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SEMI-DIFFERENTIABILITY OF THE MARGINAL MAPPING IN VECTOR OPTIMIZATION*

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Abstract. We consider a parametric multiobjective optimization problem whose objective function and constraint set are not necessarily convex. We introduce the concept of uniform efficiency, characterize it and compare it with the well-known concepts of proper efficiency and normal efficiency. Then we establish the semi-differentiability of the marginal (efficient value) mapping and a formula to compute its semi-derivative at a uniformly efficient value. As an application we derive semi-differentiability of the marginal function for a problem in which the constraint set is given by a system of inequalities, and for a problem whose constraint set is a union of polyhedral convex sets. The results of this paper are not only new in the case of nonconvex multiobjective problems, but they also deepen some existing ones for the convex case.

Key words. sensitivity analysis, lower semi-derivative, marginal mapping, parametric multiobjective optimization

AMS subject classifications. 90C29, 90C31, 90C30

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1. Introduction.

Stability and sensitivity analysis constitutes an important part in the theory of mathematical optimization. It studies continuity and differentiability properties of the optimal solution mapping and the optimal value function (called also the marginal function) of a parametric optimization problem. These properties are indispensable in post-optimal analysis and the convergence of algorithms. Excellent books on this topic already exist, for instance [1, 2, 4, 8, 18, 21] and many others, without mentioning numerous research papers on it. As to multiobjective optimization, there are also a lot of works devoted to continuity properties of the efficient solution and efficient frontier mappings in a very general setting of infinite-dimensional spaces, sometimes with moving ordering cones (see [3, 10, 13, 14, 16, 22] and the references given therein). Tanino's paper [26] seems to be the first work dealing with differentiability of the marginal mapping (called also efficient value mapping). Tanino used contingent derivatives in the framework of convex problems and established some estimates for the contingent derivative of the marginal mapping by way of the efficient set of the derivative of the value mapping. Similar results were then extended for the proto-derivative, Clarke derivative, coderivative, and epiderivatives in [5, 6, 10, 12] and some others. Among derivatives of set-valued mappings, the semi-derivative and strict semi-derivative introduced in [19] and largely studied in [4, 10, 20, 27] seem to be most interesting because they are small and enjoy quite nice calculus. Of course,

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semi-differentiability is a strong property and its validity can be expected only under a certain constraint qualification. To the best of our knowledge, sensitivity analysis in terms of semi-derivatives for vector optimization was first addressed in [24, 25, 28]. The authors of these works gave a formula of the so-called mth order lower Studniarski derivative, which coincides with the lower semi-derivative when $m = 1$, for the marginal mapping in relation with that of the value mapping (see Remark 4). Our goal is to establish conditions for semi-differentiability of the efficient solution and the marginal mappings. A formula to compute their derivatives will be obtained as a by-product. To this end, we introduce the concept of uniform efficiency and compare it with the concept of normal efficiency introduced in [26] for convex sets. This concept and the concept of an asymptotic function are key ingredients of our analysis. They allow us to establish nice formulas to compute semi-derivatives of efficient solutions and efficient value mappings of nonconvex problems. In the convex case, our results strengthen a number of existing ones on contingent and proto-derivatives. To simplify the presentation we consider Pareto efficiency, which is determined by the positive orthant in a finite-dimensional Euclidean space. The results of the present paper can easily be translated to efficiency with respect to any partial order generated by a closed, convex, and pointed cone.

The paper is structured as follows. In section 2, notation and definitions are given for tangent cones of sets and for derivatives of set-valued mappings. Section 3 is devoted to the concept of uniform efficiency, which substitutes proper efficiency in the presence of a parameter. We present some characterizations of uniformly efficient points and their links with normally efficient points of convex sets. In section 4, we study semi-differentiability of the marginal mapping of a parametric vector problem and establish a formula to compute its semi-derivative. It is the first time we obtain a condition for differentiability of the efficient solution mapping via the differentiability of the feasible set and the objective function. We discuss certain conditions, frequently invoked in the literature for contingent derivatives and proto-derivatives, and show that some of them are too restrictive and some others may lead to a stronger conclusion by using our approach. In the last two sections, we apply the general results of section 4 to a problem with inequality and set constraints and to a problem over a finite union of polyhedral convex sets.

2. Preliminaries. Let $X \subseteq \mathbb{R}^n$ be a nonempty set. Throughout this paper we shall make use of the standard notation $\mathrm{int}(X)$, $\mathrm{cl}(X)$, $cone(X)$, $pos(X)$, and X_∞ for the interior, the closure, the conic hull, the positive hull, and the asymptotic cone of X, respectively. Given $\bar{x} \in X$, the contingent cone, the adjacent cone, the Clarke tangent cone, the convex analysis normal cone, the ϵ-normal set for $\epsilon \geqq 0$, and the limiting (Mordukhovich) normal cone to X at $\bar{x}$ are respectively denoted by $T_X(\bar{x})$, $T_X^{adj}(\bar{x})$, $T_X^{cl}(\bar{x})$, $N_X^{co}(\bar{x})$, $\hat{N}_X^{\epsilon}(\bar{x})$, and $N_X(\bar{x})$. The closed unit ball of $\mathbb{R}^n$ is denoted by B_n. For a set-valued mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$, the domain and the graph of F are denoted by $domF$ and $gphF$, respectively. The outer limit, the inner limit and the outer horizon limit of F at $\bar{x} \in domF$ are respectively defined by

$$\limsup_{x\to\bar{x}} F(x) := \{y \in \mathbb{R}^p : \exists (x_\nu, y_\nu) \in gphF \text{ such that } (x_\nu, y_\nu) \to (\bar{x}, y) \text{ as } \nu \to +\infty\},$$
$$\liminf_{x\to\bar{x}} F(x) := \{y \in \mathbb{R}^p : \forall x_\nu \in domF \text{ with } x_\nu \to \bar{x}, \exists y_\nu \in F(x_\nu), y_\nu \to y \text{ as } \nu \to +\infty\},$$
$$F^\infty(\bar{x}) := \limsup_{t\downarrow 0, x\to\bar{x}} tF(x).$$

We know that in terms of set limits, $X_\infty = \limsup_{t\downarrow 0} tX$. The tangent cones and the normal cones we mentioned before are expressed as follows:

$$T_X(\bar{x}) = \limsup_{t\downarrow 0} \frac{X-\bar{x}}{t}, \qquad N_X^{co}(\bar{x}) = \{v \in \mathbb{R}^n : \langle v, x-\bar{x}\rangle \leqq 0 \text{ for all } x \in X\},$$

$$T_X^{adj}(\bar{x}) = \liminf_{t\downarrow 0} \frac{X-\bar{x}}{t}, \qquad \hat{N}_X^{\epsilon}(\bar{x}) = \left\{ v \in \mathbb{R}^n : \limsup_{x \xrightarrow{X} \bar{x}} \frac{\langle v, x-\bar{x}\rangle}{\|x-\bar{x}\|} \leqq \epsilon \right\},$$

$$T_X^{cl}(\bar{x}) = \liminf_{t\downarrow 0, x \xrightarrow{X} \bar{x}} \frac{X-x}{t}, \qquad N_X(\bar{x}) = \limsup_{x \xrightarrow{X} \bar{x}, \epsilon \downarrow 0} \hat{N}_X^{\epsilon}(x),$$

where $x \xrightarrow{X} \bar{x}$ signifies that x tends to $\bar{x}$ while lying within X, $\langle .,. \rangle$ is the inner product and $\|.\|$ is the Euclidean norm in $\mathbb{R}^n$.

