



Continuous Optimization

Robustness in nonsmooth nonlinear multi-objective programming

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ARTICLE INFO

Article history:

Received 21 May 2014

Accepted 11 June 2015

Available online 19 June 2015

Keywords:

Multiple objective programming

Robust solution

Nonsmooth optimization

Clarke generalized gradient

Proper efficiency

ABSTRACT

Recently Georgiev, Luc, and Pardalos (2013), [Robust aspects of solutions in deterministic multiple objective linear programming, *European Journal of Operational Research*, 229(1), 29–36] introduced the notion of robust efficient solutions for linear multi-objective optimization problems. In this paper, we extend this notion to nonlinear case. It is shown that, under the compactness of the feasible set or convexity, each robust efficient solution is a proper efficient solution. Some necessary and sufficient conditions for robustness, with respect to the tangent cone and the non-ascent directions, are proved. An optimization problem for calculating a robustness radius followed by a comparison between the newly-defined robustness notion and two existing ones is presented. Moreover, some alterations of objective functions preserving weak/proper/robust efficiency are studied.

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1. Introduction

Due to perturbations and partial knowledge, in most practical optimization problems we are faced with uncertainty. Popular approaches for dealing with uncertainty are stochastic optimization, robust optimization and sensitivity analysis. Each approach has its own advantages and disadvantages (Ben-Tal, Ghaoui, & Nemirovski, 2009; Bertsimas, Brown, & Caramanis, 2011). From the late 1990s, robust single-objective optimization has been being investigated. In robust single-objective optimization, we try to find a feasible point that optimizes the worst case counterpart of the objective function, see Ben-Tal et al. (2009) and Bertsimas et al. (2011) for more formal definition and more details about robustness in single-objective optimization.

The robustness of single-objective optimization has received considerable attention, but the robustness of multi-objective optimization has not been frequently considered. See e.g. Deb and Gupta (2006), Ehrgott, Ide, and Schöbel (2014), Kuroiwa and Lee (2012) and Georgiev, Luc, and Pardalos (2013), where this notion is studied from different standpoints. Deb and Gupta (2006) focused on two definitions for robustness. In the first definition, they call an efficient solution to be robust if it optimizes the mean of all objective functions. In the second definition, the objectives do not change, but a constraint is added which restricts the absolute difference between the mean

and original objective functions. There, the efficient solutions of the modified problem are called robust. Ehrgott et al. (2014) extended the worst case robustness notion from single-objective optimization to multi-objective programming. They further introduced some scalarization methods which are able to produce a robust solution in worst-case sense. Bokrantz and Fredriksson (2014) studied more methods of scalarization with respect to the definition given by Ehrgott et al. (2014). Moreover, Fliege and Werner (2014) utilized a special robustness concept in portfolio optimization.

In a recent work, Georgiev et al. (2013) have defined the robustness for linear multi-objective programming problems from a different point of view. They have considered a perturbation standpoint, and have defined an efficient solution as a robust solution if it remains efficient for small perturbations in the coefficients of the objective functions. Georgiev et al. (2013) studied their definition considering different kinds of perturbations, including changing the objectives' coefficients and adding a new objective function. They obtained necessary and sufficient conditions and presented various nice properties of the robust solutions in linear cases. Goberna, Jeyakumar, Li, and Vicente-Pérez (2015) extended Georgiev et al.'s definition for linear multi-objective optimization problems under perturbations of the coefficients of both objective functions and constraints.

In this paper, we extend the definition given by Georgiev et al. (2013) to nonlinear multi-objective programming problems. We show that, under the compactness of the feasible set or convexity, the set of robust efficient solutions is a subset of the set of proper efficient solutions. Some necessary and sufficient conditions for robust solutions with respect to the tangent cone and non-ascent directions, under appropriate assumptions, are given. A robustness radius is

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calculated. The relationships between the robustness notion considered in the present paper and two worst case-based definitions, studied by Fliege and Werner (2014) and Ehrgott et al. (2014), are highlighted. Two kinds of modifications in the objective functions are dealt with and the relationships between the weak/proper/robust efficient solutions of the problems, before and after the perturbation, are established. Some examples, to clarify the theoretical results, are given.

The rest of the paper unfolds as follows. In Section 2, some preliminaries are given. In Section 3, robustness is defined, its relationship with proper efficiency is established, and some necessary and sufficient conditions are proved. Section 4 is devoted to the robustness radius calculation. Section 5 contains some results about connections between the new and previously defined robustness definitions. In Section 6, we study some alterations of the objective functions that preserve weak/proper/robust efficiency.

2. Preliminaries

This section is devoted to some preliminaries. For a set $\Omega \subseteq \mathbb{R}^n$, we use the notations $\text{co}\Omega$, $\text{int}\Omega$ and $\text{cl}\Omega$ to denote the convex hull, the interior and the closure of Ω , respectively. For a vector $d \in \mathbb{R}^n$, d^T stands for transpose of d , and for two vectors x and y , $x^T y$ denotes the standard inner product in \mathbb{R}^n . For $\Omega \subseteq \mathbb{R}^n$,

$$\text{Pos}(\Omega) := \left\{ y \in \mathbb{R}^n : \exists m \in \mathbb{N}; y = \sum_{i=1}^m \lambda_i y_i, \lambda_i \geq 0, \right. \\ \left. y_i \in \Omega, i = 1, 2, \dots, m \right\}.$$

If $\Omega_1, \dots, \Omega_l \subseteq \mathbb{R}^n$ are convex sets, it can be shown that

$$\text{Pos}\left(\bigcup_{i=1}^l \Omega_i\right) = \left\{ \sum_{i=1}^l \lambda_i d_i : d_i \in \Omega_i, \lambda_i \geq 0, i = 1, 2, \dots, l \right\}.$$

For $\Omega \subseteq \mathbb{R}^n$ and $x \in \text{cl}\Omega$, the tangent cone of Ω at x , denoted by $T_\Omega(\bar{x})$, is defined by

$$T_\Omega(\bar{x}) = \left\{ d \in \mathbb{R}^n : \exists \{x_i\} \subseteq \Omega, \{t_i\} \subseteq \mathbb{R}; t_i \downarrow 0, \frac{x_i - \bar{x}}{t_i} \rightarrow d \right\}.$$

For $x, y \in \mathbb{R}^n$, the vector inequality $x < y$ means $x_i < y_i$ for all $i = 1, 2, \dots, n$. Analogously, \leq , $>$, and \geq are defined componentwise.

Consider the following multi-objective optimization problem:

$$\begin{aligned} \min f(x) \\ \text{s.t. } x \in \Omega, \end{aligned} \quad (1)$$

that $\Omega \subseteq \mathbb{R}^n$ is nonempty and $f : \Omega \rightarrow \mathbb{R}^p$ with $p \geq 2$. In fact, $f(x) = (f_1(x), f_2(x), \dots, f_p(x))$ for each x . In the whole of this paper, we assume that f_i is locally Lipschitz for each i , though some of the results given in this paper are valid without this assumption.

Definition 2.1. The vector $\bar{x} \in \Omega$ is called an efficient solution of Problem (1) if there exists no $x \in \Omega$ such that $f(x) \leq f(\bar{x})$ and $f(x) \neq f(\bar{x})$.

Definition 2.2. The vector $\bar{x} \in \Omega$ is called a weak efficient solution of Problem (1) if there exists no $x \in \Omega$ such that $f(x) < f(\bar{x})$.

In order to obtain efficient solutions with bounded trade-offs, Geoffrion (1968) suggested restricting attention to efficient solutions that are proper in the sense of the following definition.

Definition 2.3. (Geoffrion 1968): The vector $\bar{x} \in \Omega$ is called a proper efficient solution of Problem (1) if it is efficient and there is a real number $M > 0$ such that for all i and $x \in \Omega$ satisfying $f_i(x) < f_i(\bar{x})$ there exists an index j such that $f_j(\bar{x}) < f_j(x)$ and

$$\frac{f_i(\bar{x}) - f_i(x)}{f_j(x) - f_j(\bar{x})} \leq M.$$

Proper efficiency has been defined in different senses and has been studied in several publications, including Benson (1979), Borwein (1977), Geoffrion (1968), Henig (1982) and Kuhn and Tucker (1951). In this paper, we use the above definition.

Our robustness notion in the present work is based on the matrix perturbations. Hence, we need a matrix norm. For an $m \times n$ matrix $C = [c_{ij}]$, the Frobenius norm is as $\|C\| = (\sum_{i,j} |c_{ij}|^2)^{1/2}$. Although we use this norm, almost all of the provided results are valid with any matrix norm.

In this paper, Clarke generalized gradient (Clarke (2013)) is used in the presence of nonsmooth data.

Definition 2.4. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be Lipschitz near a given point $x \in \mathbb{R}^n$. The generalized directional derivative of f at x in the direction v , denoted by $f^\circ(x; v)$, is defined as $f^\circ(x; v) := \limsup_{y \rightarrow x, t \downarrow 0} \frac{f(y+tv) - f(y)}{t}$, where y is a vector in \mathbb{R}^n and t is a positive scalar.

