

BOGOMOLOV MULTIPLIER AND THE LAZARD CORRESPONDENCE

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ABSTRACT. In this paper we extend the notion of CP covers for groups to the class of Lie algebras, and show that despite the case of groups, all CP covers of a Lie algebra are isomorphic. Moreover we show that CP covers of groups and Lie rings which are in Lazard correspondence, are in Lazard correspondence too, and the Bogomolov multipliers of the group and the Lie ring are isomorphic.

1. INTRODUCTION

In [10] the Bogomolov multiplier and the commutativity preserving cover (CP cover) were first studied by Moravec for the class of finite groups. In the class of groups, the Bogomolov multiplier of a group is unique up to isomorphism but the corresponding CP cover is not necessarily unique. In our recent work [1], we defined the Bogomolov multiplier for Lie algebras. Here, we will define CP covers of Lie algebras, then we will show that all CP covers of a Lie algebra are isomorphic. The Lazard correspondence that was introduced by Lazard in [15], builds an equivalence of categories between finite p -groups of nilpotency class at most $p-1$ and the finite p -Lie rings of the same order and nilpotency class. Recall that a p -Lie ring is a Lie algebra over $\mathbb{Z}/p^k\mathbb{Z}$ for some positive integer k , see [21] for more information. There is a close connection between many invariants of an arbitrary group and a Lie ring that is its Lazard correspondent. For example, the centers, the Schur multipliers and the epicenters of them are isomorphic as abelian groups (see for instance [7]). By a same motivation we will prove that if G is a group and L is its Lazard correspondent, then the Bogomolov multipliers of them are isomorphic as abelian groups.

Key words and phrases. Bogomolov Multiplier, Commutativity-preserving defining pair, CP cover, Baker-Campbell-Hausdorff formula, Lazard correspondence.

Mathematics Subject Classification 2010. Primary 17B05 Secondary 17B99.

2. THE BOGOMOLOV MULTIPLIER AND THE CP COVER OF LIE ALGEBRAS

This section is devoted to introducing CP covers of Lie algebras and then we will show (unlike the situation in finite groups), all CP covers for a Lie algebra are isomorphic. We recall the following definition.

Definition 2.1. *Let R be a commutative unital ring. A Lie algebra over R is an R -module L equipped with an R -bilinear map $[\cdot, \cdot] : L \times L \rightarrow L$ which is called the Lie bracket provided that the following conditions hold.*

- $[x, x] = 0$,
- $[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$ (Jacobi identity), and $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$,
- $[ax + by, z] = a[x, z] + b[y, z]$ and $[z, ax + by] = a[z, x] + b[z, y]$,
- $[x, y] = -[y, x]$,

for all $x, y, z \in L$ and $a, b \in R$.

The Lie bracket $([x, y])$ is called the commutator of x and y .

Throughout this section, L will represent a Lie algebra over a field. Also, it is easy to see that the dimension of a Lie algebra is its dimension as a vector space over the field, and the cardinality of a minimal generating set of a Lie algebra is always less than or equal to its dimension.

The Bogomolov multiplier. The Bogomolov multiplier is a group-theoretical invariant introduced as an obstruction to the rationality problem in algebraic geometry. Let K be a field, G be a finite group and V be a faithful representation of G over K . Then there is natural action of G upon the field of rational functions $K(V)$. The rationality problem (also known as Noether's problem) asks whether the field of G -invariant functions $K(V)^G$ is rational (purely transcendental) over K ? A question related to the above mentioned problem is whether there exist independent variables x_1, \dots, x_r such that $K(V)^G(x_1, \dots, x_r)$ becomes a pure transcendental extension of K ? Saltman in [22] found examples of groups of order p^9 with a negative answer to the Noether's problem, even when taking $K = \mathbb{C}$. His main method was the application of the unramified cohomology group $H_{nr}^2(\mathbb{C}(V)^G, \mathbb{Q}/\mathbb{Z})$ as an obstruction. Bogomolov in [4] proved that it is canonically isomorphic to

$$B_0(G) = \bigcap \ker \{ \text{res}_G^A : H^2(G, \mathbb{Q}/\mathbb{Z}) \rightarrow H^2(A, \mathbb{Q}/\mathbb{Z}) \},$$

where A is an abelian subgroup of G . The group $B_0(G)$ is a subgroup of the Schur multiplier $\mathcal{M}(G) = H^2(G, \mathbb{Q}/\mathbb{Z})$ and Kunyavskii in [14] named it the *Bogomolov Multiplier* of G . Thus vanishing the Bogomolov multiplier leads to a positive answer

to Noether's problem. But it's not always easy to calculate Bogomolov multipliers of groups. Moravec in [19] introduced an equivalent definition of the Bogomolov multiplier. In this sense, he used a notion of the nonabelian exterior square $G \wedge G$ of a group G to obtain a new description of the Bogomolov multiplier. He showed that if G is a finite group, then $B_0(G)$ is non-canonically isomorphic to $\text{Hom}(\tilde{B}_0(G), \mathbb{Q}/\mathbb{Z})$, where the group $\tilde{B}_0(G)$ can be described as a section of the nonabelian exterior square of a group G . Also, he proved that $\tilde{B}_0(G) = \mathcal{M}(G)/\mathcal{M}_0(G)$, such that the Schur multiplier $\mathcal{M}(G)$ or the same $H^2(G, \mathbb{Q}/\mathbb{Z})$ interpreted as the kernel of the commutator homomorphism $G \wedge G \rightarrow [G, G]$ given by $x \wedge y \rightarrow [x, y]$, and $\mathcal{M}_0(G)$ is the subgroup of $\mathcal{M}(G)$ defined as $\mathcal{M}_0(G) = \langle x \wedge y \mid [x, y] = 0, x, y \in G \rangle$. Thus in the class of finite groups, $\tilde{B}_0(G)$ is non-canonically isomorphic to $B_0(G)$. With this definition all truly nontrivial nonuniversal commutator relations is collected into an abelian group that is called the Bogomolov multiplier. Furthermore, Moravec's method relates the Bogomolov multiplier to the concept of commuting probability of a group and shows that the Bogomolov multiplier plays an important role in commutativity preserving central extensions of groups, that are famous cases in K -theory. So, there are some papers to compute this multiplier for some groups. See for example [4, 6, 12, 13, 14, 16, 17, 19]. In the recent work [1], as a close relationship between groups and Lie algebras, we developed the analogous theory of commutativity preserving exterior product and then the Bogomolov multiplier for the class of Lie algebras.