The set-valued mapping F is said to be *closed* at $\bar{x} \in domF$ if $\limsup_{x\to\bar{x}} F(x) \subseteq F(\bar{x})$ and *closed around* $\bar{x}$ if it is closed at every point in a neighborhood of $\bar{x}$. It is *locally bounded* at $\bar{x}$ if there is a neighborhood V of $\bar{x}$ such that $F(V)$ is bounded. It is *upper (respectively, lower) semi-continuous* at $\bar{x} \in domF$ if for every open set W with $F(\bar{x}) \subseteq W$ (respectively, $F(\bar{x}) \cap W \neq \emptyset$) there is some neighborhood U of $\bar{x}$ such that $F(x) \subseteq W$ (respectively, $F(x) \cap W \neq \emptyset$) for all $x \in U$. It is *pseudo-Lipschitz* at $(\bar{x}, \bar{y}) \in gphF$ if there are neighborhoods U of $\bar{x}$, W of $\bar{y}$, and a constant $\ell \geqq 0$ such that

$$F(x_1) \cap W \subseteq F(x_2) + \ell \|x_1 - x_2\| B_p \quad \forall x_1, x_2 \in U,$$

and it is *calm* at $(\bar{x}, \bar{y}) \in gphF$ if there are neighborhoods U of $\bar{x}$, W of $\bar{y}$, and a constant $\ell \geqq 0$ such that

$$F(x) \cap W \subseteq F(\bar{x}) + \ell \|x - \bar{x}\| B_p \quad \forall x \in U.$$

If the latter inclusion holds for $W = \mathbb{R}^p$, one says that F is calm at $\bar{x}$.

The following derivatives of F at $(\bar{x}, \bar{y}) \in gphF$ in the direction $d \in \mathbb{R}^p$ will be used throughout this paper.

- The contingent, the adjacent, and the Clarke derivatives:

$$v \in DF(\bar{x}, \bar{y})(d) \Leftrightarrow (d, v) \in \limsup_{t\downarrow 0} \frac{gphF - (\bar{x}, \bar{y})}{t},$$

$$v \in D_{adj}F(\bar{x}, \bar{y})(d) \Leftrightarrow (d, v) \in \liminf_{t\downarrow 0} \frac{gphF - (\bar{x}, \bar{y})}{t},$$

$$v \in D_{cl}F(\bar{x}, \bar{y})(d) \Leftrightarrow (d, v) \in \liminf_{\substack{t\downarrow 0 \\ (x,y) \xrightarrow{gphF} (\bar{x}, \bar{y})}} \frac{gphF - (x, y)}{t}.$$

- The upper and lower Dini derivatives:

$$D_{upp}F(\bar{x}, \bar{y})(d) := \limsup_{t\downarrow 0, d'\to d} \frac{F(\bar{x} + td') - \bar{y}}{t},$$

$$D_{low}F(\bar{x}, \bar{y})(d) := \liminf_{t\downarrow 0, d'\to d} \frac{F(\bar{x} + td') - \bar{y}}{t}.$$

- The strictly lower Dini derivative:

$$D_{s-low}F(\bar{x}, \bar{y})(d) := \liminf_{\substack{t\downarrow 0, d'\to d \\ (x,y) \xrightarrow{gphF} (\bar{x}, \bar{y})}} \frac{F(x + td') - y}{t}.$$

The mapping F is said to be *proto-differentiable* (respectively, *Clarke differentiable, semi-differentiable*, and *strictly semi-differentiable*) at $(\bar{x}, \bar{y}) \in gphF$ if $D_{upp}F(\bar{x}, \bar{y}) = D_{adj}F(\bar{x}, \bar{y})$ (respectively, $D_{upp}F(\bar{x}, \bar{y}) = D_{cl}F(\bar{x}, \bar{y})$, $D_{upp}F(\bar{x}, \bar{y}) = D_{low}F(\bar{x}, \bar{y})$, and $D_{upp}F(\bar{x}, \bar{y}) = D_{s-low}F(\bar{x}, \bar{y})$.)

We note that the contingent derivative coincides with the upper Dini derivative and the strict lower Dini derivative is graphically the smallest one among the above-mentioned derivatives. Calculus rules and relations between them can be found in [10, 18, 21]. Finally, we mention below two known facts on pseudo-Lipschitz continuity and semi-differentiability that we will use in our proofs.

(R1) Assume $gphF$ is locally closed at $(\bar{x}, \bar{y})$ (its intersection with some closed neighborhood of $(\bar{x}, \bar{y})$ is closed). Then F is pseudo-Lipschitz at $(\bar{x}, \bar{y}) \in gphF$ if and only if $(\sigma, 0) \in N_{gphF}(\bar{x}, \bar{y})$ implies $\sigma = 0$ (see [18, Theorem 4.10]).

(R2) If F is pseudo-Lipschitz and proto-differentiable (respectively, Clarke differentiable) at $(\bar{x}, \bar{y})$, then it is semi-differentiable (respectively, strictly semi-differentiable) at $(\bar{x}, \bar{y})$ (see [21, Proposition 9.50]).