Definition 2.5. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be Lipschitz near a given point $x \in \mathbb{R}^n$. The generalized gradient of f at x , denoted by $\partial f(x)$, is defined as

$$\partial f(x) := \{\zeta \in \mathbb{R}^n : f^\circ(x; v) \geq \zeta^T v, \forall v \in \mathbb{R}^n\}.$$

If $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is convex, then $\partial f(x)$ reduces to the subgradient set in classic convex analysis (Clarke, 2013):

$$\{\zeta \in \mathbb{R}^n : f(y) - f(x) \geq \zeta^T (y - x), \forall y \in \mathbb{R}^n\}.$$

Definition 2.6. (Clarke 2013): A function $h : \mathbb{R}^n \rightarrow \mathbb{R}$ is called regular at \bar{x} if $h^\circ(\bar{x}; d)$ exists and $h^\circ(\bar{x}; d) = \lim_{t \downarrow 0} \frac{h(\bar{x}+td) - h(\bar{x})}{t}$ for each $d \in \mathbb{R}^n$.

Each convex function is a regular function (Clarke, 2013).

3. Robust solutions and proper efficiency

We start this section by introducing the concept of robust solution for nonlinear multi-objective optimization Problem (1). This definition extends Definition 3.1 in Georgiev et al. (2013).

Definition 3.1. Let $\bar{x} \in \Omega$ be an efficient solution of Problem (1). \bar{x} is called a robust efficient solution if there exists $\epsilon > 0$ such that for any $p \times n$ matrix C with $\|C\| < \epsilon$, the vector \bar{x} is an efficient solution for

$$\begin{aligned} \min f(x) + Cx \\ \text{s.t. } x \in \Omega. \end{aligned} \quad (2)$$

The following theorem presents a nice property of the robust efficient solutions. It proves that the set of robust efficient solutions is a subset of properly efficient solutions under the compactness of the feasible set.

Theorem 3.1. Let Ω be compact. If \bar{x} is a robust efficient solution of Problem (1), then \bar{x} is a proper efficient solution of Problem (1).

Proof. Suppose \bar{x} is not a proper efficient solution. Then, there exist $\{x_i\} \subseteq \Omega$, increasing sequence $\{M_i\}$ of positive real numbers, and $k \in \{1, \dots, p\}$, such that $M_i \rightarrow +\infty$,

$$f_k(x_i) < f_k(\bar{x}) \quad \forall i, \quad (3)$$

and

$$\frac{f_k(\bar{x}) - f_k(x_i)}{f_j(x_i) - f_j(\bar{x})} > M_i \text{ for each } j \in \{1, \dots, p\} \text{ with } f_j(x_i) > f_j(\bar{x}). \quad (4)$$

Since Ω is compact, without loss of generality, we may assume that $\{x_i\}$ converges to some $\hat{x} \in \Omega$. Also, we define $Q_i = \{j : f_j(x_i) > f_j(\bar{x})\}$. This set is nonempty because \bar{x} is efficient. Without loss of generality, by choosing an appropriate subsequence, Q_i is a constant set for all i indices. So, we denote it by Q . Two cases may occur for \hat{x} ; either it is equal to \bar{x} or not. We consider these two possible cases and we get a contradiction in each case.

As robustness of \bar{x} , there exists some $\epsilon > 0$ such that \bar{x} is an efficient solution of Problem (2) for any matrix $C_{p \times n}$ with $\|C\| < \epsilon$. Let $\bar{x} \neq \hat{x}$. We can choose matrix $\|\tilde{C}_{p \times n}\| < \epsilon$ such that

$$\tilde{C}^j(\bar{x} - \hat{x}) < -2\delta, \quad \forall j \in Q, \quad (5)$$

$$\tilde{C}^j = 0, \quad \forall j \in \{1, \dots, p\} \setminus Q, \quad (6)$$

for some $\delta > 0$ (\tilde{C}^j denotes the j th row of \tilde{C}). Since f is bounded on Ω , from (4), we have $f_j(x_i) \rightarrow f_j(\bar{x})$ for each $j \in Q$ as $i \rightarrow +\infty$. Therefore, for sufficiently large i values, we have $f_j(x_i) - f_j(\bar{x}) - \delta < 0$. Hence, by (5), for sufficiently large i values, we get

$$f_j(x_i) + \tilde{C}^j x_i < f_j(\bar{x}) + \tilde{C}^j \bar{x} - \delta < f_j(\bar{x}) + \tilde{C}^j \bar{x}, \quad \forall j \in Q. \quad (7)$$

Also, by (6) and the definition of Q , for sufficiently large i values, we have

$$f_j(x_i) + \tilde{C}^j x_i \leq f_j(\bar{x}) + \tilde{C}^j \bar{x}, \quad \forall j \in \{1, \dots, p\} \setminus Q. \quad (8)$$

Inequalities (7) and (8) contradict the robustness of \bar{x} .

Now, we consider the latter case, i.e. $\bar{x} = \hat{x}$. We assume that the sequence $\{\frac{x_i - \bar{x}}{\|x_i - \bar{x}\|}\}$ converges to some nonzero vector d . We choose $\tilde{C}_{p \times n}$ satisfying $\|\tilde{C}_{p \times n}\| < \epsilon$ and

$$\tilde{C}^j d < -2\delta, \quad \forall j \in Q, \quad (9)$$

$$\tilde{C}^j = 0, \quad \forall j \in \{1, \dots, p\} \setminus Q, \quad (10)$$

for some $\delta > 0$. Assume that L_j is the Lipschitz constant of f_j on a neighbourhood of \bar{x} . By (4), for sufficiently large i values, we get

$$f_j(x_i) - f_j(\bar{x}) < \frac{L_j \|\bar{x} - x_i\|}{M_i} < \delta \|\bar{x} - x_i\|. \quad (11)$$

Therefore, by (9)–(11) we get inequalities (7) and (8) in this case as well. These contradict the robustness of \bar{x} and the proof is complete. \square

The converse of the above theorem does not hold necessarily, even for the linear case; see Example 3.2 in Georgiev et al. (2013).

The following example shows that the compactness assumption of Ω in Theorem 3.1 is essential.

Example 3.1. Consider the multi-objective optimization problem

$$\begin{aligned} \min & (-x, x^3) \\ \text{s.t. } & x \in \mathbb{R}. \end{aligned}$$

It is not difficult to see that $\bar{x} = 1$ is a robust efficient solution (consider $\epsilon = 0.1$), while the problem does not have any properly efficient solution.

Now we are going to provide a characterization for robust efficient solutions with respect to the non-ascent directions of the objective function and the tangent cone of the feasible set.

Definition 3.2. $d \in \mathbb{R}^n$ is called a non-ascent direction of f at \bar{x} if $d^T \xi \leq 0$, for each $\xi \in \partial f_i(\bar{x})$ and each $i \in \{1, 2, \dots, p\}$. Hereafter, $G(\bar{x})$ denotes the set of all non-ascent directions of f at \bar{x} .

The following theorem presents a necessary condition for robustness.

Theorem 3.2. If \bar{x} is a robust efficient solution to Problem (1), then $T_\Omega(\bar{x}) \cap G(\bar{x}) = \{0\}$.

Proof. We prove it by contradiction. Suppose that $0 \neq d \in G(\bar{x}) \cap T_\Omega(\bar{x})$. By robustness of \bar{x} , there exists an $\epsilon > 0$ such that \bar{x} is an efficient solution to problem (2) for any matrix $C_{p \times n}$, with $\|C\| < \epsilon$. We choose a matrix $\tilde{C}_{p \times n}$ such that

$$\|\tilde{C}\| < \epsilon \quad \text{and} \quad \tilde{C}d < -2\delta e \quad (12)$$

for some $\delta > 0$ (e is a column vector with all components equal to one). Since $d \in T_\Omega(\bar{x})$,

$$\exists (\{x_i\} \subseteq \Omega, t_i \downarrow 0); \quad \frac{x_i - \bar{x}}{t_i} \rightarrow d. \quad (13)$$

Therefore, from (12) and (13), for sufficiently large i , we have

$$\tilde{C} \left(\frac{x_i - \bar{x}}{t_i} \right) < -\delta e \quad (14)$$

which implies $\tilde{C}x_i + t_i \delta e < \tilde{C}\bar{x}$. Using the mean value theorem (Theorem 10.17 in Clarke, 2013), for each i ,

$$f(x_i) = f(\bar{x}) + \xi_i^T (x_i - \bar{x}) \quad (15)$$

where ξ_i is an $n \times p$ matrix whose j th column belongs to $\partial f_j(\bar{x}_i^j)$ for some $\bar{x}_i^j \in (\bar{x}, x_i)$. Thus,

$$\begin{aligned} f(x_i) + \tilde{C}x_i + t_i \delta e &< f(\bar{x}) + \xi_i^T (x_i - \bar{x}) + \tilde{C}\bar{x} \\ &\Rightarrow f(x_i) + \tilde{C}x_i + t_i \left(\delta e - \xi_i^T \left(\frac{x_i - \bar{x}}{t_i} \right) \right) < f(\bar{x}) + \tilde{C}\bar{x}. \end{aligned}$$