The Hopf-type formula for the Bogomolov multiplier

We recall the Hopf-type formula for groups and Lie algebras as follows. Let $K(F)$ denote $\{[x, y] \mid x, y \in F\}$.

Theorem 2.2. *Let G be a group and L be a Lie algebra. Then*

- (i) *If $G \cong \frac{F_1}{R_1}$ is a presentation for G , then $\tilde{B}_0(G) \cong \frac{R_1 \cap \gamma_2(F_1)}{\langle K(F_1) \cap R_1 \rangle}$,*
- (ii) *If $L \cong \frac{F_2}{R_2}$ is a presentation for L , then $\tilde{B}_0(L) \cong \frac{R_2 \cap F_2^2}{\langle K(F_2) \cap R_2 \rangle}$.*

Proof. (i) See [19, Proposition 3.8]. (ii) from [8], $L \wedge L \cong F_2^2/[R_2, F_2]$ and $L^2 \cong F_2^2/(R_2 \cap F_2^2)$. Moreover the map $\tilde{\kappa} : L \wedge L \rightarrow L^2$ given by $x \wedge y \rightarrow [x, y]$ is an epimorphism. Thus, $\ker \tilde{\kappa} = \mathcal{M}(L) \cong (R_2 \cap F_2^2)/[R_2, F_2]$ and $\mathcal{M}_0(L)$ can be determined with the subalgebra of $F_2/[R_2, F_2]$ generated by all the commutators in $F_2/[R_2, F_2]$ that belong to $\mathcal{M}(L)$. Thus we have the following isomorphism for $\mathcal{M}_0(L)$,

$$\langle K(\frac{F_2}{[R_2, F_2]}) \cap \frac{R_2}{[R_2, F_2]} \rangle \cong \frac{\langle K(F_2) \cap R_2 \rangle + [R_2, F_2]}{[R_2, F_2]} \cong \frac{\langle K(F_2) \cap R_2 \rangle}{[R_2, F_2]}.$$

Therefore $\tilde{B}_0(L) = \mathcal{M}(L)/\mathcal{M}_0(L) \cong R_2 \cap F_2^2 / \langle K(F_2) \cap R_2 \rangle$, as required. \square

Commutativity preserving extension of groups. For groups, in parallel to the classical theory of central extension, Jezernik and Moravec in [10] developed a version of extension that preserve commutativity. Let G , N and Q be groups. An exact sequence of group $1 \rightarrow N \xrightarrow{\chi} G \xrightarrow{\pi} Q \rightarrow 1$, is called a commutativity preserving extension (CP extension) of N by Q , if commuting pairs of elements of Q have commuting lifts in G . A special type of CP extensions with the central kernel is named a central CP extension. Jezernik and Moravec in [10] proved that the central exact sequence $1 \rightarrow N \xrightarrow{\chi} G \xrightarrow{\pi} Q \rightarrow 1$ is a CP extension if and only if $\chi(N) \cap K(G) = 1$. Also, a central CP extension $1 \rightarrow N \xrightarrow{\chi} G \xrightarrow{\pi} Q \rightarrow 1$ is termed stem, whenever $\chi(N) \leq G'$, and every stem central CP extension with $|N| = |\tilde{B}_0(Q)|$, is called a CP cover. It is proved in [10, Theorem 4.2], every finite group has a CP cover and every stem central CP extension is a quotient of a CP cover, and if G is a CP cover of Q with kernel N , then $N \cong \tilde{B}_0(Q)$. Now, we investigate an analogues statement for Lie algebras, and then we show all CP covers for a finite dimensional Lie algebra are isomorphic.

The following definition is used in the next lemma.

Definition 2.3. Let C and \tilde{B}_0 be Lie algebras. We call a pair of Lie algebras (C, \tilde{B}_0) , a commutativity preserving defining pair (CP defining pair) for L , if

- (i) $L \cong C/\tilde{B}_0$
- (ii) $\tilde{B}_0 \subseteq Z(C) \cap C^2$
- (iii) $\tilde{B}_0 \cap K(C) = 0$.

In the other words, for every stem central CP extension $0 \rightarrow \tilde{B}_0 \rightarrow C \xrightarrow{\pi} L \rightarrow 0$ with $L \cong C/\tilde{B}_0$, (C, \tilde{B}_0) is termed a CP defining pair.

Lemma 2.4. Let L be a Lie algebra of finite dimension n and C be the first term in a CP defining pair for L . Then $\dim C \leq n(n+1)/2$.

Proof. Clearly, $\dim C/Z(C) \leq \dim C/\tilde{B}_0 = \dim L = n$. Now, if $\{x_1 + Z(C), \dots, x_n + Z(C)\}$ is a basis for $C/Z(C)$, then $\{[x_i, x_j]; 1 \leq i < j \leq n\}$ is a generating set for C^2 and since $[x_i, x_i] = 0$ and $[x_i, x_j] = -[x_j, x_i]$, we have $\dim C^2 \leq (n^2 - n)/2 = n(n-1)/2$. Thus $\dim \tilde{B}_0 \leq n(n-1)/2$ and $\dim C = n + \dim \tilde{B}_0 \leq n + n(n-1)/2 = n(n+1)/2$. \square

A pair (C, \tilde{B}_0) is called a maximal CP defining pair if the dimension of C is maximal.

