu}^{\prime(2)} \\
B_{\mu \nu} & \rightarrow B_{\mu \nu}+\alpha^{\prime} \delta B_{\mu \nu}^{(1)}+\alpha^{\prime 2} \delta B_{\mu \nu}^{(2)} \\
\Phi & \rightarrow \Phi+\alpha^{\prime} \delta \Phi^{(1)}+\alpha^{\prime 2} \delta \Phi^{\prime(2)}, \tag{30}
\end{align*}
$$

in which the deformations at order $\alpha^{\prime}$ are all already fixed to find the action (16) or (19), and the deformations at order $\alpha^{\prime 2}$ are yet arbitrary.

The most general deformations at order $\alpha^{\prime 2}$ are:

$$
\begin{align*}
\delta \Phi^{\prime(2)} & =b_{1} H_{\alpha}{ }^{\delta \epsilon} H^{\alpha \beta \gamma} H_{\beta \delta}{ }^{\zeta} H_{\gamma \epsilon \zeta}+b_{2} H_{\alpha \beta}{ }^{\delta} H^{\alpha \beta \gamma} H_{\gamma}{ }^{\epsilon \zeta} H_{\delta \epsilon \zeta}+\cdots \\
\delta B_{\mu \nu}^{(2)} & =c_{1} R_{\mu \nu \beta \gamma} \nabla_{\alpha} H^{\alpha \beta \gamma}+c_{2} H^{\beta \gamma \delta} H_{\mu \nu}^{\alpha} \nabla_{\alpha} H_{\beta \gamma \delta}+\cdots \\
\delta g^{\prime(2)} & =d_{1} H_{\gamma \delta \epsilon} H^{\gamma \delta \epsilon} H_{\mu}^{\alpha \beta} H_{\nu \alpha \beta}+d_{2} H_{\beta}{ }^{\delta \epsilon} H_{\gamma \delta \epsilon} H_{\mu}^{\alpha \beta} H_{\nu \alpha}^{\gamma}+\cdots, \tag{31}
\end{align*}
$$

where $b_{1}, \ldots, b_{41}, c_{1}, \ldots, c_{81}$, and $d_{1}, \ldots, d_{121}$ are arbitrary parameters.

To find the independent couplings, we write the final form of the couplings as $\mathcal{L}_{2}^{\prime}$ which is the same as the couplings (21) with different parameters $C_{1}^{\prime}, \ldots, C_{705}^{\prime}$. Then writing

$$
\begin{equation*}
\mathcal{L}_{2}^{\prime}-\mathcal{L}_{2}=\mathcal{J}_{2}+\mathcal{K}_{2} \tag{32}
\end{equation*}
$$

in the local frame, one finds some relations between the arbitrary parameters of $\mathcal{L}_{2}, \mathcal{K}_{2}$, and $\delta C_{1}, \ldots, \delta C_{705}$ in which we are not interested, and also 60 relations between only $\delta C_{1}, \ldots, \delta C_{705}$ in which we are interested. Note that these relations are independent of the form of the fixed action at order $\alpha^{\prime}$, whether it is a minimal action or not (see [38] for the case that $B$-field is zero).

To find the minimum number of couplings in $\mathcal{L}_{2}^{\prime}$, one may choose 645 parameters in $\mathcal{L}_{2}^{\prime}$ to be zero. These parameters, however, should change the 60 equations among $\delta C_{1}, \ldots, \delta C_{705}$ to 60 equations $\delta C_{i}=g_{i}\left(C_{1}, \ldots, C_{705}\right)$, where $i$ is 60 numbers among $1, \ldots, 705$ depending on the scheme that one uses for the terms in $\mathcal{L}_{2}^{\prime}$ to be zero. As in the previous section we choose two schemes.

In one scheme, we choose a set of zero parameters in $\mathcal{L}_{2}^{\prime}$ to be those that their corresponding couplings have terms with more than two derivatives or have $R, R_{\mu \nu}, \nabla_{\mu} H^{\mu \alpha \beta}$, $\nabla_{\mu} \nabla^{\mu} \Phi$. There are 543 such couplings. We have found that it is consistent to set these parameters to zero, i.e., after setting these parameters to zero, there are still 60 equations between the reaming $\delta C$ s. There are still different schemes for choosing the remaining 102 parameters in $\mathcal{L}_{2}^{\prime}$ to be zero. We choose the scheme in which the minimum number of couplings in $\mathcal{L}_{2}^{\prime}$ to be:

$$
\begin{equation*}
\mathcal{L}_{2}^{\prime}=\mathcal{L}_{2}^{1}+\mathcal{L}_{2}^{2} \tag{33}
\end{equation*}
$$