Since $\bar{x}_i^j \rightarrow \bar{x}$ as $i \rightarrow +\infty$ and f is locally Lipschitz at \bar{x} , by Proposition 10.2 in Clarke (2013), the sequence $\{\xi_i\}$ is bounded. Hence, $\xi_i \rightarrow \xi$ for some $\xi \in \partial f(\bar{x})$, because of Proposition 10.10 in Clarke (2013). Thus, $\xi^T d \leq 0$. Therefore, for sufficiently large i values, $\xi_i^T (\frac{x_i - \bar{x}}{t_i}) < \delta e$. Thus, we get

$$f(x_i) + \tilde{C}x_i < f(\bar{x}) + \tilde{C}\bar{x},$$

which contradicts the robustness of \bar{x} , and completes the proof. \square

The condition given in the above theorem is necessary for robustness and it is not sufficient in general case. The following example clarifies this:

Example 3.2. Consider the multi-objective optimization problem

$$\begin{aligned} \min & (f_1(x), f_2(x)) \\ \text{s.t. } & x \in \mathbb{R}, \end{aligned}$$

in which

$$f_1(x) := x, \quad f_2(x) := \begin{cases} -x & |x| < 1, \\ -x^{(\frac{1}{3})} & |x| \geq 1. \end{cases}$$

Let $\bar{x} = 2$. At this point we have $T_\Omega(\bar{x}) = \mathbb{R}$ and $G(\bar{x}) = \{0\}$. It is not difficult to see that $\bar{x} = 2$ is an efficient solution to the above problem, while for any $\epsilon > 0$ it is not an efficient solution to

$$\begin{aligned} \min & (f_1(x), f_2(x) + \frac{\epsilon}{2}x) \\ \text{s.t. } & x \in \mathbb{R}, \end{aligned}$$

because for each $\epsilon > 0$, by setting $x_\epsilon = \min\{-125, \frac{-1}{\epsilon^3}\}$, we have $f_1(x_\epsilon) < f_1(2)$ and $f_2(x_\epsilon) \leq f_2(2)$.

As shown by the above example, the necessary condition given in Theorem 3.2 may not be sufficient for robustness in general. Theorem 3.3 establishes that this condition is sufficient under the convexity assumption.

Theorem 3.3. Let Ω be a closed and convex set and $f_i (i = 1, \dots, p)$ be convex. Assume that \bar{x} is an efficient solution to Problem (1). \bar{x} is a robust efficient solution to Problem (1) if and only if $T_\Omega(\bar{x}) \cap G(\bar{x}) = \{0\}$.

Proof. The “only if” part is derived from Theorem 3.2. For “if” part, suppose that \bar{x} is not a robust efficient solution. Thus there exist a sequence $\{C_i\}$ of $p \times n$ matrices and a sequence $\{x_i\} \subseteq \Omega$ such that $C_i \rightarrow 0$,

$$f(x_i) + C_i x_i \leq f(\bar{x}) + C_i \bar{x}, \quad \text{and} \quad f(x_i) + C_i x_i \neq f(\bar{x}) + C_i \bar{x}. \quad (16)$$

Set

$$d_i := \frac{x_i - \bar{x}}{\|x_i - \bar{x}\|}. \quad (17)$$

Two cases may occur for the sequence $\{x_i\}$. Either it has a subsequence convergent to \bar{x} or it does not have any subsequence convergent to \bar{x} . We consider these two possible cases and we get a contradiction in each case.

In the first case, without loss of generality, we assume that $x_i \rightarrow \bar{x}$. From the convexity of f , for any $\xi \in \partial f(\bar{x})$ we have,

$$f(x_i) \geq f(\bar{x}) + \xi^T (x_i - \bar{x}), \quad (18)$$

that ξ is an $n \times p$ matrix whose j th column belongs to $\partial f_j(\bar{x})$. Therefore, due to (16), we have

$$\|x_i - \bar{x}\|^{-1} (\xi^T (x_i - \bar{x}) + C_i (x_i - \bar{x})) \leq 0. \quad (19)$$

Without loss of generality, we can assume that, $d_i \rightarrow d$ for some $d \in \mathbb{R}^p$ with $\|d\| = 1$ and it is obvious that $d \in T_{\Omega}(\bar{x})$. Moreover, from (19) we conclude that $d \in G(\bar{x})$. Thus $d \in G(\bar{x}) \cap T_{\Omega}(\bar{x})$. This makes a contradiction.

Now, we consider the latter case: $\{x_i\}$ does not have any subsequence convergent to \bar{x} . Therefore, without loss of generality, there exist an $r > 0$ such that $\|x_i - \bar{x}\| > r$. On the other hand, $d_i \rightarrow d$ for some nonzero $d \in T_{\Omega}(\bar{x})$. Since Ω is convex and closed, for each i

$$td_i + \bar{x} \in \Omega, \quad \forall t \in [0, r],$$

$$td + \bar{x} \in \Omega, \quad \forall t \in [0, r].$$

Thus, $0 \neq d \in T_{\Omega}(\bar{x})$. Suppose that $\{t_i\}$ is a sequence of scalars in $[0, r]$ that converges to zero. By convexity of f and due to (16) and (17), we get

$$\begin{aligned} f(\bar{x} + t_i d_i) &\leq \left(1 - \frac{t_i}{\|x_i - \bar{x}\|}\right) f(\bar{x}) + \frac{t_i}{\|x_i - \bar{x}\|} f(x_i) \\ &\leq f(\bar{x}) + \frac{t_i}{\|x_i - \bar{x}\|} C_i (\bar{x} - x_i). \end{aligned}$$

Since $C_i \rightarrow 0$, from the convexity of f and the above statement, we have $\xi^T d \leq 0$ that ξ is an $n \times p$ matrix whose j th column belongs to $\partial f_j(\bar{x})$. Therefore $0 \neq d \in G(\bar{x}) \cap T_{\Omega}(\bar{x})$. This makes a contradiction and completes the proof. \square

In the rest of this section, we consider a multi-objective optimization problem whose feasible set is defined by some constraint functions. Consider

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & g_i(x) \leq 0, \quad i = 1, 2, \dots, m, \end{aligned} \quad (20)$$

where $f: \mathbb{R}^n \rightarrow \mathbb{R}^p$ is the objective function (i.e. $f(x) = (f_1(x), \dots, f_p(x))$) and g_j functions define the constraints. Hereafter, whenever we use the Clarke subdifferential for g_j functions, we assume that these functions are locally Lipschitz.

For a feasible point \bar{x} , the index set $A(\bar{x})$ is defined by

$$A(\bar{x}) = \{j \in \{1, 2, \dots, m\} : g_j(\bar{x}) = 0\}.$$

In the following, we are going to provide a characterization for robust efficient solutions of Problem (20). The following constraint qualification (CQ) helps us in the sequel.

Definition 3.3. We say that the constraint qualification (CQ) holds at \bar{x} if

$$0 \notin \text{co} \left\{ \bigcup_{j \in A(\bar{x})} \partial g_j(\bar{x}) \right\}.$$

Theorem 3.4. If \bar{x} is a robust efficient solution which satisfies (CQ), then

$$\text{Pos} \left(\bigcup_{i=1}^p \partial f_i(\bar{x}) \right) + \text{Pos} \left(\bigcup_{i \in A(\bar{x})} \partial g_i(\bar{x}) \right) = \mathbb{R}^n.$$

Proof. For simplicity, we set $A_{\bar{x}} = \text{Pos}(\bigcup_{i=1}^p \partial f_i(\bar{x})) + \text{Pos}(\bigcup_{i \in A(\bar{x})} \partial g_i(\bar{x}))$. It can be seen that under the assumptions of the theorem,

$$\{d : g_i^{\circ}(\bar{x}; d) \leq 0, \quad \forall i \in A(\bar{x})\} \subseteq T_{\Omega}(\bar{x});$$

see Theorem 10.42 in Clarke (2013). Therefore, according to Theorem 3.2, the system below has no solution $d \in \mathbb{R}^n$:

$$\xi^T d \leq 0, \quad \forall \xi \in \partial f_i(\bar{x}), \quad \forall i \in \{1, \dots, p\}$$

$$\xi^T d \leq 0, \quad \forall \xi \in \partial g_i(\bar{x}), \quad \forall i \in A(\bar{x})$$

$$d \neq 0.$$

Hence, the following system has no solution $d \in \mathbb{R}^n$:

$$\xi^T d \leq 0, \quad \forall \xi \in \partial f_i(\bar{x}), \quad \forall i \in \{1, \dots, p\}$$

$$\xi^T d \leq 0, \quad \forall \xi \in \partial g_i(\bar{x}), \quad \forall i \in A(\bar{x})$$

$$d_1 > 0.$$

Using the semi-infinite Farkas' theorem (see Corollary 3.1.3 in Goberna and Lopez, 1998), we have $e_1 \in \text{cl}(A_{\bar{x}})$. Similarly, it can be shown that $\pm e_i \in \text{cl}(A_{\bar{x}})$ for each $i \in \{1, 2, \dots, p\}$. Here, e_i denotes the i -th unit vector. Therefore, $\text{cl}(A_{\bar{x}}) = \mathbb{R}^n$. Since $A_{\bar{x}}$ is a convex set whose closure is equal to \mathbb{R}^n , we have $A_{\bar{x}} = \mathbb{R}^n$ and the proof is completed. \square

Corollary 3.5. Assume that $f_i(i = 1, \dots, p)$ and $g_j(j = 1, \dots, m)$ in Problem (20) are continuously differentiable. If \bar{x} is a robust efficient solution of Problem (20) which satisfies (CQ), then

$$\text{Pos}\{\nabla f_1(\bar{x}), \dots, \nabla f_p(\bar{x})\} + \text{Pos}\{\nabla g_i(\bar{x}) : i \in A(\bar{x})\} = \mathbb{R}^n.$$

Theorem 3.6 provides a converse version of Theorem 3.4. Theorems 3.4 and 3.6 extend Theorem 3.4 in Georgiev et al. (2013).