3. Uniformly efficient points.

Consider a parametric multiobjective optimization problem, denoted by $\mathrm{P}(u)$,

$$\begin{aligned} &\min\ f(u, x) \\ &\text{s.t. } x \in X(u), \end{aligned} \tag{1}$$

where "s.t." indicates "subject to", $f : \mathbb{R}^q \times \mathbb{R}^n \to \mathbb{R}^p$ is a continuous vector function and $X : \mathbb{R}^q \rightrightarrows \mathbb{R}^n$ is a set-valued mapping, called the *feasible solution mapping*. In this problem $u \in \mathbb{R}^q$ is a parameter and $x \in \mathbb{R}^n$ is a decision variable. The feasible value mapping $Y : \mathbb{R}^q \rightrightarrows \mathbb{R}^p$ is defined by

$$Y(u) := \{f(u, x) : x \in X(u)\}.$$

We recall that a feasible solution $\bar{x} \in X(u)$ is an *efficient solution* of $\mathrm{P}(u)$ if $f(\bar{x})$ is an efficient point of $Y(u)$, that is, if

$$Y(u) \cap (f(u, \bar{x}) - \mathbb{R}^p_+ \setminus \{0\}) = \emptyset, \tag{2}$$

where $\mathbb{R}^p_+$ denotes the positive orthant of $\mathbb{R}^p$. The set of all efficient points of $Y(u)$ is denoted by $MinY(u)$ and the set of all efficient solutions of $\mathrm{P}(u)$ is denoted by $S(u)$. The mapping $u \mapsto S(u)$ is called the *efficient solution mapping* and $u \mapsto V(u) := MinY(u)$ is called the *efficient value mapping* or the *marginal mapping*. We say that Y has the *domination property* around $(\bar{u}, \bar{y})$ if there is a neighborhood Q of $(\bar{u}, \bar{y})$ such that $y \in MinY(u) + \mathbb{R}^p_+$ for every $(u, y) \in Q \cap gphY$.

Sometimes one is given a convex and pointed cone K in $\mathbb{R}^p$ and defines K-efficient solutions by substituting K instead of $\mathbb{R}^p_+$ in (2). A feasible solution $x \in X(u)$ is called a *weakly efficient solution* of $\mathrm{P}(u)$ if it is K-efficient for $K = \mathrm{int}(\mathbb{R}^p_+) \cup \{0\}$, and it is called a *properly efficient solution* of $\mathrm{P}(u)$ if it is K-efficient for some K that contains $\mathbb{R}^p_+ \setminus \{0\}$ in its interior. Let e_i denote the ith unit coordinate vector and e the vector of ones in $\mathbb{R}^p$. For $\epsilon \geqq 0$, set

$$K_\epsilon := pos\{e_i + \epsilon e : i = 1, \dots, p\}.$$

The positive polar cone of K_ϵ is denoted by K^+_ϵ, that is,

$$K^+_\epsilon := \{v \in \mathbb{R}^p : \langle v, y \rangle \geqq 0 \quad \text{for all } y \in K_\epsilon\}.$$

We have $K_\epsilon \setminus \{0\} \subset \mathrm{int}(\mathbb{R}^p_+) \subset \mathbb{R}^p_+ \setminus \{0\} \subset \mathrm{int}(K^+_\epsilon)$ for every $\epsilon > 0$. It is clear that a feasible solution $x \in X(u)$ is weakly efficient if and only if it is K_ϵ-efficient for all $\epsilon > 0$, and it is properly efficient if and only if it is K^+_ϵ-efficient for some $\epsilon > 0$. We shall also use the following characterization of proper efficiency (see [9, Proposition 2.2] for a general result).

(R3) A point $\bar{y} \in Y(u)$ is properly efficient if and only if there is some $\epsilon > 0$ such that

$$\mathrm{cl}(cone(Y(u) - \bar{y})) \cap -K^+_\epsilon = \{0\}.$$

The concept of uniform efficiency is a key ingredient of our analysis.

DEFINITION 1. *Let $\bar{u}$ be given. An efficient point $\bar{y}$ of $Y(\bar{u})$ is said to be a uniformly efficient point of Y at $\bar{u}$ if there are some $\epsilon > 0$ and neighborhood Q of $(\bar{u}, \bar{y})$ such that y is a K^+_ϵ-efficient point of $Y(u)$ whenever $(u, y) \in gphV \cap Q$.*

Needless to say that when $p = 1$, a uniformly efficient point of Y at $\bar{u}$ or an efficient point of $Y(\bar{u})$ simply signifies the optimal value of $\mathrm{P}(\bar{u})$. Moreover, if $\bar{y} \in Y(\bar{u})$ is a uniformly efficient point of Y at $\bar{u}$, then it is a properly efficient point of $Y(\bar{u})$. In general, not every properly efficient point of $Y(\bar{u})$ is uniformly efficient (see Example 5 when $\bar{u} = 0$, $\bar{y} = (0, 0)$, and $Y = Y_1 \cup Y_2$). Below we provide some characterizations of uniform efficiency.

LEMMA 2. *Let $\bar{y} \in V(\bar{u})$. The following statements are equivalent:*

(i) *$\bar{y}$ is a uniformly efficient point of Y at $\bar{u}$;*

(ii) *there is a closed, convex, and pointed cone K containing $\mathbb{R}^p_+ \setminus \{0\}$ in its interior such that for every sequence $\{(u_k, y_k)\}_k \subseteq gphV$ converging to $(\bar{u}, \bar{y})$ the points y_k are all K-efficient when k is sufficiently large;*

(iii)

$$\left(\limsup_{(u,y) \xrightarrow{gphV} (\bar{u},\bar{y})} cone(Y(u) - y) \right) \cap -\mathbb{R}^p_+ = \{0\}.$$

Moreover, if $Y^\infty(\bar{u}) \cap -\mathbb{R}^p_+ = \{0\}$ and Y is closed at $\bar{u}$, then $\bar{y}$ is uniformly efficient if and only if

$$\left(\limsup_{\substack{t \downarrow 0 \\ (u,y) \xrightarrow{gphV} (\bar{u},\bar{y})}} \frac{Y(u) - y}{t} \right) \cap -\mathbb{R}^p_+ = \{0\}. \tag{3}$$

Proof. The implication (i) $\Rightarrow$ (ii) is evident. To prove the implication (ii) $\Rightarrow$ (iii), let v be a nonzero element of

$$\limsup_{(u,y) \xrightarrow{gphV} (\bar{u},\bar{y})} cone(Y(u) - y).$$

We find a sequence $\{u_k\}_k$ converging to $\bar{u}$, $y_k \in V(u_k)$, $y'_k \in Y(u_k)$, and real numbers $t_k \geqq 0$ such that $v = \lim_{k \to \infty} t_k(y'_k - y_k)$. According to (ii) and because $v \neq 0$, we may assume that y_k is K-efficient and $y'_k \neq y_k$ for all $k \geqq 0$. Then $y'_k - y_k \notin -K$, which implies that $v \notin -\mathrm{int}(K)$. We deduce that $v \notin -\mathbb{R}^p_+ \setminus \{0\}$. This proves (iii). Now assume (iii). Because both of the cones in the intersection of (iii) are closed, there is some $\epsilon > 0$ such that

$$\limsup_{(u,y) \xrightarrow{gphV} (\bar{u},\bar{y})} cone(Y(u) - y) \cap -K^+_{2\epsilon} = \{0\}.$$

Then there is a neighborhood Q of $(\bar{u},\bar{y})$ such that $cone(Y(u)-y)\cap -K_\epsilon^+ = \{0\}$ for $(u,y)\in Q\cap gphV$. This shows, in particular, that y is K_ϵ^+-efficient of $Y(u)$ for every $(u,y)\in Q\cap gphV$. By definition, $\bar{y}$ is a uniformly efficient point of Y at $\bar{u}$.