where $\mathcal{L}_{2}^{1}$ has the minimum number of couplings in which the dilaton does not appear, i.e.,

$$
\begin{align*}
& \mathcal{L}_{2}^{1}=C_{1} R_{\alpha}{ }^{\epsilon} \gamma^{\zeta} R^{\alpha \beta \gamma \delta} R_{\beta \zeta \delta \epsilon}+C_{2} R_{\alpha \beta}{ }^{\epsilon \zeta} R^{\alpha \beta \gamma \delta} R_{\gamma \epsilon \delta \zeta}+C_{3} H_{\alpha}{ }^{\delta \epsilon} H^{\alpha \beta \gamma} H_{\beta \delta}{ }^{\zeta} H_{\gamma}{ }^{\imath \kappa} H_{\epsilon l}{ }^{\mu} H_{\zeta \kappa \mu} \\
& +C_{4} H_{\alpha \beta}{ }^{\delta} H^{\alpha \beta \gamma} H_{\gamma}{ }^{\epsilon \zeta} H_{\delta}{ }^{\imath \kappa} H_{\epsilon \zeta}{ }^{\mu} H_{\iota \kappa \mu}+C_{5} H_{\alpha \beta}{ }^{\delta} H^{\alpha \beta \gamma} H_{\gamma}{ }^{\epsilon \zeta} H_{\delta \epsilon}{ }^{l} H_{\zeta}{ }^{\kappa \mu} H_{\text {וкц }} \\
& +C_{6} H_{\alpha}{ }^{\delta \epsilon} H^{\alpha \beta \gamma} H_{\beta}{ }^{\zeta \iota} H_{\delta \zeta}{ }^{\kappa} R_{\gamma \epsilon I \kappa}+C_{7} H_{\alpha}{ }^{\delta \epsilon} H^{\alpha \beta \gamma} R_{\beta}{ }^{\zeta}{ }_{\delta}{ }^{\imath} R_{\gamma \zeta \epsilon \iota}+C_{8} H_{\alpha \beta}{ }^{\delta} H^{\alpha \beta \gamma} H_{\epsilon \zeta}{ }^{\kappa} H^{\epsilon \zeta \iota} R_{\gamma \iota \delta \kappa} \\
& +C_{9} H^{\alpha \beta \gamma} H^{\delta \epsilon \zeta} R_{\alpha \beta \delta}{ }^{l} R_{\gamma l \epsilon \zeta}+C_{10} H_{\alpha}{ }^{\delta \epsilon} H^{\alpha \beta \gamma} R_{\beta}{ }^{\zeta}{ }_{\delta}{ }^{l} R_{\gamma l \epsilon \zeta}+C_{11} H_{\alpha}{ }^{\delta \epsilon} H^{\alpha \beta \gamma} R_{\beta}{ }^{\zeta}{ }_{\gamma}{ }^{l} R_{\delta \zeta \epsilon l} \\
& +C_{12} H_{\alpha \beta}{ }^{\delta} H^{\alpha \beta \gamma} R_{\gamma}{ }^{\epsilon \zeta \iota} R_{\delta \zeta \epsilon \iota}+C_{13} H_{\alpha \beta}{ }^{\delta} H^{\alpha \beta \gamma} H_{\gamma}{ }^{\epsilon \zeta} H_{\epsilon}{ }^{\imath \kappa} R_{\delta \iota \zeta \kappa}+C_{14} H_{\alpha}{ }^{\delta \epsilon} H^{\alpha \beta \gamma} H_{\beta \delta}{ }^{\zeta} H_{\gamma}{ }^{\imath \kappa} R_{\epsilon \iota \zeta \kappa} \\
& +C_{15} H_{\alpha \beta}{ }^{\delta} H^{\alpha \beta \gamma} H_{\gamma}{ }^{\epsilon \zeta} H_{\delta}{ }^{\imath \kappa} R_{\epsilon \iota \zeta \kappa}+C_{16} H_{\alpha}^{\delta \epsilon} H^{\alpha \beta \gamma} \nabla_{l} H_{\delta \epsilon \zeta} \nabla^{\imath} H_{\beta \gamma}{ }^{\zeta}+C_{17} H_{\alpha}{ }^{\delta \epsilon} H^{\alpha \beta \gamma} \nabla_{\zeta} H_{\gamma \epsilon l} \nabla^{\imath} H_{\beta \delta}{ }^{\zeta} \\
& +C_{18} H_{\alpha}{ }^{\delta \epsilon} H^{\alpha \beta \gamma} \nabla_{l} H_{\gamma \epsilon \zeta} \nabla^{\imath} H_{\beta \delta^{\zeta}}+C_{19} H_{\alpha \beta}{ }^{\delta} H^{\alpha \beta \gamma} \nabla_{\zeta} H_{\delta \epsilon l} \nabla^{\imath} H_{\gamma}{ }^{\epsilon \zeta}+C_{20} H_{\alpha \beta}{ }^{\delta} H^{\alpha \beta \gamma} \nabla_{l} H_{\delta \epsilon \zeta} \nabla^{l} H_{\gamma}{ }^{\epsilon \zeta}, \tag{34}
\end{align*}
$$