Theorem 3.6. Let $f_i(i = 1, \dots, p)$ and $g_j(j = 1, \dots, m)$ in Problem (20) be convex. If \bar{x} is an efficient solution and

$$\text{Pos} \left(\bigcup_{i=1}^p \partial f_i(\bar{x}) \right) + \text{Pos} \left(\bigcup_{i \in A(\bar{x})} \partial g_i(\bar{x}) \right) = \mathbb{R}^n,$$

then \bar{x} is a robust efficient solution to Problem (20).

Proof. We prove the theorem by contradiction. Suppose that \bar{x} is not robust. Then, according to Theorem 3.3, there exists a nonzero vector $\bar{d} \in T_{\Omega}(\bar{x}) \cap G(\bar{x})$. From the convexity assumption, we get $\xi^T \bar{d} \leq 0$ for each $\xi \in \text{Pos}(\partial g_i(\bar{x}))$ and each $i \in A(\bar{x})$. Also, $\xi^T \bar{d} \leq 0$ for each $\xi \in \text{Pos}(\partial f_i(\bar{x}))$ and each $i \in \{1, 2, \dots, p\}$, because of $\bar{d} \in G(\bar{x})$. On the other hand, by the assumption of the theorem, $\bar{d} = \sum_{i=1}^p u_i \xi_i + \sum_{j \in A(\bar{x})} v_j \zeta_j$ for some $u_i, v_j \geq 0$, $\xi_i \in \text{Pos}(\partial f_i(\bar{x}))$, and $\zeta_j \in \text{Pos}(\partial g_j(\bar{x}))$. Therefore, $\bar{d}^T \bar{d} \leq 0$. Hence we get $\bar{d} = 0$ which makes a contradiction. \square

Corollary 3.7. Assume that $f_i(i = 1, \dots, p)$ and $g_j(j = 1, \dots, m)$ in Problem (20) are differentiable and convex. If \bar{x} is an efficient solution and

$$\text{Pos}\{\nabla f_1(\bar{x}), \dots, \nabla f_p(\bar{x})\} + \text{Pos}\{\nabla g_i(\bar{x}) : i \in A(\bar{x})\} = \mathbb{R}^n,$$

then \bar{x} is a robust efficient solution of Problem (20).

Although the compactness assumption is essential in Theorem 3.1 (see Example 3.1), the following result shows that Theorem 3.1 remains valid without compactness of the feasible set for convex programming problems with an appropriate (CQ).

Theorem 3.8. Let $f_i(i = 1, 2, \dots, p)$ and $g_j(j = 1, 2, \dots, m)$ be convex in Problem (20). If \bar{x} is a robust efficient solution of Problem (20) which satisfies (CQ), then \bar{x} is a proper efficient solution of Problem (20).

Proof. Suppose that \bar{x} is not a proper efficient solution. Therefore, there exist $\{x_i\} \subseteq \Omega$, increasing sequence $\{M_i\}$ of positive real numbers, and $k \in \{1, \dots, p\}$, such that $M_i \rightarrow +\infty$,

$$f_k(x_i) < f_k(\bar{x}) \quad \forall i, \quad (21)$$

and

$$\frac{f_k(\bar{x}) - f_k(x_i)}{f_j(x_i) - f_j(\bar{x})} > M_i \text{ for each } j \in \{1, \dots, p\} \text{ with } f_j(x_i) > f_j(\bar{x}). \quad (22)$$

Define $Q_i = \{j : f_j(x_i) > f_j(\bar{x})\}$. This set is nonempty because \bar{x} is efficient. Without loss of generality, by choosing an appropriate subsequence, Q_i is a constant set for all i indices. So, we denote it by Q . Also, define the feasible set of Problem (20) as $\Omega = \{x \in \mathbb{R}^n : g_j(x) \leq 0, j = 1, 2, \dots, m\}$.

Without loss of generality, we assume that the sequence $\{\frac{x_i - \bar{x}}{\|x_i - \bar{x}\|}\}$ converges to some nonzero vector d . Setting $t_i = \min\{\frac{1}{\eta}, \|x_i - \bar{x}\|\}$ and $d_i = \frac{x_i - \bar{x}}{\|x_i - \bar{x}\|}$, we have $t_i \downarrow 0$ and $\bar{x} + t_i d_i \in \Omega$, according to the convexity assumptions. Hence $d \in T_\Omega(\bar{x})$.

Due to the convexity assumption, we get

$$\begin{aligned} f_j(x_i) &\geq f_j(\bar{x}) + \xi^T(x_i - \bar{x}), \quad \forall \xi \in \partial f_j(\bar{x}), \quad \forall j \in \{1, \dots, p\} \setminus Q, \\ &\Rightarrow \xi^T(x_i - \bar{x}) \leq 0, \quad \forall \xi \in \partial f_j(\bar{x}), \quad \forall j \in \{1, \dots, p\} \setminus Q, \\ &\Rightarrow \xi^T d \leq 0, \quad \forall \xi \in \partial f_j(\bar{x}), \quad \forall j \in \{1, \dots, p\} \setminus Q. \end{aligned}$$

Moreover from (22) and the convexity of the objective functions, we have

$$\begin{aligned} \xi^T(x_i - \bar{x}) &\leq f_j(x_i) - f_j(\bar{x}), \quad \forall \xi \in \partial f_j(\bar{x}), \quad \forall j \in Q, \\ &< \frac{f_k(\bar{x}) - f_k(x_i)}{M_i}, \quad \forall \xi \in \partial f_j(\bar{x}), \quad \forall j \in Q, \\ &\leq \frac{1}{M_i} \eta^T(\bar{x} - x_i), \quad \forall \eta \in \partial f_k(\bar{x}). \end{aligned}$$

Thus

$$\xi^T d \leq 0, \quad \forall \xi \in \partial f_j(\bar{x}), \quad \forall j \in Q.$$

Therefore, $d \in T_\Omega(\bar{x}) \cap G(\bar{x})$. This is a contradiction because of Theorem 3.2, and the proof is complete. \square

Remark 3.1. This remark indicates that the robust solution studied in the present paper may not exist in some special cases, though these solutions (if exist) have nice properties as compared to non-robust points. An efficient point is robust if it stays efficient under small linear perturbations. Let us assume that f_i and g_j functions are differentiable here. Under some CQs and appropriate assumptions, the KKT/FJ condition

$$\sum_{i=1}^p \lambda_i \nabla f_i(\bar{x}) + \sum_{j \in A(\bar{x})} \mu_j \nabla g_j(\bar{x}) = 0$$

for some nonnegative μ_j 's and some nonnegative λ_i 's (not all zero), is necessary for the efficiency of \bar{x} . When some objective function, say f_1 , is perturbed, then $\nabla f_1(\bar{x})$ is alerted and hence to preserve the KKT/FJ condition (efficiency of \bar{x}), the Lagrangian multiplier(s) of some other objective function(s) or some constraint function(s) should be changed. Hence, at least one other objective function or at least one constraint function is required for robustness, i.e. $m + p \geq 2$. Thus, there is not any robust solution for unconstrained single-objective problems. To show this analytically, let \bar{x} be an arbitrary optimal solution to $\min_{x \in \mathbb{R}^n} h(x)$ where $h : \mathbb{R}^n \rightarrow \mathbb{R}$. Then $\nabla h(\bar{x}) = 0$ which implies $\nabla h(\bar{x}) + C \neq 0$ for each $C \neq 0$. Therefore, \bar{x} is not optimal to $\min_{x \in \mathbb{R}^n} h(x) + Cx$ for each $C \neq 0$. Hence, \bar{x} is not robust for $\min_{x \in \mathbb{R}^n} h(x)$. Thus this unconstrained problem does not have any robust solution.

Now, consider an unconstrained multi-objective programming problem $\min_{x \in \mathbb{R}^n} f(x)$ with $f : \mathbb{R}^n \rightarrow \mathbb{R}^p$ and $p \geq 2$. Here, $m = 0$. If \bar{x} is a robust solution, then by Corollary 3.5, $\text{Pos}\{\nabla f_1(\bar{x}), \dots, \nabla f_p(\bar{x})\} = \mathbb{R}^n$, and hence $p \geq n + 1$.

For constrained problem (20) satisfying the assumptions of Corollary 3.5, if \bar{x} is a robust solution, then $p + m \geq n + 1$. It is not

restrictive for practical cases, because in practice the problem has at least $2n$ constraints due to the lower and upper bounds of variables.

Remark 3.2. The necessary condition introduced in Theorem 3.2 provides a tie-in to the gradient-like descent methods existing in the literature for solving vector optimization problems; see Drummond and Iusem (2004) and Fliege and Svaiter (2000). Extending these numerical tools to generate robust solution(s) can be worth studying in future.

4. Robustness radius

In this short section, we compute a radius of robustness. For a given vector $a \in \mathbb{R}^p$, the vector a^+ is obtained from a by substituting all negative components by zero. It is not difficult to show that $\|a^+\|$ is equal to the distance from a to $-\mathbb{R}_+^p = \{x \in \mathbb{R}^p : x \leq 0\}$.