For the second part of the lemma, let $\bar{y}\in Y(\bar{u})$ be a uniformly efficient point of Y at $\bar{u}$. According to the first part, (iii) holds, which implies (3) because $(Y(u)-y)/t\subseteq cone(Y(u)-y)$. For the converse, suppose that $\bar{y}$ is not uniformly efficient. For each $\epsilon = 1/k$ we find u_k and $y_k\in V(u_k)$ and $z_k\in Y(u_k)$ such that $\lim_{k\to\infty}(u_k,y_k)=(\bar{u},\bar{y})$ and $z_k-y_k\in -K_{1/k}^+$. It can be seen that $\{z_k\}_k$ is bounded, because otherwise we would obtain from this sequence a nonzero vector $v\in Y^\infty(\bar{u})\cap -\mathbb{R}_+^p$, which is a contradiction. Then, we may assume either it converges to some vector $z\in Y(\bar{u}), z\neq\bar{y}$ because Y is closed at $\bar{u}$, or it converges to $\bar{y}$. The first case is impossible because $\bar{y}$ is an efficient point of $Y(\bar{u})$. In the second case, we may assume that $\{(z_k-y_k)/\|z_k-y_k\|\}_k$ converges to some nonzero vector w. It is clear that (3) does not hold, for w lies in the intersection on its left-hand side. The proof is complete. ◻

Another concept of efficient points introduced in [26] is also essential in the study of parametric convex problems. An efficient point $\bar{y}$ of $Y(\bar{u})$ is called *normally efficient* if

$$N^{co}_{Y(\bar{u})+\mathbb{R}_+^p}(\bar{y})\subset -\operatorname{int}(\mathbb{R}_+^p)\cup\{0\}.$$

Of course, normally efficient points are properly efficient, but the converse is generally not true. Here is a useful link between normally efficient points and uniformly efficient points when the set-valued mapping $Y+\mathbb{R}_+^p:\mathbb{R}^q\rightrightarrows\mathbb{R}^p$, defined by $(Y+\mathbb{R}_+^p)(u) = Y(u)+\mathbb{R}_+^p$ for $u\in\mathbb{R}^q$, is lower semi-continuous.

LEMMA 3. *Assume that $Y+\mathbb{R}_+^p$ has convex values around $\bar{u}$ and is lower semi-continuous at $\bar{u}$. Then every normally efficient point of $Y(\bar{u})$ is a uniformly efficient point of Y at $\bar{u}$.*

Proof. Let $\bar{y}\in V(\bar{u})$ be normally efficient. We claim that there is some $\epsilon>0$ such that

$$N^{co}_{Y(\bar{u})+\mathbb{R}_+^p}(\bar{y})\subseteq -K_\epsilon. \tag{4}$$

Indeed, let $\Delta := co\{e_1,\dots,e_p\}$ and $\Delta_0 := \Delta\cap(-N^{co}_{Y(\bar{u})+\mathbb{R}_+^p}(\bar{y}))$. Since the cone $-N^{co}_{Y(\bar{u})+\mathbb{R}_+^p}(\bar{y})$ is closed and convex and lies in $-\operatorname{int}(\mathbb{R}_+^p)\cup\{0\}$, the set Δ_0 is a compact set and lies in the relative interior of Δ. There is a real number $\epsilon>0$ such that every $y\in\Delta_0$ has components y^i, $i=1,\dots,p$, greater than or equal to ϵ. Consequently, we may express y as

$$y=\sum_{i=1}^p y^i e_i = \sum_{i=1}^p\left(y^i-\frac{\epsilon}{1+p\epsilon}\right)(e_i+\epsilon e)$$

with $y^i-\epsilon/(1+p\epsilon)\geq 0$, by which y belongs to K_ϵ. Because Δ_0 is a base of $-N^{co}_{Y(\bar{u})+\mathbb{R}_+^p}(\bar{y})$, we deduce that $N^{co}_{Y(\bar{u})+\mathbb{R}_+^p}(\bar{y}) = -cone(\Delta_0)\subseteq -K_\epsilon$, as requested. We claim further that

$$\limsup_{(u,y)\xrightarrow{gphY}(\bar{u},\bar{y})} N^{co}_{Y(u)+\mathbb{R}_+^p}(y)\subseteq N^{co}_{Y(\bar{u})+\mathbb{R}_+^p}(\bar{y}). \tag{5}$$

In fact, let

$$v\in \limsup_{(u,y)\xrightarrow{gphY}(\bar{u},\bar{y})} N^{co}_{Y(u)+\mathbb{R}_+^p}(y),$$

and say $v = \lim_{k\to\infty} v_k$, where $v_k \in N^{co}_{Y(u_k)+\mathbb{R}^p_+}(y_k)$ for some $(u_k, y_k) \in gphY$ that converges to $(\bar{u}, \bar{y})$ as k tends to ∞. Let y be an arbitrary element of $Y(\bar{u}) + \mathbb{R}^p_+$. Due to the semi-continuity hypothesis, we find some $z_k \in Y(u_k) + \mathbb{R}^p_+$ such that $\lim_{k\to\infty} z_k = y$. Then $\langle v_k, z_k - y_k\rangle \le 0$ for every $k \ge 1$. When k tends to ∞, we obtain $\langle v, y - \bar{y}\rangle \le 0$, which proves $v \in N^{co}_{Y(\bar{u})+\mathbb{R}^p_+}(\bar{y})$. In view of (4) and (5), there is a neigborhood Q of $(\bar{u}, \bar{y})$ such that

$$N^{co}_{Y(u)+\mathbb{R}^p_+}(y) \subseteq -K_{\epsilon/2} \quad \text{for all } (u, y) \in Q \cap gphY.$$

Because $Y+\mathbb{R}^p_+$ has convex values around $\bar{u}$, we may choose Q so small that $Y(u)+\mathbb{R}^p_+$ is convex when $(u, y) \in Q\cap gphV$. For such (u, y), the cone $N^{co}_{Y(u)+\mathbb{R}^p_+}(y)$ is nontrivial because y is a boundary point of $Y(u) + \mathbb{R}^p_+$. Choose a nonzero vector ξ from this cone. We have $\xi \in -\operatorname{int}(K_{\epsilon/4})$ and find some strict positive numbers $\alpha_i, i = 1, \dots, p$, such that $\xi = \sum_{i=1}^p -\alpha_i(e_i + (\epsilon/4)e)$. It follows that

$$pos(Y(u) - y) \subseteq -[N^{co}_{Y(u)+\mathbb{R}^p_+}(y)]^+ \subseteq -\{cone(\xi)\}^+ \subseteq \bigcup_{i=1}^{p} \{cone(e_i + (\epsilon/4)e)\}^+.$$