and $\mathcal{L}_{2}^{2}$ has the other couplings which all include derivatives of the dilaton, i.e.,

$$
\begin{aligned}
\mathcal{L}_{2}^{2}= & C_{21} H_{\beta}{ }^{\epsilon \zeta} H^{\beta \gamma \delta} H_{\gamma \epsilon}{ }^{l} H_{\delta \zeta l} \nabla_{\alpha} \Phi \nabla^{\alpha} \Phi+C_{22} R_{\beta \gamma \delta \epsilon} R^{\beta \gamma \delta \epsilon} \nabla_{\alpha} \Phi \nabla^{\alpha} \Phi+C_{23} H_{\beta}{ }^{\epsilon \zeta} H^{\beta \gamma \delta} R_{\gamma \epsilon \delta \zeta} \nabla_{\alpha} \Phi \nabla^{\alpha} \Phi \\
& +C_{24} H_{\alpha}{ }^{\gamma \delta} H_{\beta}{ }^{\epsilon \zeta} H_{\gamma \epsilon}{ }^{l} H_{\delta \zeta l} \nabla^{\alpha} \Phi \nabla^{\beta} \Phi+C_{25} R_{\alpha}{ }^{\gamma \delta \epsilon} R_{\beta \delta \gamma \epsilon} \nabla^{\alpha} \Phi \nabla^{\beta} \Phi \\
& +C_{26} H_{\alpha}{ }^{\gamma \delta} H_{\beta}{ }^{\epsilon \zeta} R_{\gamma \epsilon \delta \zeta} \nabla^{\alpha} \Phi \nabla^{\beta} \Phi+C_{27} H_{\gamma \delta \epsilon} H^{\gamma \delta \epsilon} \nabla^{\alpha} \Phi \nabla_{\beta} \nabla_{\alpha} \Phi \nabla^{\beta} \Phi \\
& +C_{28} H_{\alpha}{ }^{\gamma \delta} H_{\beta}{ }^{\epsilon \zeta} H_{\gamma \epsilon}{ }^{l} H_{\delta \zeta l} \nabla^{\beta} \nabla^{\alpha} \Phi+C_{29} H_{\alpha}{ }^{\gamma \delta} H_{\beta}^{\epsilon \zeta} H_{\gamma \delta}{ }^{l} H_{\epsilon \zeta l} \nabla^{\beta} \nabla^{\alpha} \Phi \\
& +C_{30} H_{\alpha}{ }^{\gamma \delta} H_{\beta \gamma}{ }^{\epsilon} H_{\delta}^{\zeta l} H_{\epsilon \zeta l} \nabla^{\beta} \nabla^{\alpha} \Phi+C_{31} H_{\gamma \delta}{ }^{\zeta} H^{\gamma \delta \epsilon} R_{\alpha \epsilon \beta \zeta} \nabla^{\beta} \nabla^{\alpha} \Phi+C_{32} R_{\alpha}{ }^{\gamma \delta \epsilon} R_{\beta \delta \gamma \epsilon} \nabla^{\beta} \nabla^{\alpha} \Phi \\
& +C_{33} H_{\alpha}{ }^{\gamma \delta} H_{\gamma}{ }^{\epsilon \zeta} R_{\beta \epsilon \delta \zeta} \nabla^{\beta} \nabla^{\alpha} \Phi+C_{34} H_{\alpha}{ }^{\gamma \delta} H_{\beta}{ }^{\epsilon \zeta} R_{\gamma \epsilon \epsilon \zeta} \nabla^{\beta} \nabla^{\alpha} \Phi+C_{35} H_{\alpha}{ }^{\delta \epsilon} \nabla^{\alpha} \Phi \nabla^{\beta} \Phi \nabla_{\gamma} H_{\beta \delta \epsilon} \nabla^{\gamma} \Phi \\
& +C_{36} \nabla_{\alpha} \Phi \nabla^{\alpha} \Phi \nabla_{\beta} \Phi \nabla^{\beta} \Phi \nabla_{\gamma} \Phi \nabla^{\gamma} \Phi+C_{37} \nabla_{\alpha} \Phi \nabla^{\alpha} \Phi \nabla^{\beta} \Phi \nabla_{\gamma} \nabla_{\beta} \Phi \nabla^{\gamma} \Phi
\end{aligned}
$$