Lemma 4.1. Let Ω be a closed and convex set and f_i ($i = 1, \dots, p$) be convex. Let $d \in T_\Omega(\bar{x})$ with $\|d\| = 1$. If \bar{x} is a robust solution of Problem (1), then $\|(f'(\bar{x}; d))^+\| > 0$ and it is equal to the optimal value of the following problem:

$$\sup\{t : f'(\bar{x}; d) + tCd \notin -\mathbb{R}_+^p, \quad \forall \|C\| \leq 1\}.$$

Proof. First we show that $f'(\bar{x}; d) \notin -\mathbb{R}_+^p$. If $f'(\bar{x}; d) \leq 0$, then due to the convexity of f , we have $d \in G(\bar{x})$, which makes a contradiction according to Theorem 3.2. The proof of the second part is similar to that of Lemma 4.2 in Georgiev et al. (2013). \square

Theorem 4.2. Under the assumptions of Lemma 4.1, the optimal value of the following problem is positive and it is a robustness radius for \bar{x} .

$$\begin{aligned} \min \quad & \|(f'(\bar{x}; d))^+\| \\ \text{s.t.} \quad & d \in T_\Omega(\bar{x}), \\ & \|d\| = 1 \end{aligned}$$

Proof. Let ρ be the optimal value of the given problem. Thus, by Lemma 4.1, $\rho > 0$. Now, we show that ρ is a robustness radius for \bar{x} . If this is not a robustness radius, then there exist some $x^0 \in X$ and some matrix C^0 such that $\|C^0\| < \rho$, and

$$f(x^0) + C^0 x^0 \leq f(\bar{x}) + C^0 \bar{x}, \quad f(x^0) + C^0 x^0 \neq f(\bar{x}) + C^0 \bar{x}. \quad (23)$$

Setting $d^0 = \frac{x^0 - \bar{x}}{\|x^0 - \bar{x}\|}$, we have $\|d^0\| = 1$ and $d^0 \in T_\Omega(\bar{x})$ due to the convexity of Ω . Furthermore, by convexity of f , we get

$$f'(\bar{x}; d^0) + C^0 d^0 = \frac{f'(\bar{x}; x^0 - \bar{x})}{\|x^0 - \bar{x}\|} + C^0 d^0 \leq \frac{f(x^0) - f(\bar{x})}{\|x^0 - \bar{x}\|} + \frac{C^0(x^0 - \bar{x})}{\|x^0 - \bar{x}\|}.$$

Therefore, according to (23), we get

$$f'(\bar{x}; d^0) + C^0 d^0 \in -\mathbb{R}_+^p. \quad (24)$$

Defining

$$\rho^0 = \sup\{t : f'(\bar{x}; d^0) + tCd^0 \notin -\mathbb{R}_+^p, \quad \forall \|C\| \leq 1\}, \quad (25)$$

we have $\rho \leq \rho^0$. Furthermore, for each $t \in (0, \rho^0)$ and each C with $\|C\| \leq 1$, we have $f'(\bar{x}; d^0) + tCd^0 \notin -\mathbb{R}_+^p$. This is in contradiction with (24) by setting $t = \|C^0\|$ and $C = \frac{C^0}{\|C^0\|}$, and the proof is complete. \square

It can be seen that, the optimal value of the optimization problem considered in the above theorem is equal to the maximum robustness radius if one furthermore assumes the equality of the tangent cone and the cone of feasible directions.

5. Comparison with worst case-based notions

There are some definitions for robustness in the multi-objective programming literature that optimize the worst case of the objective functions. In the following, we highlight the relationships between the robustness notion considered in the present paper and two worst case-based definitions studied by Fliege and Werner (2014) (FW in

brief) and Ehrgott et al. (2014) (EIS in brief). See also Georgiev et al. (2013) for some comparison.

Let U be an uncertain set and $\Omega \subseteq \mathbb{R}^n$ be the set of feasible solutions. Also, let $f_i : \Omega \times U \rightarrow \mathbb{R}$ for $i = 1, 2, \dots, p$ be objective functions. For a feasible decision variable vector $x \in \Omega$ and a $u \in U$, the value of objective function is denoted by $f(x, u)$. Define $F : \Omega \rightarrow \mathbb{R}^p$ by

$$F_i(x) = \max_{u \in U} f_i(x, u), \quad i = 1, 2, \dots, p.$$

A feasible vector $\bar{x} \in \Omega$ is called a robust solution in the sense of FW if it is an efficient solution to the following multi-objective problem

$$\begin{aligned} \min F(x) \\ \text{s.t. } x \in \Omega \end{aligned}$$

The following proposition provides a connection between the robustness notion studied in the present paper and that in the sense of FW.

Proposition 5.1. *Let \bar{x} be a robust solution of Problem (1), in the sense of Definition 3.1, with radius ϵ . Then considering any $\tilde{\epsilon} \in (0, \epsilon)$, the vector \bar{x} is a robust solution in the sense of FW with $U = \{C_{p \times n} : \|C^i\| \leq \frac{\tilde{\epsilon}}{\sqrt{p}}, \forall i = 1, 2, \dots, p\}$ and $f(x, C) = f(x) + Cx$.*

Proof. By contradiction, assume that there exists some $x^0 \in \Omega$ such that $F(x^0) \leq F(\bar{x})$ and $F(x^0) \neq F(\bar{x})$. If $\bar{x} = 0$, then $F(\bar{x}) = f(0)$ and hence by considering a $p \times n$ matrix C with $\|C^i\| \leq \frac{\tilde{\epsilon}}{\sqrt{p}}, i = 1, 2, \dots, p$,

$$\text{we get } \|C\| = \left(\sum_{i=1}^p \|C^i\|^2 \right)^{\frac{1}{2}} < \epsilon, \text{ and}$$

$$f(x^0) + Cx^0 \leq f(0), \quad f(x^0) + Cx^0 \neq f(0).$$

These relations contradict the robustness of \bar{x} (in the sense of Definition 3.1). Now, assume that $\bar{x} \neq 0$. Then $F(\bar{x}) = f(\bar{x}) + \frac{\epsilon \|\bar{x}\|}{\sqrt{p}} e$, where e is a vector with all components equal to one. Now, we consider a $p \times n$ matrix \tilde{C} , with $\tilde{C}^i = \frac{\tilde{\epsilon}}{\sqrt{p} \|\bar{x}\|} \bar{x}^T$. We get $\tilde{C} \in U$ and

$$f(x^0) + \tilde{C}x^0 \leq F(x^0) \leq F(\bar{x}) = f(\bar{x}) + \frac{\tilde{\epsilon} \|\bar{x}\|}{\sqrt{p}} e = f(\bar{x}) + \tilde{C}\bar{x}$$

$$\text{Furthermore, } \|\tilde{C}\| = \sqrt{\sum_{i=1}^p \|\tilde{C}^i\|^2} < \epsilon. \text{ Hence, } \|\tilde{C}\| < \epsilon, \text{ and}$$

$$f(x^0) + \tilde{C}x^0 \leq f(\bar{x}) + \tilde{C}\bar{x}, \quad f(x^0) + \tilde{C}x^0 \neq f(\bar{x}) + \tilde{C}\bar{x}.$$

These relations contradict the robustness of \bar{x} (in the sense of Definition 3.1) and the proof is complete. \square

In a recently published paper, Ehrgott et al. (2014) (EIS in brief) have defined a feasible point $\bar{x} \in \Omega$ as a robust solution if there is no $x \in \Omega$ such that

$$f_U(x) \subseteq f_U(\bar{x}) - (\mathbb{R}_{\geq}^p \setminus \{0\})$$

that $f_U(x) = \{f(x, u) : u \in U\}$ and U is an uncertain set.

Theorem 5.2. *Let \bar{x} be a robust solution of (1), in the sense of Definition 3.1, with radius ϵ . Then \bar{x} is a robust solution in the sense of EIS with $f_U(x) = \{f(x) + Cx : \|C\| \leq 0.5\epsilon\}$.*

Proof. By contradiction assume that

$$f_U(x^0) \subseteq f_U(\bar{x}) - (\mathbb{R}_{\geq}^p \setminus \{0\}) \quad (26)$$

for some $x^0 \in \Omega$. This implies that

$$\begin{aligned} \forall C \in U \exists \tilde{C} \in U \text{ s.t. } f(x^0) + Cx^0 \leq f(\bar{x}) + \tilde{C}\bar{x}, \quad f(x^0) + Cx^0 \\ \neq f(\bar{x}) + \tilde{C}\bar{x}. \end{aligned} \quad (27)$$

Two vectors x^0 and \bar{x} cannot be zero. If $x^0 = 0$, then by (27),

$$f(0) + \tilde{C}(0) \leq f(\bar{x}) + \tilde{C}\bar{x}, \quad f(x^0) + \tilde{C}x^0 \neq f(\bar{x}) + \tilde{C}\bar{x}$$

for some $\tilde{C} \in U$. This contradicts the robustness assumption. Moreover, if $\bar{x} = 0$, then by considering $C = 0$ in (27), there exists some \tilde{C}

with $\|\tilde{C}\| \leq 0.5\epsilon$ such that $f(x^0) \leq f(\bar{x})$ and $f(x^0) \neq f(\bar{x})$. This contradicts the efficiency of \bar{x} . Hence, $x^0 \neq 0$ and $\bar{x} \neq 0$.