We deduce that $pos(Y(u)-y)\cap -\operatorname{int}(K^+_{\epsilon/4}) = \emptyset$ for all $(u, y) \in Q\cap gphV$. This implies (iii) of Lemma 2, by which $\bar{y}$ is uniformly efficient. ∎

We note that the conclusion of Lemma 3 may fail without convexity or the lower semi-continuity hypothesis. This is because normal efficiency reflects the position of a point with respect to a given value set, while uniform efficiency involves all value sets around a point. On the other hand, a uniformly efficient point is not necessarily normally efficient. For instance, the constant set-valued mapping $u \mapsto \mathbb{R}^p_+$ has a unique uniformly efficient point $\bar{y} = 0$, which is not normally efficient. Now let us establish a sufficient condition of uniform efficiency for union of convex sets.

THEOREM 4. *Let $Y = \bigcup_{i=1}^m Y_i$ and let $\bar{y} \in V(\bar{u})$. Assume the following conditions hold for every $i \in \{1, \dots, m\}$:*
(i) *the mapping $Y_i+\mathbb{R}^p_+$ has nonempty convex values and the domination property around $\bar{u}$, and is closed and lower semi-continuous at $\bar{u}$;*
(ii) $N^{co}_{Y_i(\bar{u})+\mathbb{R}^p_+}(\bar{y}) \subseteq -\operatorname{int}(\mathbb{R}^p_+) \cup \{0\}$ *if* $\bar{y} \in Y_i(\bar{u})$.
Then $\bar{y}$ is a uniformly efficient point of Y at $\bar{u}$.

Proof. We prove this theorem for $m = 2$. The proof for $m > 2$ follows the same lines. We suppose to the contrary that $\bar{y}$ is not uniformly efficient for Y at $\bar{u}$. In view of Lemma 2, there exist $(u_k, y_k) \in gphV$ converging to $(\bar{u}, \bar{y})$, $t_k > 0$ and $y'_k \in Y(u_k)$ such that

$$\lim_{k\to\infty} t_k(y'_k - y_k) = v \in -\mathbb{R}^p_+ \setminus \{0\}. \tag{6}$$

Without loss of generality we may assume that $(u_k, y_k) \in gphY_1$ for all $k \geqq 1$ and then $(\bar{u}, \bar{y}) \in gphY_1$ because $Y_1 + \mathbb{R}^p_+$ is closed at $\bar{u}$ and $\bar{y}$ is efficient. We distinguish two cases: (a) $(\bar{u}, \bar{y}) \notin gphY_2$, and (b) $(\bar{u}, \bar{y}) \in gphY_2$. In the first case, there is a neighborhood Q of $(\bar{u}, \bar{y})$ such that $Q \cap gph(Y_2 + \mathbb{R}^p_+) = \emptyset$.

Claim 1. $y'_k \in Y_1(u_k)$ for all k sufficiently large.

Indeed, if not, there exists a subsequence $\{(u_{\nu_k}, y'_{\nu_k})\}_k \subset gphY_2$. One may assume either (a1) $\lim_{k\to\infty} y'_{\nu_k} = y' \in Y_2(\bar{u}) + \mathbb{R}^p_+$ with $(\bar{u}, y') \notin Q$ because $Y_2 + \mathbb{R}^p_+$ is closed at $\bar{u}$, in which case $\lim_{k\to\infty} t_{\nu_k} = t_0$ for some $t_0 > 0$, or (a2) $\lim_{k\to\infty} \|y'_{\nu_k}\| = +\infty$, in which case $\lim_{k\to\infty} t_{\nu_k} = 0$. In the (a1) case, we obtain $y' = \bar{y} + v/t_0$, which is in contradiction with the fact that $\bar{y}$ is an efficient point of $Y(\bar{u})$. In the (a2) case, we choose an efficient point $\hat{y}$ of $Y_2(\bar{u})$. Due to the lower semi-continuity hypothesis, there exists $\hat{y}_{\nu_k} \in Y_2(u_{\nu_k}) + \mathbb{R}^p_+$ converging to $\hat{y}$. Now, for each positive number α we consider the convex combinations $(1-\alpha t_{\nu_k})\hat{y}_{\nu_k} + \alpha t_{\nu_k} y'_{\nu_k}$, which lie in $Y_2(u_{\nu_k}) + \mathbb{R}^p_+$ for k sufficiently large. Then

$$\hat{y} + \alpha v = \lim_{k\to\infty} \left((1-\alpha t_{\nu_k})\hat{y}_{\nu_k} + \alpha t_{\nu_k} y'_{\nu_k}\right) \in Y_2(\bar{u}) + \mathbb{R}^p_+$$

because $Y_2 + \mathbb{R}^p_+$ is closed at $\bar{u}$. This contradicts the choice of $\hat{y}$ and proves Claim 1. We return to relation (6). By Claim 1, both y'_k and y_k belong to $Y_1(u_k)$. Due to Lemma 3 and (ii), $\bar{y}$ is a uniformly efficient point of Y_1 at $\bar{u}$. Relation (6) is then in contradiction with (iii) of Lemma 2.

We now consider (b) the case in which $(\bar{u}, \bar{y}) \in gph(Y_2 + \mathbb{R}^p_+)$. It follows from the hypotheses and from Lemma 3 that $\bar{y}$ is uniformly efficient for both Y_1 and Y_2 at $\bar{u}$. In view of Lemma 2 and because of (6), we may assume that

$$y_k \in Y_1(u_k) \setminus Y_2(u_k), \quad y'_k \in Y_2(u_k) \setminus Y_1(u_k) \quad \text{for } k \geqq 1.$$

By considering $z'_k \in Y_2(u_k) + \mathbb{R}^p_+$ instead of y'_k if necessary, where z'_k is a unique point on the segment joining y_k and y'_k such that $[y_k, y'_k] \cap (Y_2(u_k) + \mathbb{R}^p_+) = [z'_k, y'_k]$, we may assume that

$$[y_k, y'_k] \cap (Y_2(u_k) + \mathbb{R}^p_+) = \{y'_k\}. \tag{7}$$

Note that when $z'_k \neq y'_k$, by a suitable change in t_k, we still have (6) with z'_k replacing y'_k. Moreover, due to the domination property, there are some $y''_k \in V_2(u_k)$ (the efficient set of $Y_2(u_k)$) and $r_k \in \mathbb{R}^p_+$ such that $y'_k = y''_k + r_k$.

Claim 2. $\lim_{k\to\infty} y''_k = \bar{y}$.