Definition 2.5. For a maximal CP defining pair (C, \tilde{B}_0) , C is called a commutativity preserving cover or (CP cover) for L .

The following definition is used for finding the Hopf-type formula for \tilde{B}_0 , where (C, \tilde{B}_0) is a maximal CP defining pair, and it is used to prove the uniqueness of the CP covers of a Lie algebra.

Definition 2.6. Let

$$c(L) = \{(C, \lambda) \mid \lambda \in \text{Hom}(C, L), \lambda \text{ surjective and } \ker \lambda \subseteq C^2 \cap Z(C), \ker \lambda \cap K(C) = 0\}$$

(T, τ) is called a universal member in $c(L)$ if for each $(C, \lambda) \in c(L)$, there exists $h' \in \text{Hom}(T, C)$ such that $\lambda \circ h' = \tau$, in the other words the following diagram commutes.

$$\begin{array}{ccc} T & \xrightarrow{\tau} & L \\ h' \downarrow & \nearrow \lambda & \\ C & & \end{array}$$

It can be shown that, CP defining pairs and elements of $c(L)$ are related in the following sense.

Let $(C, \sigma) \in c(L)$, so $\ker \sigma \subseteq Z(C) \cap C^2$, $\ker \sigma \cap K(C) = 0$ and $L \cong C/\ker \sigma$. Therefore $(C, \ker \sigma)$ is a CP defining pair for L . Conversely, if (C, N) is a CP defining pair for L , then there is a surjective homomorphism $\sigma : C \rightarrow L$ with $\ker \sigma = N \subseteq Z(C) \cap C^2$ and $N \cap K(C) = 0$. Thus $(C, \sigma) \in c(L)$.

Now, we want to show that all CP covers of a Lie algebra are isomorphic. First, we recall the following lemma.

Lemma 2.7. [3, Lemma 1.4] Let $(N, \mu) \in c(L)$ and $\lambda \in \text{Hom}(C, L)$ where λ is surjective. Suppose that $\beta \in \text{Hom}(C, N)$ such that $\mu \circ \beta = \lambda$, then β is surjective.

We are going to show that $c(L)$ has a universal element and they are precisely those elements (T, τ) where T is a CP cover of L .

Proposition 2.8. Let L be a finite dimensional Lie algebra. Then (T, τ) is a universal element of $c(L)$ if and only if T is a CP cover.

Proof. Let $(T, \tau) \in c(L)$ such that for each $(C, \lambda) \in c(L)$, there is $\rho \in \text{Hom}(T, C)$ such that $\lambda \circ \rho = \tau$. By using Lemma 2.7, ρ is surjective and $\dim C \leq \dim T$. Thus T is a CP cover of L . Also, since every CP cover of L is a homomorphic image of T and has the same dimension as T , so it is isomorphic to T . Moreover by using Lemma 2.7, any CP cover of L , gives a universal element in $c(L)$. \square

Proposition 2.9. *Let L be a finite dimensional Lie algebra, then all CP covers of L are isomorphic.*

Proof. By using Proposition 2.8, since there is a universal element in $c(L)$, all CP covers of L are isomorphic. \square

To find the Hopf-type formula for \tilde{B}_0 , when (C, \tilde{B}_0) is a maximal CP defining pair of L , let $L \cong F/R$ be a free presentation of a finite dimensional Lie algebra L , $A_L = R/\langle K(F) \cap R \rangle$, $B_L = F/\langle K(F) \cap R \rangle$ and $D_L = (F^2 \cap R)/\langle K(F) \cap R \rangle$. We will show that there is a central ideal E_L of B_L complement to D_L in A_L , also there are λ , $\tilde{\sigma}$ and $\tilde{\pi}$, such that $(B_L/E_L, \tilde{\pi})$ is a universal element of $c(L)$, $\tilde{\sigma} \in \text{Hom}(B_L/E_L, C)$ and $\tilde{\pi} = \lambda \tilde{\sigma}$. Also B_L/E_L is a CP cover of L and $(B_L/E_L, \ker \tilde{\pi})$ is a maximal CP defining pair for L .

Since (C, \tilde{B}_0) is a maximal CP defining pair, there is a surjective map $\lambda : C \rightarrow L$ such that $\ker \lambda = \tilde{B}_0 \subseteq Z(C) \cap C^2$, $\tilde{B}_0 \cap K(C) = 0$ and $(C, \lambda) \in c(L)$. By using Lemma 2.7, σ is surjective. On the other hand, we have the following commutative diagram.

$$\begin{array}{ccc} F & \xrightarrow{\pi} & L \\ \sigma \downarrow & \nearrow \lambda & \\ C & & \end{array}$$

In the following lemmas, we show that σ induces $\sigma_1 \in \text{Hom}(B_L, C)$.