$$
\begin{align*}
& +C_{38} H_{\beta}{ }^{\delta \epsilon} H_{\gamma \delta \epsilon} \nabla^{\alpha} \Phi \nabla^{\beta} \Phi \nabla^{\gamma} \nabla_{\alpha} \Phi+C_{39} H_{\beta}{ }^{\delta \epsilon} H_{\gamma \delta \epsilon} \nabla^{\beta} \nabla^{\alpha} \Phi \nabla^{\gamma} \nabla_{\alpha} \Phi \\
& +C_{40} \nabla^{\alpha} \Phi \nabla^{\beta} \Phi \nabla_{\gamma} \nabla_{\beta} \Phi \nabla^{\gamma} \nabla_{\alpha} \Phi+C_{41} \nabla^{\beta} \nabla^{\alpha} \Phi \nabla_{\gamma} \nabla_{\beta} \Phi \nabla^{\gamma} \nabla_{\alpha} \Phi \\
& +C_{42} H_{\beta}{ }^{\delta \epsilon} H_{\gamma \delta \epsilon} \nabla_{\alpha} \Phi \nabla^{\alpha} \Phi \nabla^{\gamma} \nabla^{\beta} \Phi+C_{43} H_{\alpha}^{\delta \epsilon} \nabla^{\alpha} \Phi \nabla_{\gamma} H_{\beta \delta \epsilon} \nabla^{\gamma} \nabla^{\beta} \Phi \\
& +C_{44} \nabla_{\alpha} \Phi \nabla^{\alpha} \Phi \nabla_{\gamma} \nabla_{\beta} \Phi \nabla^{\gamma} \nabla^{\beta} \Phi+C_{45} H_{\alpha \gamma}^{\epsilon} H_{\beta \delta \epsilon} \nabla^{\alpha} \Phi \nabla^{\beta} \Phi \nabla^{\delta} \nabla^{\gamma} \Phi \\
& +C_{46} R_{\alpha \gamma \beta \delta} \nabla^{\alpha} \Phi \nabla^{\beta} \Phi \nabla^{\delta} \nabla^{\gamma} \Phi+C_{47} H_{\alpha \gamma}{ }^{\epsilon} H_{\beta \delta \epsilon} \nabla^{\beta} \nabla^{\alpha} \Phi \nabla^{\delta} \nabla^{\gamma} \Phi+C_{48} R_{\alpha \gamma \beta \delta} \nabla^{\beta} \nabla^{\alpha} \Phi \nabla^{\delta} \nabla^{\gamma} \Phi \\
& +C_{49} H_{\beta}{ }^{\delta \epsilon} \nabla^{\alpha} \Phi \nabla^{\gamma} \nabla^{\beta} \Phi \nabla_{\epsilon} H_{\alpha \gamma \delta}+C_{50} H^{\gamma \delta \epsilon} \nabla^{\alpha} \Phi \nabla^{\beta} \nabla_{\alpha} \Phi \nabla_{\epsilon} H_{\beta \gamma \delta} \\
& +C_{51} \nabla^{\alpha} \Phi \nabla^{\beta} \Phi \nabla_{\delta} H_{\beta \gamma \epsilon} \nabla^{\epsilon} H_{\alpha}^{\gamma \delta}+C_{52} \nabla^{\beta} \nabla^{\alpha} \Phi \nabla_{\delta} H_{\beta \gamma \epsilon} \nabla^{\epsilon} H_{\alpha}^{\gamma \delta} \\
& +C_{53} \nabla^{\alpha} \Phi \nabla^{\beta} \Phi \nabla_{\epsilon} H_{\beta \gamma \delta} \nabla^{\epsilon} H_{\alpha}^{\gamma \delta}+C_{54} \nabla^{\beta} \nabla^{\alpha} \Phi \nabla_{\epsilon} H_{\beta \gamma \delta} \nabla^{\epsilon} H_{\alpha}^{\gamma \delta} \\
& +C_{55} \nabla_{\alpha} \Phi \nabla^{\alpha} \Phi \nabla_{\epsilon} H_{\beta \gamma \delta} \nabla^{\epsilon} H^{\beta \gamma \delta}+C_{56} H_{\alpha}^{\beta \gamma} R_{\gamma \zeta \delta \epsilon} \nabla^{\alpha} \Phi \nabla^{\zeta} H_{\beta}{ }^{\delta \epsilon} \\
& +C_{57} H_{\beta \gamma}{ }^{\epsilon} H^{\beta \gamma \delta} H_{\delta}^{\zeta \iota} \nabla^{\alpha} \Phi \nabla_{l} H_{\alpha \epsilon \zeta}+C_{58} H_{\alpha}^{\beta \gamma} H_{\delta \epsilon}{ }^{l} H^{\delta \epsilon \zeta} \nabla^{\alpha} \Phi \nabla_{l} H_{\beta \gamma \zeta} \\
& +C_{59} H_{\alpha}{ }^{\beta \gamma} H_{\beta}{ }^{\delta \epsilon} H_{\delta}{ }^{\zeta l} \nabla^{\alpha} \Phi \nabla_{l} H_{\gamma \epsilon \zeta}+C_{60} H_{\alpha}^{\beta \gamma} H_{\beta}{ }^{\delta \epsilon} H_{\gamma}^{\zeta l} \nabla^{\alpha} \Phi \nabla_{l} H_{\delta \epsilon \zeta} . \tag{35}
\end{align*}
$$