Define

$$M = \{\lambda \in \mathbb{R}_{\geq}^p : \|\lambda\| \leq 1, \sum_{j=1}^p \lambda_j \geq 1\}.$$

It is clear that, M is a nonempty compact convex set. Now, let $F : M \rightrightarrows M$ be a set-valued mapping defined by

$$F(\lambda) = \left\{ \lambda' \in M : f(x^0) + \frac{\epsilon \|x^0\|}{2\|\lambda\|} \lambda \leq f(\bar{x}) + \frac{\epsilon \|\bar{x}\|}{2\|\lambda'\|} \lambda' \right\}.$$

We show that $F(\lambda)$ is nonempty and convex for each $\lambda \in M$.

Let $\lambda \in M$. Defining $p \times n$ matrix C^0 by $C^0 := \frac{\epsilon}{2\|\lambda\| \|x^0\|} \lambda x^{0T}$, we have $\|C^0\| \leq 0.5\epsilon$, and hence by (27), there exists some $p \times n$ matrix \tilde{C} such that $\|\tilde{C}\| \leq 0.5\epsilon$, and

$$f(x^0) + \frac{\epsilon \|x^0\|}{2\|\lambda\|} \lambda \leq f(\bar{x}) + \tilde{C}\bar{x}. \quad (28)$$

Consider $\bar{\lambda}$ with $\bar{\lambda}_i = \|\tilde{C}^i\|$. Define $\lambda' := \frac{\bar{\lambda}}{\|\bar{\lambda}\|}$. By considering Cauchy–Schwarz inequality and $\frac{\epsilon}{2\|\bar{\lambda}\|} \geq 1$, we have

$$\begin{aligned} f(x^0) + \frac{\epsilon \|x^0\|}{2\|\lambda\|} \lambda &\leq f(\bar{x}) + \tilde{C}\bar{x} \\ &\leq f(\bar{x}) + \|\bar{x}\| \bar{\lambda} \\ &\leq f(\bar{x}) + \frac{\epsilon \|\bar{x}\|}{2\|\bar{\lambda}\|} \bar{\lambda} \\ &\leq f(\bar{x}) + \frac{\epsilon \|\bar{x}\|}{2\|\lambda'\|} \lambda' \end{aligned}$$

Therefore, due to $\lambda' \in M$, we have $\lambda' \in F(\lambda)$, and hence $F(\lambda)$ is nonempty.

To prove the convexity, let $\lambda_1, \lambda_2 \in F(\lambda)$ and $v \in (0, 1)$. First we assume that $\|\lambda_1\| = \|\lambda_2\| = 1$. Then, by definition of $F(\lambda)$, we get

$$f(x^0) + \frac{\epsilon \|x^0\|}{2\|\lambda\|} \lambda \leq f(\bar{x}) + \frac{\epsilon \|\bar{x}\|}{2} (v\lambda_1 + (1-v)\lambda_2)$$

Due to $\|v\lambda_1 + (1-v)\lambda_2\| \leq 1$ and $v\lambda_1 + (1-v)\lambda_2 \geq 0$, we can infer that

$$f(x^0) + \frac{\epsilon \|x^0\|}{2\|\lambda\|} \lambda \leq f(\bar{x}) + \frac{\epsilon \|\bar{x}\|}{2\|v\lambda_1 + (1-v)\lambda_2\|} (v\lambda_1 + (1-v)\lambda_2).$$

Hence, $v\lambda_1 + (1-v)\lambda_2 \in F(\lambda)$ when $\|\lambda_1\| = \|\lambda_2\| = 1$. Now, considering two arbitrary vectors $\lambda_1, \lambda_2 \in F(\lambda)$ and $v \in (0, 1)$, there are $\gamma > 0$ and $\mu \in (0, 1)$ such that

$$v\lambda_1 + (1-v)\lambda_2 = \gamma \left(\mu \frac{\lambda_1}{\|\lambda_1\|} + (1-\mu) \frac{\lambda_2}{\|\lambda_2\|} \right) \quad (29)$$

Notice that $0 < \|\lambda_1\|, \|\lambda_2\| \leq 1$. By definition of $F(\lambda)$, it is clear that $\frac{\lambda_1}{\|\lambda_1\|}, \frac{\lambda_2}{\|\lambda_2\|} \in F(\lambda)$. Furthermore, if $\lambda' \in F(\lambda)$ and $\gamma\lambda' \in M$ for some $\gamma > 0$, then $\gamma\lambda' \in F(\lambda)$. Therefore, according to (29), we have $v\lambda_1 + (1-v)\lambda_2 \in F(\lambda)$. Hence, F is a convex-valued mapping. It is clear that F is a closed mapping. Therefore, by Kakutani fixed-point theorem (see Franklin, 2003), there exists some $\lambda^* \in M$ such that

$$f(x^0) + \frac{\epsilon \|x^0\|}{2\|\lambda^*\|} \lambda^* \leq f(\bar{x}) + \frac{\epsilon \|\bar{x}\|}{2\|\lambda^*\|} \lambda^*. \quad (30)$$

The above inequality does hold as equality, otherwise due to (27) we have

$$f(\bar{x}) + \frac{\epsilon \|\bar{x}\|}{2\|\lambda^*\|} \lambda^* = f(x^0) + \frac{\epsilon \|x^0\|}{2\|\lambda^*\|} \lambda^* \leq f(\bar{x}) + \tilde{C}\bar{x},$$

$$f(x^0) + \frac{\epsilon \|x^0\|}{2\|\lambda^*\|} \lambda^* \neq f(\bar{x}) + \tilde{C}\bar{x},$$

for some \tilde{C} with $\|\tilde{C}\| \leq 0.5\epsilon$. Then $\frac{\epsilon\|\tilde{x}\|}{2\|\lambda^*\|}\lambda^* \leq \tilde{C}\tilde{x}$ and $\frac{\epsilon\|\tilde{x}\|}{2\|\lambda^*\|}\lambda^* \neq \tilde{C}\tilde{x}$. By Cauchy–Schwarz inequality, we get $\frac{\epsilon\|\tilde{x}\|}{2\|\lambda^*\|}\lambda^* \leq \|\tilde{x}\|d$ and $\frac{\epsilon\|\tilde{x}\|}{2\|\lambda^*\|}\lambda^* \neq \|\tilde{x}\|d$ in which $d \in \mathbb{R}^p$ with $d_i = \|\tilde{C}^T\|$. Therefore, $\|\tilde{C}\| = \|d\| > 0.5\epsilon$ which is a contradiction. Thus, inequality (30) holds and it does not hold as equality. On the other hand, by Cauchy–Schwarz inequality,

$$f(x^0) + \frac{\epsilon\tilde{x}^T x^0}{2\|\lambda^*\|\|\tilde{x}\|}\lambda^* \leq f(x^0) + \frac{\epsilon x^{0T} x^0}{2\|\lambda^*\|\|x^0\|}\lambda^*$$

Hence, according to (30),

$$f(x^0) + \frac{\epsilon\tilde{x}^T x^0}{2\|\lambda^*\|\|\tilde{x}\|}\lambda^* \leq f(\tilde{x}) + \frac{\epsilon\tilde{x}^T \tilde{x}}{2\|\lambda^*\|\|\tilde{x}\|}\lambda^*,$$

$$f(x^0) + \frac{\epsilon\tilde{x}^T x^0}{2\|\lambda^*\|\|\tilde{x}\|}\lambda^* \neq f(\tilde{x}) + \frac{\epsilon\tilde{x}^T \tilde{x}}{2\|\lambda^*\|\|\tilde{x}\|}\lambda^*.$$

Therefore, setting $C^0 = \frac{\epsilon}{2\|\lambda^*\|\|\tilde{x}\|}\lambda^*\tilde{x}^T$, we have $\|C^0\| < \epsilon$ and

$$f(x^0) + C^0 x^0 \leq f(\tilde{x}) + C^0 \tilde{x}, \quad f(x^0) + C^0 x^0 \neq f(\tilde{x}) + C^0 \tilde{x}.$$

Two last relations contradict the robustness of \tilde{x} (in the sense of Definition 3.1) and the proof is complete. \square

It is not difficult to see that, Theorem 5.2 will be valid if one replaces 0.5ϵ , in the considered uncertainty set, with some $\bar{\epsilon} \in (0, \epsilon)$.

In fact, the robustness notion considered in the present paper is sufficient for two above mentioned (worst case-based) notions. In definition studied in the present paper, the robustness is coming from a linear perturbation instead of an arbitrary perturbation and this leads to Proper efficiency as proved in Section 3. See also, Section 3 of Georgiev et al. (2013) for some comparison.

6. Modification of the objective function

In this section, we consider two robustness aspects of (weakly/properly) efficient solutions. In the first one, we consider a convex combination of the objective function of Problem (20) with a new special function. The second robustness aspect is due to adding a new objective function to the problem. In both cases, we examine preserving the weak/proper/robust efficiency.