Lemma 2.10. *Let $L \cong F/R$ be a free presentation of L , then for every $x \in F$, we have $x \in R$ if and only if $\sigma(x) \in \ker \lambda$. Moreover $\langle K(F) \cap R \rangle \subseteq \ker \sigma$, and σ induces a surjective homomorphism $\sigma_1 \in \text{Hom}(B_L, C)$ such that $\lambda \sigma_1 = \pi_1$.*

Proof. Let $x \in R$. Then $0 = \pi(x) = \lambda \sigma(x)$. Thus $\sigma(x) \in \ker \lambda$. On the other hand, let $\sigma(x) \in \ker \lambda$, then $\lambda(\sigma(x)) = 0$. It implies that $\pi(x) = 0$. So, $x \in \ker \pi = R$. Now, since $\sigma(r) \in \ker \lambda \subseteq Z(C) \cap C^2 \subseteq Z(C)$ and $\sigma(f) \in C$, for all $r_1, r_2, r \in R$ and $f \in F$, we have $\sigma([r, f]) = [\sigma(r), \sigma(f)] = 0$ and $\sigma([r_1, r_2]) = [\sigma(r_1), \sigma(r_2)] = 0$. Thus $\langle K(F) \cap R \rangle \subseteq \ker \sigma$. Hence σ induces $\sigma_1 \in \text{Hom}(B_L, C)$ and $\lambda \sigma_1(f + \langle K(F) \cap R \rangle) = \lambda \sigma(f) = \pi(f) = \pi_1(f + \langle K(F) \cap R \rangle)$. Therefore $\lambda \sigma_1 = \pi_1$. One can see that σ_1 is surjective. So, we have the following commutative diagram

$$\begin{array}{ccc} B_L & \xrightarrow{\pi_1} & L \\ \sigma_1 \downarrow & \nearrow \lambda & \\ C & & \end{array}.$$

\square

Lemma 2.11. *Using the notations and assumptions in Lemma 2.12, we have*

- (i) $\sigma_1(A_L) = \ker \lambda$
- (ii) $\sigma_1(D_L) = \ker \lambda$
- (iii) $A_L = D_L + \ker \sigma_1$.

Proof. (i) Let $y \in \sigma_1(A_L)$, then $y = \sigma_1(a)$ for some $a \in A_L$. We have $\ker \pi_1 = A_L$, so, $\lambda \sigma_1(a) = \pi_1(a) = 0$. Hence $y \in \ker \lambda$. Now, let $m \in \ker \lambda$, then there is $b \in B_L$ such that $\sigma_1(b) = m$, since σ_1 is surjective and $0 = \lambda(m) = \lambda \sigma_1(b) = \pi_1(b)$, $b \in \ker \pi_1 = A_L$. Thus, $\sigma_1(A_L) = \ker \lambda$.

(ii) Clearly $D_L \subseteq A_L$, and (i) imply $\sigma_1(D_L) \subseteq \ker \lambda$. Let $y \in \ker \lambda$. Since $(C, \lambda) \in c(L)$ and $y \in C^2 = \sigma(F^2) = \sigma_1(B_L^2)$, there is $z \in B_L^2$ such that $y = \sigma_1(z)$. Since $z \in A_L$, $z \in B_L^2 \cap A_L = D_L$. Hence, $\ker \lambda \subseteq \sigma_1(D_L)$.

(iii) Let $a \in A_L$. By using (i) and (ii), $\sigma_1(a) \in \ker \lambda = \sigma_1(D_L)$. Therefore there is $d \in D_L$ such that $\sigma_1(a) = \sigma_1(d)$. So, $\sigma_1(a - d) = 0$, and $a = d + e$ for some $e \in \ker \sigma_1$. Thus $A_L \subseteq D_L + \ker \sigma_1$. On the other hand, we have $\sigma_1(x) = 0$, for all $x \in \ker \sigma_1$. Therefore $\pi_1(x) = \lambda \sigma_1(x) = 0$. So, $x \in \ker \pi_1 = A_L$. Hence $\ker \sigma_1 \subseteq A_L$. Since $D_L \subseteq A_L$, $D_L + \ker \sigma_1 \subseteq A_L$. So $A_L = D_L + \ker \sigma_1$. \square

Note that since $A_L = D_L + \ker \sigma_1$ and $\ker \sigma_1 = \ker \sigma / \langle K(F) \cap R \rangle$, $\ker \sigma_1$ has $(\ker \sigma \cap R \cap F^2) / \langle K(F) \cap R \rangle$ as a finite dimensional Lie subalgebra.

Also, $\ker \sigma / (\ker \sigma \cap R \cap F^2) \cong R/R \cap F^2$ is abelian, $(L/L^2 \cong F/(R + F^2))$ and $(F/F^2)/(F/(R + F^2)) \cong (R + F^2)/F^2 \cong R/R \cap F^2$. Put $E_L = R/R \cap F^2$. Clearly it is a central ideal of A_L . Therefore A_L is a central extension of D_L by the abelian Lie algebra E_L , and this extension splits. So $A_L = D_L \oplus E_L$.

On the other hand $[R, F] \leq \langle K(F) \cap R \rangle$ and A_L and D_L are central ideals of B_L . Thus, $[D_L + \ker \sigma_1, B_L] = 0$ and $[\ker \sigma_1, B_L] = 0$. Now since $E_L \leq \ker \sigma_1$, $[E_L, B_L] = 0$. So, E_L is a central ideal of B_L . Thus, σ_1 and π_1 induce $\bar{\sigma} \in \text{Hom}(B_L/E_L, C)$ and $\bar{\pi} \in \text{Hom}(B_L/E_L, L)$, respectively. Moreover the following diagram is commutative.

$$\begin{array}{ccc} \frac{B_L}{E_L} & \xrightarrow{\bar{\pi}} & L \\ \bar{\sigma} \downarrow & \nearrow \lambda & \\ C & & \end{array}$$

Using the previous notations, the following lemmas show that $(B_L/E_L, A_L/E_L)$ is a maximal CP defining pair for L .