We have also dropped the prime on the parameters and relabeled them from 1 to 60 . We have found the minimum number of couplings which have no dilaton, i.e., $\mathcal{L}_{2}^{1}$, by solving the equation (32) for the constant dilaton which produces 20 relations between $\delta C \mathrm{~s}$.

In the second scheme, we are going to write all 60 couplings such that there is no dilaton in any of them. We
have found that there are many such schemes. We choose the following scheme:

$$
\begin{equation*}
\mathcal{L}_{2}^{\prime}=\mathcal{L}_{2}^{1}+\mathcal{L}_{2}^{3} \tag{36}
\end{equation*}
$$

where $\mathcal{L}_{2}^{1}$ is the same as (34) and $\mathcal{L}_{2}^{3}$ contains the following 40 couplings:

$$
\begin{align*}
& \mathcal{L}_{2}^{3}=C_{21} H_{\alpha \beta \gamma} H^{\alpha \beta \gamma} H_{\delta \epsilon}{ }^{l} H^{\delta \epsilon \zeta} H_{\zeta}{ }^{\kappa \mu} H_{1 \kappa \mu}+C_{22} H_{\alpha \beta \gamma} H^{\alpha \beta \gamma} H_{\delta \epsilon \zeta} H^{\delta \epsilon \zeta} H_{l \kappa \mu} H^{ו \kappa \mu} \\
& +C_{23} H_{\alpha}{ }^{\gamma \delta} H_{\beta}{ }^{\epsilon \zeta} H_{\gamma \epsilon}{ }^{l} H_{\delta \zeta_{l}} R^{\alpha \beta}+C_{24} H_{\alpha}{ }^{\gamma \delta} H_{\beta \gamma \delta} H_{\epsilon \zeta_{l}} H^{\epsilon \zeta l} R^{\alpha \beta}+C_{25} R_{\alpha}{ }^{\gamma} R^{\alpha \beta} R_{\beta \gamma} \\
& +C_{26} H_{\alpha}{ }^{\delta \epsilon} H^{\alpha \beta \gamma} H_{\beta \delta}{ }^{\zeta} H_{\gamma \epsilon \zeta} R+C_{27} H_{\alpha \beta}{ }^{\delta} H^{\alpha \beta \gamma} H_{\gamma}{ }^{\epsilon \zeta} H_{\delta \epsilon \zeta} R+C_{28} H_{\alpha \beta \gamma} H^{\alpha \beta \gamma} H_{\delta \epsilon \zeta} H^{\delta \epsilon \zeta} R \\
& +C_{29} R_{\alpha \beta} R^{\alpha \beta} R+C_{30} H_{\alpha \beta \gamma} H^{\alpha \beta \gamma} R^{2}+C_{31} R^{3}+C_{32} R R_{\alpha \gamma \beta \delta} R^{\alpha \beta \gamma \delta}+C_{33} H_{\alpha}{ }^{\delta \epsilon} H^{\alpha \beta \gamma} R R_{\beta \delta \gamma \epsilon} \\
& +C_{34} H_{\alpha \beta \gamma} H^{\alpha \beta \gamma} R_{\delta \zeta \epsilon l} R^{\delta \epsilon \zeta \imath}+C_{35} H_{\alpha \beta \gamma} H^{\alpha \beta \gamma} H_{\delta}{ }^{\iota \kappa} H^{\delta \epsilon \zeta} R_{\epsilon l \zeta \kappa}+C_{36} H^{\gamma \delta \epsilon} R^{\alpha \beta} \nabla_{\beta} \nabla_{\epsilon} H_{\alpha \gamma \delta} \\
& +C_{37} H_{\alpha}{ }^{\gamma \delta} R^{\alpha \beta} \nabla_{\beta} \nabla_{\epsilon} H_{\gamma \delta}{ }^{\epsilon}+C_{38} \nabla_{\alpha} R^{\alpha \beta} \nabla_{\gamma} R_{\beta}{ }^{\gamma}+C_{39} H^{\alpha \beta \gamma} R \nabla_{\gamma} \nabla_{\delta} H_{\alpha \beta}{ }^{\delta} \\