Consider the following problem for $\alpha \in [0, 1]$,

$$\begin{aligned} \min & f(x) + (1 - \alpha)h(x)q \\ \text{s.t.} & g_i(x) \leq 0, \quad i \in \{1, 2, \dots, m\}, \end{aligned} \quad (31)$$

where $h: \mathbb{R}^n \rightarrow \mathbb{R}$ is a convex function and $0 \neq q \in \mathbb{R}_+^p$ is a p -vector with nonnegative components. We denote this program by $(MOP)_\alpha$, and this program coincides with (20) when $\alpha = 1$.

Note: In the whole of this section, we assume that the functions h_i ($i = 1, \dots, p$) and g_j ($j = 1, \dots, m$) are convex.

Theorem 6.1 presents a sufficient condition for properly efficient solutions of problems (20) and $(MOP)_0$ to remain properly efficient for $(MOP)_\alpha$.

Theorem 6.1. Let \tilde{x} be a proper efficient solution to both problems (20) and $(MOP)_0$. If (CQ) holds at \tilde{x} , then \tilde{x} is a proper efficient solution of $(MOP)_\alpha$ for each $\alpha \in (0, 1)$.

Proof. Since \tilde{x} is a properly efficient solution to Problem (20), then there exists $\lambda \in \mathbb{R}^p$ and $w \in \mathbb{R}^m$ such that (see Clarke, 2013):

$$0 \in \sum_{i=1}^p \lambda_i \partial f_i(\tilde{x}) + \sum_{j=1}^m w_j \partial g_j(\tilde{x}),$$

$$w_j g_j(\tilde{x}) = 0, \quad j = 1, \dots, m, \quad \lambda > 0, \quad w \geq 0.$$

Also, since \tilde{x} is a proper efficient solution to Problem $(MOP)_0$, there exist $\mu \in \mathbb{R}^p$ and $v \in \mathbb{R}^m$ such that $\mu > 0$, $v \geq 0$, and

$$0 \in \sum_{i=1}^p \mu_i \partial f_i(\tilde{x}) + q^T \mu \partial h(\tilde{x}) + \sum_{j=1}^m v_j \partial g_j(\tilde{x}), \quad (32)$$

$$v_j g_j(\tilde{x}) = 0, \quad j = 1, \dots, m. \quad (33)$$

Notice that the convexity of h is crucial in obtaining (32).

Let $\alpha \in (0, 1)$. We define t and γ as follows

$$t := \frac{\alpha q^T \mu}{\alpha q^T \mu + (1 - \alpha)q^T \lambda},$$

$$\gamma := t\lambda + (1 - t)\mu.$$

It is clear that $0 < t < 1$ and $\gamma > 0$. Also,

$$(1 - t)q^T \mu = (1 - \alpha)q^T \gamma. \quad (34)$$

Thus,

$$\begin{aligned} & \sum_{i=1}^p \gamma_i \partial f_i(\tilde{x}) + (1 - \alpha)q^T \gamma \partial h(\tilde{x}) \\ &= t \sum_{i=1}^p \lambda_i \partial f_i(\tilde{x}) + (1 - t) \sum_{i=1}^p \mu_i \partial f_i(\tilde{x}) + (1 - t)q^T \mu \partial h(\tilde{x}). \end{aligned}$$

Therefore, setting $z = tw + (1 - t)v$, we get

$$\begin{aligned} 0 & \in \sum_{i=1}^p (t\lambda_i + (1 - t)\mu_i) \partial f_i(\tilde{x}) + (1 - t)q^T \mu \partial h(\tilde{x}) \\ & \quad + \sum_{j=1}^m (tw_j + (1 - t)v_j) \partial g_j(\tilde{x}) \\ &= \sum_{i=1}^p \gamma_i \partial f_i(\tilde{x}) + (1 - \alpha)q^T \gamma \partial h(\tilde{x}) + \sum_{j=1}^m z_j \partial g_j(\tilde{x}), \end{aligned}$$

where $\gamma > 0$ and $z \geq 0$. Therefore \tilde{x} is a global minimum for

$$\min \sum_{i=1}^p \gamma_i f_i(x) + (1 - \alpha)q^T \gamma h(x)$$

$$\text{s.t. } g_j(x) \leq 0, \quad j = 1, \dots, m.$$

This implies that \tilde{x} is a proper efficient solution of $(MOP)_\alpha$, according to Theorem 3.11 in Ehrgott (2005). \square

The following two results give sufficient conditions for efficient (resp. weakly efficient) solutions of problems (20) and $(MOP)_0$ to remain efficient (resp. weakly efficient) for $(MOP)_\alpha$. These results extend Proposition 2.2 in Georgiev et al. (2013).

Theorem 6.2. Let \tilde{x} be an efficient solution to both Problems (20) and $(MOP)_0$. Then \tilde{x} is an efficient solution of $(MOP)_\alpha$ for each $\alpha \in (0, 1)$.

Proof. Let $\alpha \in (0, 1)$. By contradiction assume that there exists a feasible point, \hat{x} , such that

$$f(\hat{x}) + (1 - \alpha)qh(\hat{x}) \leq f(\tilde{x}) + (1 - \alpha)qh(\tilde{x}),$$

$$f(\hat{x}) + (1 - \alpha)qh(\hat{x}) \neq f(\tilde{x}) + (1 - \alpha)qh(\tilde{x}).$$

If $h(\tilde{x}) < h(\hat{x})$, then

$$f(\hat{x}) - f(\tilde{x}) \leq (1 - \alpha)q(h(\tilde{x}) - h(\hat{x})) \leq (\text{and } \neq) 0.$$

This contradicts the efficiency of \tilde{x} for (20). Hence, we assume that $h(\tilde{x}) \geq h(\hat{x})$. Due to the convexity assumption, we have

$$\begin{aligned} & f\left(\frac{1}{2}\hat{x} + \frac{1}{2}\tilde{x}\right) + qh\left(\frac{1}{2}\hat{x} + \frac{1}{2}\tilde{x}\right) \\ & \leq \frac{1}{2}f(\hat{x}) + \frac{1}{2}f(\tilde{x}) + \frac{1}{2}qh(\hat{x}) + \frac{1}{2}qh(\tilde{x}) \\ & \leq (\text{and } \neq) f(\tilde{x}) + \frac{1}{2}(1 - \alpha)q(h(\tilde{x}) - h(\hat{x})) + \frac{1}{2}qh(\hat{x}) + \frac{1}{2}qh(\tilde{x}) \\ & = f(\tilde{x}) + q(h(\tilde{x}) + \frac{\alpha}{2}(h(\hat{x}) - h(\tilde{x}))) \leq f(\tilde{x}) + qh(\tilde{x}). \end{aligned}$$

Hence, setting $z = \frac{1}{2}\hat{x} + \frac{1}{2}\tilde{x}$, the vector z is feasible and

$$f(z) + qh(z) \leq f(\tilde{x}) + qh(\tilde{x}) \quad \text{and} \quad f(z) + qh(z) \neq f(\tilde{x}) + qh(\tilde{x}).$$

This contradicts the efficiency of \tilde{x} for $(MOP)_0$, and the proof is complete. \square

Theorem 6.3. Let \bar{x} be a weak efficient solution to both Problems (20) and $(MOP)_0$. Then \bar{x} is a weak efficient solution of $(MOP)_\alpha$ for each $\alpha \in (0, 1)$.

Proof. The proof of this theorem is similar to that of Theorem 6.2 and is hence omitted. \square

Theorem 6.4 gives a sufficient condition for robust efficient solutions of Problems (20) and $(MOP)_0$ to remain robust efficient for $(MOP)_\alpha$.

Theorem 6.4. Let \bar{x} be a robust efficient solution for both Problems (20) and $(MOP)_0$. If (CQ) holds at \bar{x} , then \bar{x} is a robust efficient solution for $(MOP)_\alpha$ for each $\alpha \in [0, 1]$.

Proof. Let $\alpha \in [0, 1]$. By Theorem 6.2, \bar{x} is efficient for Problem $(MOP)_\alpha$. Now, we show that \bar{x} is robust for $(MOP)_\alpha$. By Theorem 3.4,

$$\text{Pos}\left(\bigcup_{i=1}^p \partial f_i(\bar{x})\right) + \text{Pos}\left(\bigcup_{i \in A(\bar{x})} \partial g_i(\bar{x})\right) = \mathbb{R}^n$$

and

$$\text{Pos}\left(\bigcup_{i=1}^p \partial(f_i + q_i h)(\bar{x})\right) + \text{Pos}\left(\bigcup_{i \in A(\bar{x})} \partial g_i(\bar{x})\right) = \mathbb{R}^n.$$

By the above two equalities, and since all of the ∂ -sets are convex here, we have $\text{Pos}\left(\bigcup_{i=1}^p \partial(f_i + (1-\alpha)q_i h)(\bar{x})\right) + \text{Pos}\left(\bigcup_{i \in A(\bar{x})} \partial g_i(\bar{x})\right) = \mathbb{R}^n$. Therefore, \bar{x} is a robust efficient solution for $(MOP)_\alpha$, because of Theorem 3.6. \square

In the rest of this section, we examine adding a new objective function to Problem (20). Consider the following multi-objective optimization problem, denoted by $(MOPh)$:

$$\begin{aligned} \min \quad & \begin{pmatrix} f(x) \\ h(x) \end{pmatrix} \\ \text{s.t.} \quad & g_i(x) \leq 0 \quad i = 1, \dots, m, \end{aligned} \quad (35)$$

where $h: \mathbb{R}^n \rightarrow \mathbb{R}$. The following two theorems address some connections between the proper efficient solutions of two problems (20) and $(MOPh)$. Recall that the functions h, f_i , and g_j are convex.