Lemma 2.12. *Let L be a finite dimensional Lie algebra, and $L \cong F/R$ for a free Lie algebra F . Then $(B_L/E_L, A_L/E_L)$ is a CP defining pair for L , where E_L is any complementary subspace to D_L in A_L .*

Proof. Since $A_L \subseteq Z(B_L)$ and $A_L/E_L \subseteq Z(B_L/E_L)$, we have

$$\frac{B_L/E_L}{A_L/E_L} \cong \frac{B_L}{A_L} \cong \frac{F}{R} \cong L,$$

and

$$D_L = \frac{F^2 \cap R}{\langle K(F) \cap R \rangle} \subseteq \frac{F^2}{\langle K(F) \cap R \rangle} \cong \left(\frac{F}{\langle K(F) \cap R \rangle} \right)^2 = B_L^2.$$

Hence

$$\frac{A_L}{E_L} \cong \frac{D_L + E_L}{E_L} \subseteq \frac{B_L^2 + E_L}{E_L} = \left(\frac{B_L}{E_L} \right)^2.$$

Thus, $A_L/E_L \subseteq Z(B_L/E_L) \cap (B_L/E_L)^2$ and $(A_L/E_L) \cap K(B_L/E_L) = 0$. \square

Lemma 2.13. *B_L/E_L is a CP cover of L and $\tilde{B}_0 \cong (F^2 \cap R)/\langle K(F) \cap R \rangle$.*

Proof. C is a CP cover of L , so $\dim C \geq \dim(B_L/E_L)$. Since $\bar{\sigma}$ is surjective, C is the homomorphic image of B_L/E_L and $\dim C \leq \dim(B_L/E_L)$. Therefore $\dim C = \dim(B_L/E_L)$ and B_L/E_L is a CP cover of L . Now by using Propositions 2.8 and 2.9, $(B_L/E_L, \tilde{\pi})$ is a universal element of $c(L)$ and $C \cong B_L/E_L$. Now since $C/\tilde{B}_0 \cong L \cong B_L/A_L$ and $D_L \cong A_L/E_L$, $\dim \tilde{B}_0 = \dim D_L$, and so $\tilde{B}_0 \cong D_L = (F^2 \cap R)/\langle K(F) \cap R \rangle$. \square

The following key lemma is used in the next investigation.

Lemma 2.14. [3, Lemma 1.11] *Let B, D, B_1, D_1 be Lie algebras and $B \oplus D = B_1 \oplus D_1$. If $B \cong B_1$ and B is finite dimensional, then $D \cong D_1$.*

Note that since $\dim L = n$ and F is generated by n elements, E_L has finite dimension.

Lemma 2.15. *With the previous notations, all CP covers of L are isomorphic to B_L/E_L .*

Proof. Let (N, \tilde{B}_0) be a maximal CP defining pair of L . So there is a surjective map $\beta : N \rightarrow L$ such that $(N, \beta) \in c(L)$. Similar to the previous statement, there is a central ideal E'_L that is complementary to D_L in A_L , and $\sigma'_1 \in \text{Hom}(B_L, N)$ such that $E'_L \leq \ker \sigma'_1$ and $\beta \circ \sigma'_1 = \pi_1$. On the other hand, $D_L \subseteq B_L^2$ and $B_L^2 \cap E_L = 0$. So we can write $Z(B_L) = B_L^2 \cap Z(B_L) \oplus E_L \oplus A$ where A is abelian and $B_L^2 \cap A = 0$. Thus $B_L \cong T \oplus E_L \oplus A$, where T is non-abelian. Therefore $B_L \cong E_L \oplus K_L$, such that $K_L = T \oplus A$. Similarly, there is a Lie algebra K'_L such that $B_L \cong E'_L \oplus K'_L$. Also, E_L and E'_L are abelian Lie algebras and both are of the same finite dimension, so $E_L \cong E'_L$. By using Lemma 2.14, $K_L \cong K'_L$. Thus $B_L/E_L \cong B_L/E'_L$. \square

Therefore we showed that for every finite dimensional Lie algebra L , there is a CP cover which in fact is $\frac{B_L}{E_L}$, and for every CP defining pair (C, \tilde{B}_0) , C is isomorphic to a quotient of $\frac{B_L}{E_L}$ ($C \cong \frac{B_L}{\ker \sigma_1} \cong \frac{\frac{B_L}{E_L}}{\frac{E_L}{\ker \sigma_1}}$). Also, since $A_L = D_L + \ker \sigma_1$ and $\tilde{B}_0 \cong \frac{A_L}{\ker \sigma_1}$, \tilde{B}_0 is isomorphic to a quotient of the Bogomolov multiplier of L .

3. THE BOGOMOLOV MULTIPLIER AND THE LAZARD CORRESPONDENCE

This section is devoted to show the Bogomolov multiplier of a Lie ring L and a group G is isomorphic, when L is Lazard correspondent of G . We recall the following definition.

Definition 3.1. *A Lie ring is defined as an abelian group L equipped with a \mathbb{Z} -bilinear map $[\cdot, \cdot] : L \times L \rightarrow L$ called the Lie bracket satisfying the following conditions*

- $[x, x] = 0$,
- $[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$ and $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$ (Jacobi identity),
- $[x + y, z] = [x, z] + [y, z]$ and $[x, y + z] = [x, y] + [x, z]$,
- $[x, y] = -[y, x]$,

for all $x, y, z \in L$.

The Lie bracket $([x, y])$ is called the commutator of x and y .

Let L be a Lie ring and M and N are subrings of it, we define $[M, N]$ as the Lie subring of L generated by all commutators $[m, n]$ with $m \in M$ and $n \in N$. This allows us to define the lower central series $L = L^1 \geq L^2 \geq L^3 \geq \dots$ via $L^i = [L^{i-1}, L]$. The Lie ring L is nilpotent if this series terminates at 0, and in this case, the class $cl(L)$ is the length of the lower central series of L .