We notice first that the argument used to prove Claim 1 can be applied to show that $\lim_{k\to\infty} y'_k = \bar{y}$. Then, consider the sequence $\{r_k\}_k$. If it is bounded, we may assume it converges to some vector $r \in \mathbb{R}^p_+$. It follows that $\lim_{k\to\infty} y''_k = \bar{y} - r$. Because $Y_2 + \mathbb{R}^p_+$ is closed at $\bar{u}$ and $\bar{y}$ is an efficient point of $Y(\bar{u})$, we have $r = 0$, and so $\lim_{k\to\infty} y''_k = \bar{y}$. If that sequence is unbounded, we may assume $\lim_{k\to\infty} \|r_k\| = +\infty$ and $\lim_{k\to\infty} r_k/\|r_k\| = \tilde{r} \in \mathbb{R}^p_+ \setminus \{0\}$. By the convexity hypothesis, we have $y'_k - r_k/\|r_k\| \in Y_2(u_k) + \mathbb{R}^p_+$ for k sufficiently large, which, in view of the closedness hypothesis, implies $\bar{y} - \tilde{r} = \lim_{k\to\infty}(y'_k - r_k/\|r_k\|) \in Y_2(\bar{u}) + \mathbb{R}^p_+$. This is a contradiction because $\bar{y}$ is an efficient point of $Y(\bar{u}) + \mathbb{R}^p_+$.

Now we consider the convex sets

$$A_k := Y_2(u_k) + \mathbb{R}^p_+,$$
$$B_k := \{y''_k + t(y_k - y''_k) : t \geqq 0\}.$$

We have $\mathrm{int}(A_k) \neq \emptyset$ and $B_k \cap \mathrm{int}(A_k) = \emptyset$. The latter empty intersection is obtained from (7), because otherwise one should find some $t \in (0,1)$ such that $y''_k + t(y_k - y''_k) \in A_k$. This implies $y'_k + t(y_k - y'_k) \in [y_k, y'_k] \cap (Y_2(u_k) + \mathbb{R}^p_+)$, which contradicts (7). We separate them by a unit norm vector $\lambda_k \in \mathbb{R}^p$ as

$$\langle \lambda_k, y - y''_k \rangle \leqq 0 \leqq \langle \lambda_k, z - y''_k \rangle \quad \text{for all } y \in A_k,\ z \in B_k.$$

We deduce, in particular, that $\lambda_k \in N^{co}_{Y_2(u_k)+\mathbb{R}^p_+}(y''_k) \cap (-\mathbb{R}^p_+)$ and

$$\langle \lambda_k, y_k - y''_k \rangle \geqq 0 \quad \text{for } k \geqq 1. \tag{8}$$

We may assume that $\lim_{k\to\infty} \lambda_k = \lambda$ for some $\lambda \in -\mathbb{R}^p_+$. Due to Claim 2, the lower semi-continuity of $Y_2 + \mathbb{R}^p_+$ at $\bar{u}$, and (8), we obtain $\lambda \in N^{co}_{Y_2(\bar{u})+\mathbb{R}^p_+}(\bar{y})$ and

$$\langle \lambda, -v\rangle = \lim_{k\to\infty} \langle \lambda_k, t_k(y_k - y'_k)\rangle = \lim_{k\to\infty} \langle \lambda_k, t_k(y_k - y''_k - r_k)\rangle \geqq 0.$$

This is a contradiction because $\lambda \in -\operatorname{int}(\mathbb{R}^p_+)$ by (ii) and $-v \in \mathbb{R}^p_+ \setminus \{0\}$ by (6). The proof is complete. ∎

One should ask whether condition (ii) can be replaced by requiring $\bar{y}$ to be uniformly efficient for Y_i whenever it belongs to $Y_i(\bar{u})$. The answer is negative, as is shown by the next example.

Example 5. Consider $Y_1, Y_2 : \mathbb{R} \rightrightarrows \mathbb{R}^2$ defined by

$$\begin{aligned} Y_1(u) &:= co\left(\left\{\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right\}\right),\\ Y_2(u) &:= Y_1(u) + \begin{pmatrix}u\\u^2\end{pmatrix} \end{aligned}$$

for every $u \in \mathbb{R}$. Then the point $\bar{y}^T = (0 \quad 0)$ is uniformly efficient for both Y_1 and Y_2 at $\bar{u} = 0$, but not for their union. In this example condition (ii) of Theorem 4 is violated.

We close this section by considering a parametric vector problem of minimizing a linear function on a finite union of polyhedral convex sets. Namely, let $f(u,x) = C(u)x$, $X_i(u) = \{x \in \mathbb{R}^n : A_i(u)x \leqq b_i(u)\}$, where $C(u)$ is a $p \times n$-matrix, $A_i(u)$ is an $m_i \times n$-matrix, and $b_i(u)$ is an m_i-vector for each $u \in \mathbb{R}^q$ and $i = 1, \dots, m$. The problem $\mathrm{P}(u)$ is written as

$$\begin{aligned} \min \quad & C(u)x\\ \text{s.t. } & x \in \bigcup_{i=1}^m X_i(u). \end{aligned}$$

In our notation, $X(u) = \bigcup_{i=1}^m X_i(u)$, $Y(u) = \bigcup_{i=1}^m Y_i(u)$ with $Y_i(u) = \{C(u)x : x \in X_i(u)\}$, and $V_i(u) = Min(Y_i(u))$ for $i = 1, \dots, m$. In the next corollary, given a point u, the kernel of the linear operator $C(u)$ is denoted by $\operatorname{Ker} C(u)$.

COROLLARY 6. *Let $\bar{y} \in V(\bar{u})$. Assume that the following conditions hold:*
(i) *for every $i \in \{1, \dots, m\}$, the functions C, A_i, b_i are continuous at $\bar{u}$ and there exists x_i with $A_i(\bar{u})x_i < b(\bar{u})$;*
(ii) *for every $i \in \{1, \dots, m\}$, $\operatorname{Ker} C(\bar{u}) \cap [X_i(\bar{u})]_\infty = \{0\}$;*
(iii) *for $i \in \{1, \dots, m\}$ such that $\bar{y} \notin Y_i(\bar{u})$, one has*

$$C(\bar{u})[X_i(\bar{u})]_\infty \cap -\mathbb{R}^p_+ = \{0\}; \tag{9}$$

(iv) *for $i \in \{1, \dots, m\}$ such that $\bar{y} \in Y_i(\bar{u})$, one has*

$$\{\lambda \in -\mathbb{R}^p_+ : C^T(\bar{u})\lambda \in N^{co}_{X_i(\bar{u})}(\bar{x}_i)\} \subseteq -\operatorname{int}(\mathbb{R}^p_+) \cup \{0\},$$

where $\bar{x}_i \in X_i(\bar{u})$ such that $\bar{y} = C(\bar{u})\bar{x}_i$, $i \in \{1, \dots, m\}$.