Theorem 6.5. Let \bar{x} be a proper efficient solution to Problem $(MOPh)$. If (CQ) holds at \bar{x} , and $\partial h(\bar{x}) \subseteq \text{Pos}\left(\bigcup_{i=1}^p \partial f_i(\bar{x})\right) + \text{Pos}\left(\bigcup_{i \in A(\bar{x})} \partial g_i(\bar{x})\right)$, then \bar{x} is a proper efficient solution to Problem (20).

Proof. Since \bar{x} is a proper efficient solution of Problem $(MOPh)$, then there exist $\lambda \in \mathbb{R}^p$ and $w \in \mathbb{R}^m$ such that (see Clarke, 2013):

$$\begin{aligned} 0 &\in \sum_{i=1}^p \lambda_i \partial f_i(\bar{x}) + \partial h(\bar{x}) + \sum_{j=1}^m w_j \partial g_j(\bar{x}), \\ w_j g_j(\bar{x}) &= 0, \quad j = 1, \dots, m \quad \lambda > 0, \quad w \geq 0. \end{aligned}$$

Therefore, by assumption of the theorem, $0 \in \sum_{i=1}^p \tilde{\lambda}_i \partial f_i(\bar{x}) + \sum_{j=1}^m \tilde{w}_j \partial g_j(\bar{x})$, for some $\tilde{\lambda} > 0, \tilde{w} \geq 0$. This implies that \bar{x} is a proper efficient solution for Problem (20). \square

By a manner similar to the proof of the above theorem, it can be shown that this theorem is valid for weak efficient solutions as well. The following example shows that this result may not be valid for efficient solutions.

Example 6.1. Let $g(x) = f(x) = x$ and $h(x) = x^2$. It is clear that $\bar{x} = 0$ is an efficient solution of $(MOPh)$ and $\{\nabla h(0)\} \subseteq \text{Pos}(\nabla f(0))$ but $\bar{x} = 0$ is not an efficient solution of Problem (20).

The following result gives more connections between the proper efficient solutions of two problems (20) and $(MOPh)$ when f_i and g_j functions are continuously differentiable.

Theorem 6.6.

- (i) Let \bar{x} be a proper efficient solution to both Problems (20) and $(MOPh)$. If (CQ) holds at \bar{x} , then there exist vectors $u \in \mathbb{R}^p$ and

$v \in \mathbb{R}^m$ such that $u > 0$, and

$$\left(\sum_{i=1}^p u_i \nabla f_i(\bar{x}) + \sum_{j=1}^m v_j \nabla g_j(\bar{x}) \right) \in \partial h(\bar{x}).$$

- (ii) Let \bar{x} be a proper efficient solution to Problem (20). If there exist vectors $u \in \mathbb{R}^p$ and $v \in \mathbb{R}^{|A(\bar{x})|}$ such that $v \geq 0$ and

$$\left(\sum_{i=1}^p u_i \nabla f_i(\bar{x}) - \sum_{j \in A(\bar{x})} v_j \nabla g_j(\bar{x}) \right) \in \partial h(\bar{x}),$$

then \bar{x} is a proper efficient solution for Problem $(MOPh)$.

Proof.

- (i) Since \bar{x} is a proper efficient solution of Problem $(MOPh)$, then there exist $(\lambda, \lambda_{p+1}) \in \mathbb{R}^p \times \mathbb{R}$ and $w \in \mathbb{R}^m$ such that

$$\begin{aligned} 0 &\in \sum_{i=1}^p \lambda_i \nabla f_i(\bar{x}) + \lambda_{p+1} \nabla h(\bar{x}) + \sum_{j=1}^m w_j \nabla g_j(\bar{x}), \\ w_j g_j(\bar{x}) &= 0, \quad j = 1, \dots, m, \quad \lambda > 0, \quad w \geq 0. \end{aligned}$$

Therefore, there exists some $d \in \partial h(\bar{x})$ such that

$$d = - \sum_{i=1}^p \frac{\lambda_i}{\lambda_{p+1}} \nabla f_i(\bar{x}) - \sum_{j=1}^m \frac{w_j}{\lambda_{p+1}} \nabla g_j(\bar{x}).$$

On other hand, \bar{x} is a proper efficient solution to Problem (20). Therefore, there exist $\lambda' \in \mathbb{R}^p$ and $w' \in \mathbb{R}^m$ such that

$$0 = \sum_{i=1}^p \lambda'_i \nabla f_i(\bar{x}) + \sum_{j=1}^m w'_j \nabla g_j(\bar{x}), \quad (36)$$

$$w'_j g_j(\bar{x}) = 0, \quad j = 1, \dots, m \quad \lambda' > 0, \quad w' \geq 0. \quad (37)$$

Let $t > \max_{1 \leq i \leq p} \left\{ \frac{\lambda_i}{\lambda'_i \lambda_{p+1}} \right\}$. We have

$$d = \sum_{i=1}^p \left(t \lambda'_i - \frac{\lambda_i}{\lambda_{p+1}} \right) \nabla f_i(\bar{x}) + \sum_{j=1}^m \left(t w'_j - \frac{w_j}{\lambda_{p+1}} \right) \nabla g_j(\bar{x}).$$

Setting $u_i := t \lambda'_i - \frac{\lambda_i}{\lambda_{p+1}}$ and $v_j = t w'_j - \frac{w_j}{\lambda_{p+1}}$, completes the proof of part (i).

- (ii) Setting $\mu_j = 0$ for each $j \notin A(\bar{x})$, by the assumption of the theorem, we have $0 = - \sum_{i=1}^p u_i \nabla f_i(\bar{x}) + d + \sum_{j=1}^m \mu_j \nabla g_j(\bar{x})$ for some $d \in \partial h(\bar{x})$. On other hand, since \bar{x} is a proper efficient solution to Problem (20), there exist $\lambda' \in \mathbb{R}^p$ and $w' \in \mathbb{R}^m$ satisfying (36) and (37). For $t > \max_{1 \leq i \leq p} \left\{ \frac{u_i}{\lambda'_i} \right\}$, we have

$$\begin{aligned} 0 &= \sum_{i=1}^p (t \lambda'_i - u_i) \nabla f_i(\bar{x}) + d + \sum_{j=1}^m (t w'_j + \mu_j) \nabla g_j(\bar{x}) \\ &\in \sum_{i=1}^p (t \lambda'_i - u_i) \nabla f_i(\bar{x}) + \partial h(\bar{x}) + \sum_{j=1}^m (t w'_j + \mu_j) \nabla g_j(\bar{x}). \end{aligned}$$

Therefore, \bar{x} is a proper efficient solution to $(MOPh)$, and the proof is complete. \square

The following example shows that part (i) of the above theorem may not hold when some f_i or g_j functions are nonsmooth. A similar example can be constructed for part (ii).

Example 6.2. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x_2 - x_1 & x_2 > 0 \\ -x_1 & x_2 \leq 0 \end{cases}$$

Consider the following optimization problem

$$\min f(x)$$

$$\text{s.t.} \quad g(x) = x_1 - x_2 \leq 0.$$

The functions f and g are convex and

$$\partial f \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} \right) = \left\{ \begin{pmatrix} -1 \\ \alpha \end{pmatrix} : \alpha \in [0, 1] \right\}, \quad \partial g \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} \right) = \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}.$$

With $\lambda = \alpha = \mu = 1$, we have:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \lambda \begin{pmatrix} -1 \\ \alpha \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Therefore, $\bar{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is an optimal solution to the above problem. Now consider the function $h(x) = x_1$ and the following problem

$$\min \begin{pmatrix} f(x) \\ h(x) \end{pmatrix} \quad (38)$$

s.t. $g(x) = x_1 - x_2 \leq 0$.

We have $\partial h(\bar{x}) = \{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \}$. Also, for $\lambda_1 = \lambda_2 = 1$ and $\alpha = \mu = 0$, we get

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \lambda_1 \begin{pmatrix} -1 \\ \alpha \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Therefore, $\bar{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a proper efficient solution to Problem (38).

Hence, in this example, $\bar{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a proper efficient solution to both problems (20) and (MOPh), while there is not any $\lambda > 0$ satisfying

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \lambda \begin{pmatrix} -1 \\ \alpha \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

for some $\mu \in \mathbb{R}$ and $\alpha \in [0, 1]$. It shows that part (i) of Theorem 6.6 may not be valid in the presence of nonsmooth f_i or g_j functions.

The last theorem of this section provides a connection between the robust solutions of two problems (20) and (MOPh).

Theorem 6.7. *If \bar{x} be a robust efficient solution to Problem (20), then \bar{x} is a robust efficient solution of (MOPh). The converse holds if*

$$\partial h(\bar{x}) \subseteq \text{Pos} \left(\bigcup_{i=1}^p \partial f_i(\bar{x}) \right) + \text{Pos} \left(\bigcup_{i \in A(\bar{x})} \partial g_i(\bar{x}) \right).$$

Proof. These are derived from Theorems 3.3 and 3.8. \square

Acknowledgement

The authors would like to express their gratitude to anonymous referees for their helpful comments on the first version of this

paper. This research was in part supported by a Grant from IPM (No. 93260124).

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