Note that a Lie ring which is also an algebra over a field (or a commutative unital ring) is termed a Lie algebra over that field (or commutative unital ring). Also a Lie ring can be defined as a \mathbb{Z} -Lie algebra (see [9]), and a p -Lie ring is a Lie algebra over $\mathbb{Z}/p^k\mathbb{Z}$ for some positive integer k (see [21]). Therefore more definitions and proofs of Lie rings can be obtained as generalizations from the Lie algebras, and there are similar results between finite Lie rings and finite dimensional Lie algebras over a field. So similar to recent work [1], we have the Bogomolov multiplier for Lie rings. Also we want to introduce CP defining pairs and CP covers of Lie rings.

In the following, for a given Lie ring L , the set $\{[x, y] | x, y \in L\}$ of all commutators of L is denoted by $K(L)$. Also, we use the same notations as in the previous section.

Definition 3.2. Let C and N be finite Lie rings, a pair (C, N) is called a *commutativity preserving defining pair* (CP defining pair) for L , provided that $L \cong C/N$, $N \subseteq Z(C) \cap C^2$ and $N \cap K(C) = 0$.

Lemma 3.3. Let L be a Lie ring of finite order n and C be the first term in a CP defining pair for L . Then $|C| \leq n^2(n-1)$.

Proof. Clearly, $|C/Z(C)| \leq |C/N| = |L| = n$. So $|N| \leq |C^2| \leq n(n-1)$. Since $|C| = |L||N|$, $|C| \leq n^2(n-1)$. \square

Therefore if L is a finite Lie ring, the order of the first coordinate of CP defining pairs for L is bounded, and a pair (C, N) is called a maximal CP defining pair, if the order of C is maximal.

Definition 3.4. For a maximal CP defining pair (C, N) , C is called a *commutativity preserving cover* (CP cover) for L .

Similar to the proofs in the previous section, it can be proven that for every finite Lie ring L , there is a unique CP cover, and for every CP defining pair (C, N) , C and N are isomorphic to a quotient of CP cover and the Bogomolov multiplier of L , respectively. Also for every maximal CP defining pair (C, N) , we have the following Lemma.

Lemma 3.5. Let (C, N) be a maximal CP defining pair for a finite Lie ring L . Then N is isomorphic to the Bogomolov multiplier of L .

Proof. Similar to Lemma 2.13 and by using the previous assumptions, $|D_L| = |N|$. Since $\lambda\sigma = \pi$, $\sigma(F) \in N = \ker \lambda$ is equivalent to $F \in R$. Also

$$\bar{\sigma}(D_L) = \sigma(R \cap F^2) = \sigma(R) \cap \sigma(F^2) = N \cap C^2 = N.$$

So, $\bar{\sigma}|_{D_L} : D_L \rightarrow N$ is a surjective, and since $|D_L| = |N|$, $N \cong D_L$. \square

In the following, we introduce the Lazard correspondence between finite p -Lie rings of nilpotency class at most $p-1$ and finite p -groups of the same order and nilpotency class. Also, to avoid confusion, in a group G , we denote the multiplication by xy and the commutator is written $[x, y]_G = x^{-1}y^{-1}xy$.

The Baker-Campbell-Hausdorff formula (B-C-H) and its inverse. Let L be a p -Lie ring of order p^n and nilpotency class c with $p-1 \geq c$ and G be a

finite p -group with order p^n and the same nilpotency class c . The Baker-Campbell-Hausdorff formula (B-C-H formula) is a group multiplication in terms of Lie ring operations

$$xy := x + y + \frac{1}{2}[x, y]_L + \frac{1}{12}[x, x, y]_L + \dots,$$

where $x, y \in L$. The inverse g^{-1} of the group element g corresponds to $-g$. and the identity 1 in the group corresponds to 0 in the Lie ring. So, the B-C-H formula is used to turn Lie ring presentations into group presentations. Conversely the inverse B-C-H formula is a Lie ring addition and Lie bracket in terms of group multiplication that is used to turn group presentations into Lie ring presentations. This have the general form

$$\begin{aligned} x + y &:= xy[x, y]_G^{\frac{-1}{2}} \dots \\ [x, y]_L &:= [x, y]_G[x, x, y]_G^{\frac{1}{2}} \dots \end{aligned}$$

when $c \leq 14$. See [5].

The Lazard correspondence. The B-C-H formula and its inverse give an isomorphism between the category of nilpotent p -Lie rings of order p^n and the nilpotency class c , provided $p-1 \geq c$ and the category of finite p -groups of the same order and nilpotency class which is known as the Lazard correspondence. By using this correspondence, in the same line of investigation, the same results on p -groups can be checked on p -Lie rings. In the following, we mention some of these correspondences that were proved by Eick in [7].

Proposition 3.6. [7, Proposition 3]

Let G be a finite p -group of class at most $p-1$, and L be its Lazard correspondent. Let X be a subset of G and hence of L . Then

- (i) *There is a Lazard correspondence between the subring $L_0 \subseteq L$ generated by X and the subgroup $G_0 \subseteq G$ generated by X .*
- (ii) *L_0 is an ideal of L if and only if G_0 is a normal subgroup of G .*

Proposition 3.7. [7, Proposition 4]

Let G be a finite p -group of class at most $p-1$, and L be its Lazard correspondent. Then

- (i) *$Z(G)$ and $Z(L)$ coincide as sets and are isomorphic as abelian groups.*
- (ii) *G' and L^2 coincide as sets and are in Lazard correspondence.*

Proposition 3.8. [7, Proposition 5]

Let G be a finite p -group of class at most $p-1$, and L be its Lazard correspondent. Let G_0 be a normal subgroup in G and L_0 be the corresponding ideal in L . Then

$\psi : G/G_0 \rightarrow L/L_0$ given by $(xG_0 \mapsto x + L_0)$, is a well-defined bijection, and it induces a Lazard correspondence between G/G_0 and L/L_0 .