Then $\bar{y}$ is a uniformly efficient point of Y at $\bar{u}$.

Proof. Our aim is to check the hypotheses of Theorem 4. First, we note that (9) is true for all $i = 1, \dots, m$. Indeed, if not, say for some i with $\bar{y} = C(\bar{u})\bar{x}_i \in Y_i(\bar{u})$, there is some nonzero vector $v = C(\bar{u})d \in C(\bar{u})[X_i(\bar{u})]_\infty \cap -\mathbb{R}^p_+$ with $d \in [X_i(\bar{u})]_\infty$. Pick any nonzero vector λ from the set on the left-hand side of the inclusion in (iv) (such a λ exists because $\bar{y}$ is an efficient point of $Y_i(\bar{u})$). Then $\langle C^T(\bar{u})\lambda, d\rangle = \langle \lambda, C(\bar{u})d\rangle \geqq 0$, which contradicts (iv). Second, we show that $Y_1, \dots, Y_m$ are closed and lower semi-continuous at $\bar{u}$. Observe that due to (i) the mappings $X_1, \dots, X_m$ are closed and lower semi-continuous at $\bar{u}$. Then the lower semi-continuity of $Y_1, \dots, Y_m$ is immediate. For the closedness of Y_i at $\bar{u}$, let $(u_k, y_k) \in gphY_i$ converge to $(\bar{u}, y)$. We find some $x_k \in X_i(u_k)$ such that $C(u_k)x_k = y_k$. If the sequence $\{x_k\}_k$ is bounded, then it admits a cluster point $x \in X_i(\bar{u})$. Then $y = C(\bar{u})x$, proving that $(\bar{u}, y) \in gphY_i$. If that sequence is unbounded, say $\lim_{k\to\infty} \|x_k\| = +\infty$, we may assume $\lim_{k\to\infty} x_k/\|x_k\| = v$ for some nonzero vector v. Then, on the one hand $C(\bar{u})v = \lim_{k\to\infty} C(u_k)x_k/\|x_k\| = \lim_{k\to\infty} y_k/\|x_k\| = 0$. On the other hand, $A_i(\bar{u})v = \lim_{k\to\infty} A_i(u_k)x_k/\|x_k\| \leqq \lim_{k\to\infty} b_i(u_k)/\|x_k\| = 0$. This contradicts (ii). Hence, $Y_1, \dots, Y_m$ are closed and semi-continuous at $\bar{u}$. Third, by definition, $Y_1(u), \dots, Y_m(u)$ are polyhedral convex sets. Moreover, due to (ii) and [15, Lemma 4.2], we have $C(\bar{u})[X_i(\bar{u})]_\infty = [Y_i(\bar{u})]_\infty$. Since (iii) is true for all $i = 1, \dots, m$, we deduce that $[Y_i(\bar{u})]_\infty \cap -\mathbb{R}^p_+ = \{0\}$. Due to the closedness of Y at $\bar{u}$, we have $[Y_i(u)]_\infty \cap -\mathbb{R}^p_+ = \{0\}$ for all u in a small neighborhood U of $\bar{u}$ too. Because $Y_i(u)$ is a convex polyhedral set, the latter equality implies its domination property around $\bar{u}$. Thus, $Y_1, \dots, Y_m$ are closed, semi-continuous at $\bar{u}$, and have convex values and the domination property around $\bar{u}$. Then so too are the mappings $Y_i + \mathbb{R}^p_+, i = 1, \dots, m$. Moreover, it follows from (iv) that the cone $N_{Y_i(\bar{u})+\mathbb{R}^p_+}(\bar{y})$ entirely lies in $-\operatorname{int}(\mathbb{R}^p_+) \cup \{0\}$ whenever $\bar{y} \in Y_i(\bar{u})$. It remains to apply Theorem 4 to complete the proof. ∎

We note that when $X_i(\bar{u})$ is bounded, conditions (ii) and (iii) of the preceding corollary are evidently satisfied. A sufficient condition for (iv) to hold is that $\bar{y}$ is a relative interior point of a $(p-1)$-dimensional face of $Y_i(\bar{u}) + \mathbb{R}^p_+$ if $\bar{y} \in Y_i(\bar{u}), i \in \{1, \dots, m\}$. The conclusion of Corollary 6 may fail if the last condition does not hold, as is shown by the next example.

Example 7. Let $X : \mathbb{R} \rightrightarrows \mathbb{R}^3$ be given by $X(u) =: \{x \in \mathbb{R}^3 : -x_1 \leqq -u, -ux_1 - x_2 \leqq 0, x_3 \leqq 0, -x_1 - x_2 - x_3 \leqq -u, x_1 + x_2 - x_3 \leqq 10\}$ and let $C(u)$ be given by

$$C(u) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

At $\bar{u} = 0$, the point $\bar{y}^T = (0 \quad 0 \quad 0)$ is not uniformly efficient, while $\hat{y}^T = (2 \quad 1 \quad -3)$ is uniformly efficient. Clearly $\bar{y}$ is not a relative interior point of a 2-dimensional face of $Y(\bar{u})$ at which condition (iv) does not hold, but $\hat{y}$ is.

4. Semi-differentiability of the marginal mapping. In this section, we shall focus only on conditions for the semi-differentiability of the marginal mapping V and find a formula to compute its semi-derivative. But, mutatis mutandis, most of the results of this section and the next ones remain true for strict semi-differentiability. Let us begin with outer and inner estimates for the semi-derivative of V.

PROPOSITION 8. *Let $\bar{y} \in V(\bar{u})$. The following statements hold for every $d \in \mathbb{R}^q$:*
(i) $D_{low}V(\bar{u}, \bar{y})(d) \subseteq WMin[D_{low}Y(\bar{u}, \bar{y})(d)]$;

(ii) $D_{low}V(\bar{u},\bar{y})(d) \subseteq Min[D_{low}Y(\bar{u},\bar{y})(d)]$ *provided that* $\bar{y}$ *is a uniformly efficient point;*

(iii) $D_{low}V(\bar{u},\bar{y})(d) \supseteq Min[D_{low}Y(\bar{u},\bar{y})(d)]$ *provided that* Y *is semi-differentiable at* $(\bar{u},\bar{y})$*, closed at* $\bar{u}$*, and has the domination property around* $(\bar{u},\bar{y})$*,* $Y^\infty(\bar{u})\cap -\mathbb{R}^p_+ = \{0\}$*, and* $DY(\bar{u},\bar{y})(0)\cap -\mathbb{R}^p_+ = \{0\}$*.*

Proof. Let $d \in \mathbb{R}^q$ be given. We prove (ii) first. Suppose to the contrary that the inclusion in (ii) does not hold, that is, there are some $w \in D_{low}V(\bar{u},\bar{y})(d)$, $w' \in D_{low}Y(\bar{u},\bar{y})(d)$ and a nonzero vector $r \in \mathbb{R}^p_+$ such that $w = w' + r$. Let $t_k > 0$ with $t_k \to 0$ and $d_k \to d$ be given. There are $z_k \in V(\bar{u} + t_k d_k)$ and $y_k \in Y(\bar{u} + t_k d_k)$ such that $\lim_{k\to\infty}(z_k - \bar{y})/t_k = w$ and $\lim_{k\to\infty}(y_k - \bar{y})/t_k = w - r$. It follows that $\lim_{k\to\infty}(y_k - z_k)/t_k = \lim_{k\to\infty}((y_k - \bar{y}) + (\bar{y} - z_k))/t_k = -r$. By Lemma 2 this contradicts the uniform efficiency hypothesis.