& +C_{40} H^{\alpha \beta \gamma} R \nabla_{\delta} \nabla^{\delta} H_{\alpha \beta \gamma}+C_{41} H_{\alpha}{ }^{\gamma \delta} R^{\alpha \beta} \nabla_{\delta} \nabla_{\epsilon} H_{\beta \gamma}{ }^{\epsilon}+C_{42} H^{\alpha \beta \gamma} R_{\beta \epsilon \gamma \zeta} \nabla_{\delta} \nabla^{\zeta} H_{\alpha}{ }^{\delta \epsilon} \\
& +C_{43} H_{\alpha \beta}{ }^{\delta} H^{\alpha \beta \gamma} H^{\epsilon \zeta \iota} \nabla_{\delta} \nabla_{l} H_{\gamma \epsilon \zeta}+C_{44} H_{\alpha \beta}{ }^{\delta} H^{\alpha \beta \gamma} H_{\gamma}{ }^{\epsilon \zeta} \nabla_{\delta} \nabla_{l} H_{\epsilon \zeta}{ }^{l}+C_{45} H^{\gamma \delta \epsilon} \nabla_{\alpha} R^{\alpha \beta} \nabla_{\epsilon} H_{\beta \gamma \delta} \\
& +C_{46} \nabla_{\alpha} R^{\alpha \beta \gamma \delta} \nabla_{\epsilon} R_{\beta}{ }^{\epsilon}{ }_{\gamma \delta}+C_{47} H^{\gamma \delta \epsilon} R^{\alpha \beta} \nabla_{\epsilon} \nabla_{\beta} H_{\alpha \gamma \delta}+C_{48} \nabla_{\alpha} H^{\alpha \beta \gamma} \nabla_{\epsilon} \nabla_{\gamma} \nabla_{\delta} H_{\beta}{ }^{\delta \epsilon} \\
& +C_{49} H_{\alpha}{ }^{\gamma \delta} R^{\alpha \beta} \nabla_{\epsilon} \nabla_{\delta} H_{\beta \gamma}{ }^{\epsilon}+C_{50} \nabla^{\delta} \nabla_{\alpha} H^{\alpha \beta \gamma} \nabla_{\epsilon} \nabla_{\delta} H_{\beta \gamma}{ }^{\epsilon}+C_{51} \nabla_{\alpha} H^{\alpha \beta \gamma} \nabla_{\epsilon} \nabla_{\delta} \nabla^{\epsilon} H_{\beta \gamma}{ }^{\delta} \\
& +C_{52} H_{\alpha}{ }^{\gamma} R^{\alpha \beta} \nabla_{\epsilon} \nabla^{\epsilon} H_{\beta \gamma \delta}+C_{53} \nabla^{\delta} \nabla_{\alpha} H^{\alpha \beta \gamma} \nabla_{\epsilon} \nabla^{\epsilon} H_{\beta \gamma \delta}+C_{54} \nabla_{\alpha} H^{\alpha \beta \gamma} \nabla_{\epsilon} \nabla^{\epsilon} \nabla_{\delta} H_{\beta \gamma}{ }^{\delta} \\
& +C_{55} H^{\alpha \beta \gamma} R_{\beta \delta \gamma \epsilon} \nabla_{\zeta} \nabla^{\zeta} H_{\alpha}{ }^{\delta \epsilon}+C_{56} H_{\alpha}{ }^{\delta \epsilon} H^{\alpha \beta \gamma} H_{\beta \delta}{ }^{\zeta} \nabla_{\zeta} \nabla_{l} H_{\gamma \epsilon}{ }^{l}+C_{57} H^{\alpha \beta \gamma} R_{\beta \epsilon \gamma \zeta} \nabla^{\zeta} \nabla_{\delta} H_{\alpha}{ }^{\delta \epsilon} \\
& +C_{58} H_{\alpha}{ }^{\delta \epsilon} H^{\alpha \beta \gamma} \nabla_{\zeta} H_{\beta \delta}{ }^{\zeta} \nabla_{l} H_{\gamma \epsilon}{ }^{l}+C_{59} H_{\alpha \beta}{ }^{\delta} H^{\alpha \beta \gamma} \nabla_{\epsilon} H_{\gamma}{ }^{\epsilon \zeta} \nabla_{l} H_{\delta \zeta}{ }^{l}+C_{60} H_{\alpha}{ }^{\delta \epsilon} H^{\alpha \beta \gamma} \nabla^{\zeta} H_{\beta \gamma \delta} \nabla_{l} H_{\epsilon \zeta}{ }^{l} . \tag{37}
\end{align*}
$$