Note that similar to the definition of CP covers for groups in [10], for every stem central CP extension $1 \rightarrow N \rightarrow G \xrightarrow{\pi} Q \rightarrow 1$ with $Q \cong G/N$, (N, G) is termed a CP defining pair of Q , and G is called a CP cover, whenever $|N| = |\tilde{B}_0(Q)|$.

Proposition 3.9. *Let G be a finite p -group of class at most $p - 1$, and L be its Lazard correspondent. Then every CP defining pair of G is in the Lazard correspondence with a CP defining pair of L and vice versa.*

Proof. Suppose (G^*, G_0) is a CP defining pair of G with $G \cong G^*/G_0$. Then $cl(G^*) < cl(G) + 1 \leq p - 1 + 1 = p$. So, $cl(G^*) \leq p - 1$ and there is a Lie ring L^* in the Lazard correspondence with G^* . Since $G_0 \subseteq (G^*)' \cap Z(G^*)$ and $G_0 \cap K(G^*) = 1$, by Proposition 3.6 and the Lazard correspondence, there is a central ideal L_0 of L^* such that G_0 and L_0 are in the Lazard correspondence, $L_0 \subseteq (L^*)^2 \cap Z(L^*)$ and $L_0 \cap K(L^*) = 0$. Now, Proposition 3.8 shows that $G \cong G^*/G_0$ and L^*/L_0 are in the Lazard correspondence. So, $L \cong L^*/L_0$. Hence, (L^*, L_0) is a unique CP defining pair of L . The proof of the converse is similar. \square

We know that there are many invariants between G and its Lazard correspondent. Now, we want to introduce another instance of these invariants.

Theorem 3.10. *Let G be a finite p -group of class at most $p - 1$, and L be its Lazard correspondent. Then*

- (i) *The isomorphism types of CP covers of G are in the Lazard correspondence with the isomorphism types of CP covers of L and vice versa.*
- (ii) *The Bogomolov multipliers of G and L are isomorphic as abelian groups.*

Proof. We know that CP covers and Bogomolov multipliers are the first and the second components of the maximal CP defining pairs of groups and Lie rings, respectively, this is a direct consequence of the Proposition 3.9 and the B-C-H formula. \square

Example 3.11. We consider a finite p -group G_{1p} of order p^5 with the nilpotency class 3 and the following presentation

$$G_{1p} = \langle g, g_1, g_2, g_3 \mid [g_1, g] = g_2, [g_2, g] = g^{p^2} = g_3, g_1^p = g_2^p = g_3^p = 1 \rangle$$

Moravec in [19] showed that $\tilde{B}_0(G_{1p}) = 0$. For $p \geq 5$ these groups are in the Lazard correspondence with the finite p -Lie ring L_{1p} of the same order and nilpotency class. For fixed prime p , the method of [7] can be used to determine the Lie ring

presentation for L_{1p} with p as parameter. Let F_{1p} be a free Lie ring on v, v_1, v_2, v_3 , and denote presentations of L_{1p} as F_{1p}/R_{1p} . So,

$$L_{1p} = \langle v, v_1, v_2, v_3 \mid [v_1, v] = v_2 - p^2v/2 = v_2 - v_3/2, [v_2, v] = p^2v + p^4v/2 = v_3, \\ pv_1 = pv_2 = pv_3 = 0 \rangle.$$

The Moravec's method in [20] which determines the Bogomolov multiplier for a polycyclic group may translate to finite p -Lie rings. Now we use this method to determine the Bogomolov multiplier of L_{1p} . Based on the above presentation, we have

$$L_{1p} \wedge L_{1p} = \langle v \wedge v_1, v \wedge v_2, v \wedge v_3, v_1 \wedge v_2, v_1 \wedge v_3, v_2 \wedge v_3 \rangle.$$

Hence for all $w \in \mathcal{M}(L_{1p}) \leq L_{1p} \wedge L_{1p}$ there exist $\alpha_1, \dots, \alpha_6 \in \mathbb{Z}_{p^k}$, such that $w = \alpha_1(v \wedge v_1) + \alpha_2(v \wedge v_2) + \alpha_3(v \wedge v_3) + \alpha_4(v_1 \wedge v_2) + \alpha_5(v_1 \wedge v_3) + \alpha_6(v_2 \wedge v_3)$. Consider $\tilde{\kappa} : L_{1p} \wedge L_{1p} \rightarrow L_{1p}^2$. $\tilde{\kappa}(w) = 0$, and

$$\alpha_1[v, v_1] + \alpha_2[v, v_2] + \alpha_3[v, v_3] + \alpha_4[v_1, v_2] + \alpha_5[v_1, v_3] + \alpha_6[v_2, v_3] = 0.$$

So, $\alpha_1v_2 + (\alpha_2 - \alpha_1/2)v_3 = 0$. Hence $\alpha_1 = \alpha_2 = 0$. Therefore

$$w = \alpha_3(v \wedge v_3) + \alpha_4(v_1 \wedge v_2) + \alpha_5(v_1 \wedge v_3) + \alpha_6(v_2 \wedge v_3).$$

On the other hand, $[v, v_3] = [v_1, v_2] = [v_1, v_3] = [v_2, v_3] = 0$. Thus $(v \wedge v_3), (v_1 \wedge v_2), (v_1 \wedge v_3), (v_2 \wedge v_3) \in \mathcal{M}_0(L_{1p})$. So, $w \in \mathcal{M}_0(L_{1p})$ and $\mathcal{M}(L_{1p}) \subseteq \mathcal{M}_0(L_{1p})$. Hence $\tilde{B}_0(L_{1p}) = 0$.

Example 3.12. We consider a finite p -group G_{2p} of order p^6 with the nilpotency class 3 given by the following presentation

$$G_{2p} = \langle g, g_1, g_2, g_3, g_4, g_5 \mid [g_1, g_2] = g_3, [g_3, g_1] = g_4, [g_3, g_2] = g_5, [g, g_1] = g_4, \\ g_1^p = g_2^p = g_3^p = g_4^p = g_5^p = g^p = 1 \rangle$$

Chen and Ma in [6] showed that $\tilde{B}_0(G_{2p}) = 0$. For $p \geq 5$ these groups are in the Lazard correspondence with the finite p -Lie ring L_{2p} of the same order and nilpotency class. For fixed prime p , the method of [7] can be used to determine the Lie ring presentation for L_{2p} with p as parameter. Let F_{2p} be a free Lie ring on v, v_1, \dots, v_5 , and denote presentation of L_{2p} as F_{2p}/R_{2p} . So,

$$L_{2p} = \langle v, v_1, v_2, v_3, v_4, v_5 \mid [v_1, v_2] = v_3 - v_4/2 - v_5/2, [v_3, v_1] = v_4, [v_3, v_2] = v_5, \\ [v, v_1] = v_4, pv_1 = pv_2 = pv_3 = pv_4 = pv_5 = 0 \rangle.$$

We use a method similar to that of Moravec to determine the Bogomolov multiplier of L_{2p} . Based on the above presentation we have

$$L_{2p} \wedge L_{2p} = \langle v \wedge v_1, v \wedge v_2, v \wedge v_3, v \wedge v_4, v \wedge v_5, v_1 \wedge v_2, \dots \rangle$$

$$v_1 \wedge v_3, v_1 \wedge v_4, v_1 \wedge v_5, v_2 \wedge v_3, v_2 \wedge v_4, v_2 \wedge v_5, v_3 \wedge v_4, v_3 \wedge v_5, v_4 \wedge v_5 \rangle.$$

For all $w \in M(L_{2p}) \leq L_{2p} \wedge L_{2p}$, there exists $\alpha_1, \dots, \alpha_{15} \in \mathbb{Z}_{p^k}$, such that

$$\begin{aligned} w = & \alpha_1(v \wedge v_1) + \alpha_2(v \wedge v_2) + \alpha_3(v \wedge v_3) + \alpha_4(v \wedge v_4) + \alpha_5(v \wedge v_5) + \alpha_6(v_1 \wedge v_2) + \alpha_7(v_1 \wedge v_3) \\ & + \alpha_8(v_1 \wedge v_4) + \alpha_9(v_1 \wedge v_5) + \alpha_{10}(v_2 \wedge v_3) + \alpha_{11}(v_2 \wedge v_4) + \alpha_{12}(v_2 \wedge v_5) + \alpha_{13}(v_3 \wedge v_4) \\ & + \alpha_{14}(v_3 \wedge v_5) + \alpha_{15}(v_4 \wedge v_5) \end{aligned}$$

Let $\tilde{\kappa} : L_{2p} \wedge L_{2p} \rightarrow L_{2p}^2$. Then we have $\tilde{\kappa}(w) = 0$ and

$$\begin{aligned} & \alpha_1[v, v_1] + \alpha_2[v, v_2] + \alpha_3[v, v_3] + \alpha_4[v, v_4] + \alpha_5[v, v_5] + \alpha_6[v_1, v_2] + \alpha_7[v_1, v_3] \\ & + \alpha_8[v_1, v_4] + \alpha_9[v_1, v_5] + \alpha_{10}[v_2, v_3] + \alpha_{11}[v_2, v_4] + \alpha_{12}[v_2, v_5] + \alpha_{13}[v_3, v_4] + \alpha_{14}[v_3, v_5] \\ & + \alpha_{15}[v_4, v_5] = 0, \end{aligned}$$

So, $(\alpha_1 - \alpha_6/2 - \alpha_7)v_4 + \alpha_6v_3 + (-\alpha_6/2 - \alpha_{10})v_5 = 0$. Thus, $\alpha_6 = \alpha_{10} = 0$ and $\alpha_1 = \alpha_7$. Therefore

$$\begin{aligned} w = & \alpha_1((v \wedge v_1) + (v_1 \wedge v_3)) + \alpha_2(v \wedge v_2) + \alpha_3(v \wedge v_3) + \alpha_4(v \wedge v_4) + \alpha_5(v \wedge v_5) + \alpha_8(v_1 \wedge v_4) + \\ & \alpha_9(v_1 \wedge v_5) + \alpha_{11}(v_2 \wedge v_4) + \alpha_{12}(v_2 \wedge v_5) + \alpha_{13}(v_3 \wedge v_4) + \alpha_{14}(v_3 \wedge v_5) + \alpha_{15}(v_4 \wedge v_5) \end{aligned}$$

On the other hand, $(v \wedge v_2)$, $(v \wedge v_3)$, $(v \wedge v_4)$, $(v \wedge v_5)$, $(v_1 \wedge v_4)$, $(v_1 \wedge v_5)$, $(v_2 \wedge v_4)$, $(v_2 \wedge v_5)$, $(v_3 \wedge v_4)$, $(v_3 \wedge v_5)$, $(v_4 \wedge v_5)$ belong to the $M_0(L_{2p})$. We can see that, $[v + v_1, v_1 + v_3] = 0$. So, $(v + v_1) \wedge (v_1 + v_3) \in \mathcal{M}_0(L_{2p})$. Hence $(v \wedge v_1) + (v \wedge v_3) + (v_1 \wedge v_1) + (v_1 \wedge v_3) \in \mathcal{M}_0(L_{2p})$. Thus, $((v \wedge v_1) + (v_1 \wedge v_3)) \in \mathcal{M}_0(L_{2p})$ and $w \in \mathcal{M}_0(L_{2p})$. Therefore $\mathcal{M}(L_{2p}) \subseteq \mathcal{M}_0(L_{2p})$ and $\tilde{B}_0(L_{2p}) = 0$, as required.

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