When the metric and $B$-field are constant, there is no relation between $\delta C$ s; hence, the minimum number of couplings between only the dilaton is zero. On the other hand, when the metric and $B$-field are constant, there is no coupling in (36).

The parameters in the field redefinitions that change the action (21) to (33) or (36), are functions of $\delta C_{1}, \ldots, \delta C_{705}$ and some of unfixed parameters $b_{i} \mathrm{~s}, c_{i} \mathrm{~s}$, and $d_{i} \mathrm{~s}$ in (31).

In the above schemes, $\delta C_{i}=g_{i}\left(C_{1}, \ldots, C_{705}\right)$, where $i$ is 60 numbers among $1, \ldots, 705$ depending on the scheme, and all others are $\delta C_{j}=-C_{j}$. The unfixed parameters $b_{i} \mathrm{~s}, c_{i} \mathrm{~s}$, and $d_{i} \mathrm{~s}$ satisfy the following relation:

$$
\begin{equation*}
\mathcal{J}_{2}+\mathcal{K}_{2}=0 \tag{38}
\end{equation*}
$$

One may ask if there are similar relations as (28) for the residual field redefinitions at order $\alpha^{\prime 2}$. To answer this question, we write the above equation in the local frame and then solve the resulting equations to find some relations between the unfixed parameters $b_{i} \mathrm{~s}, c_{i} \mathrm{~s}$, and $d_{i} \mathrm{~s}$ and the arbitrary parameters in the total derivative terms, i.e., $J_{i} \mathrm{~s}$. Then one can use the arbitrary parameters $J_{i}$ to cancel the field redefinitions which involve terms with more than two derivatives. The resulting field redefinitions which we call $\delta \hat{\Phi}^{(2)}, \delta \hat{g}_{\mu \nu}^{(2)}, \delta \hat{B}_{\mu \nu}^{(2)}$ can then be rewritten as

$$
\begin{align*}
& \int d^{D} x \sqrt{-g} e^{-2 \Phi} \delta \hat{\Phi}^{(2)}=\frac{\delta S_{0}}{\delta g_{\mu \nu}} \delta \bar{g}_{\mu \nu}^{(1)}+\frac{\delta S_{0}}{\delta B_{\mu \nu}} \delta \bar{B}_{\mu \nu}^{(1)} \\
& +\frac{\delta S_{0}}{\delta \Phi} \delta \bar{\Phi}^{(1)}+\frac{\delta S_{0}}{\delta g_{\mu \nu}} g_{\mu \nu} \delta \bar{\Phi}_{1}{ }^{(1)} \\
& \int d^{D} x \sqrt{-g} e^{-2 \Phi} \delta \hat{g}_{\mu \nu}^{(2)}=\frac{\delta S_{0}}{\delta g^{\mu \nu}} \delta \bar{\Phi}_{2}^{(1)}+\frac{\delta S_{0}}{\delta g_{\alpha \beta}} g_{\alpha \beta} \delta \bar{g}_{1 \mu \nu}^{(1)} \\
& +\frac{\delta S_{0}}{\delta g^{\alpha\{\mu}} \delta \bar{g}_{2 \nu\} \alpha}^{(1)}+\frac{\delta S_{0}}{\delta g_{\alpha \beta}} g_{\alpha \beta} \delta \bar{g}_{3 \mu \nu}^{(1)} \\
& +\frac{\delta S_{0}}{\delta \Phi} \delta \bar{g}_{4 \mu \nu}^{(1)}+\frac{\delta S_{0}}{\delta B^{\alpha\{\mu}} \delta \bar{B}_{1_{\nu\} \alpha}}^{(1)} \\
& +\frac{\delta S_{0}}{\delta B_{\alpha \beta}} \delta \bar{B}_{2 \alpha \beta \mu \nu}^{(1)} \\
& \int d^{D} x \sqrt{-g} e^{-2 \Phi} \delta \hat{B}_{\mu \nu}^{(2)}=\frac{\delta S_{0}}{\delta B^{\alpha[\mu}} \delta \bar{g}_{5 \nu] \alpha}^{(1)}+\frac{\delta S_{0}}{\delta B_{\mu \nu}} \delta \bar{\Phi}_{3}{ }^{(1)} \\
& +\frac{\delta S_{0}}{\delta \Phi} \delta \bar{B}_{3 \mu \nu}^{(1)}+\frac{\delta S_{0}}{\delta g_{\alpha \beta}} \delta \bar{B}_{4 \alpha \beta \mu \nu}^{(1)}, \tag{39}
\end{align*}
$$

where the tensors $\delta \bar{g}_{\mu \nu}^{(1)}, \cdots \delta \bar{g}_{5 \mu \nu}^{(1)}, \delta \bar{B}_{\mu \nu}^{(1)}, \ldots, \delta \bar{B}_{4 \alpha \beta \mu \nu}^{(1)}$, and $\delta \bar{\Phi}^{(1)}, \ldots, \delta \bar{\Phi}_{3}{ }^{(1)}$ are some specific functions of $R, H, \nabla \Phi$ and their derivatives at order $\alpha^{\prime}$. Using similar steps as in (25) and (26), and using the residual field redefinitions in (28) and (39), one then can write the variations that the actions $S_{0}, S_{1}, S_{2}$ produce at order $\alpha^{\prime 3}$, up to some total derivative terms, as

$$
\begin{align*}
\delta S_{0}+\delta S_{1}+\delta S_{2}= & \frac{\delta S_{0}}{\delta g_{\alpha \beta}} \delta g_{\alpha \beta}^{(3)}+\frac{\delta S_{0}}{\delta B_{\alpha \beta}} \delta B_{\alpha \beta}^{(3)} \\
& +\frac{\delta S_{0}}{\delta \Phi} \delta \Phi^{(3)} \tag{40}
\end{align*}
$$

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where the deformations $\delta g_{\mu \nu}^{(3)}, \delta B_{\mu \nu}^{(3)}, \delta \Phi^{(3)}$ are arbitrary functions of $R, H, \nabla \Phi$ and their derivatives at order $\alpha^{13}$.

It seems similar rewriting as (39) can be done for residual field redefinitions at higher orders of $\alpha^{\prime}$ as well. As a result, the variations of actions $S_{0}, \ldots, S_{n-1}$ may produce the following contributions at order $\alpha^{\prime n}$ :

$$
\begin{align*}
\delta S_{0}+\cdots+\delta S_{n-1}= & \frac{\delta S_{0}}{\delta g_{\alpha \beta}} \delta g_{\alpha \beta}^{(n)}+\frac{\delta S_{0}}{\delta B_{\alpha \beta}} \delta B_{\alpha \beta}^{(n)} \\
& +\frac{\delta S_{0}}{\delta \Phi} \delta \Phi^{(n)} \tag{41}
\end{align*}
$$

where the deformations $\delta g_{\mu \nu}^{n)}, \delta B_{\mu \nu}^{(n)}, \delta \Phi^{(n)}$ are arbitrary functions of $R, H, \nabla \Phi$ and their derivatives at order $\alpha^{\prime n}$. Therefore, as long as one considers fixed couplings at orders $\alpha^{\prime}, \ldots, \alpha^{\prime n-1}$, the contributions of the field redefinitions on the actions $S_{0}, \ldots, S_{n-1}$ at order $\alpha^{\prime n}$ are given by (41). Using a high performance computer, it is then straightforward to extend the calculations in this paper to the order $\alpha^{\prime n}$. The minimal couplings may be written in the following scheme:

$$
\begin{equation*}
\mathcal{L}_{n}^{\prime}=\mathcal{L}_{n}^{1}+\mathcal{L}_{n}^{2}, \tag{42}
\end{equation*}
$$

where $\mathcal{L}_{n}^{1}$ contains the minimum number of couplings for the metric and $B$-field, and $\mathcal{L}_{n}^{2}$ contains all other couplings for each derivative of the dilaton.

The minimal independent parameters in (42) for $n=2$ which are given in (34) and (35) or (37) may be calculated by studying in details the S-matrix element of vertex operators in string theory, or they may be found by the T-duality constraint up to an overall factor. We leave these calculations for future works.

## ACKNOWLEDGMENTS

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[^1]:    ${ }^{1}$ The authors in [40] reported that there are 21 independent couplings at order $\alpha^{2}$. We think there is a typo in writing the number of independent $H^{6}$ terms in [40]. They have written 8 independent terms with structure $H^{6}$, whereas, we have found that there are only 7 such terms. That indicates that there should be 20 independent couplings for gravity and B-field.