For the first statement, if the inclusion does not hold, then the vector r in the proof of the first statement belongs to $\operatorname{int}(\mathbb{R}^p_+)$. Hence for k sufficiently large, one has $y_k - z_k \in -\operatorname{int}(\mathbb{R}^p_+)$, which contradicts the fact that z_k is an efficient point of $Y(\bar{u} + t_k d_k)$.

For (iii), let $w \in Min[D_{low}Y(\bar{u},\bar{y})(d)]$. Let $t_k > 0$ with $t_k \to 0$ and $d_k \to d$ be arbitrarily given. Our aim is to find $z_k \in V(\bar{u} + t_k d_k)$ such that $\lim_{k\to\infty}(z_k - \bar{y})/t_k = w$. Note that, because $w \in D_{low}Y(\bar{u},\bar{y})(d)$, there are $y_k \in Y(\bar{u} + t_k d_k)$ such that $\lim_{k\to\infty}(y_k - \bar{y})/t_k = w$. Due to the domination property, we may find some $z_k \in V(\bar{u} + t_k d_k)$ and $r_k \in \mathbb{R}^p_+$ such that $y_k = z_k + r_k$. Consider the sequence $\{r_k/t_k\}_k$.

Claim 1. $\lim_{k\to\infty} r_k = 0$.

Indeed, if not, we may assume, without loss of generality, that either $\lim_{k\to\infty} r_k = r$ for some nonzero vector $r \in \mathbb{R}^p_+$, or $\lim_{k\to\infty} r_k/\|r_k\| = \bar{r}$ for some nonzero vector $\bar{r} \in \mathbb{R}^p_+$ with $\lim_{k\to\infty}\|r_k\| = +\infty$. In the first case, $\lim_{k\to\infty} z_k = \bar{y} - r \in Y(\bar{u})$ because Y is closed at $\bar{u}$, which contradicts the fact that $\bar{y}$ is an efficient point of $Y(\bar{u})$. In the second case

$$\lim_{k\to\infty}\frac{z_k}{\|r_k\|} = \lim_{k\to\infty}\left(\frac{y_k}{\|r_k\|} - \frac{r_k}{\|r_k\|}\right) = -\bar{r},$$

which shows that $-\bar{r} \in Y^\infty(\bar{u})$. This contradicts the assumption that $Y^\infty(\bar{u})\cap -\mathbb{R}^p_+ = \{0\}$.

Claim 2. The sequence $\{r_k/t_k\}_k$ is bounded.

In fact, suppose to the contrary that it is unbounded, say $\lim_{k\to\infty}\|r_{\nu_k}\|/t_{\nu_k} = +\infty$ for some subsequence $\{r_{\nu_k}/t_{\nu_k}\}_k$. Then we may assume that $\lim_{k\to\infty} r_{\nu_k}/\|r_{\nu_k}\| = r$ for some nonzero vector $r \in \mathbb{R}^p_+$ and derive

$$\lim_{k\to\infty}\frac{z_{\nu_k} - \bar{y}}{\|r_{\nu_k}\|} = \lim_{k\to\infty}\left(\frac{y_{\nu_k} - \bar{y}}{t_{\nu_k}}\frac{t_{\nu_k}}{\|r_{\nu_k}\|} - \frac{r_{\nu_k}}{\|r_{\nu_k}\|}\right) = -r,$$
$$\lim_{k\to\infty}\frac{(\bar{u} + t_{\nu_k}d_{\nu_k}) - \bar{u}}{\|r_{\nu_k}\|} = \lim_{k\to\infty} d_{\nu_k}\frac{t_{\nu_k}}{\|r_{\nu_k}\|} = 0.$$

Since $\lim_{k\to\infty} r_k = 0$ by Claim 1, these limits prove that $-r \in DY(\bar{u},\bar{y})(0)$, which contradicts the assumption that $DY(\bar{u},\bar{y})(0)\cap -\mathbb{R}^p_+ = \{0\}$.

Claim 3. $\lim_{k\to\infty} r_k/t_k = 0$.

We proceed by contradiction. In view of Claim 2, assume there is a subsequence

$\{r_{\nu_k}/t_{\nu_k}\}$ converging to some nonzero vector $\bar r \in \mathbb{R}^p_+$. We have

$$w - \bar r = \lim_{k\to\infty} \frac{(y_{\nu_k} - \bar y) + r_{\nu_k}}{t_{\nu_k}} = \lim_{k\to\infty} \frac{z_{\nu_k} - \bar y}{t_{\nu_k}},$$

which belongs to $DY(\bar u, \bar y)(d)$. The above limit also belongs to $D_{low}Y(\bar u, \bar y)(d)$ because Y is semi-differentiable at $(\bar u, \bar y)$. This contradicts the fact that w is an efficient point of $D_{low}Y(\bar u, \bar y)(d)$.

We return to the sequence $\{(\bar u + t_k d_k, z_k)\}_k \subset gphV$, which converges to $(\bar u, \bar y) \in gphV$ according to Claim 1. In view of Claim 3, we also have $\lim_{k\to\infty}(z_k - \bar y)/t_k = \lim_{k\to\infty}((y_k - \bar y) - r_k)/t_k = w$, by which $w \in D_{low}V(\bar u, \bar y)(d)$ and statement (iii) is proven. ∎

Remark 1. In general, the estimates given in the preceding proposition are not true either for contingent derivatives or for proto-derivatives (see also Remark 3). When Y is semi-differentiable, inclusion (i) is true for the contingent derivative (see also [11, Theorem 3.3]), that is,

$$DV(\bar u, \bar y)(d) \subseteq WMin[D_{low}Y(\bar u, \bar y)(d)]. \tag{10}$$

Corollary 9 shows that when $d \in domD_{low}V(\bar u, \bar y)$, the equality $D_{low}V(\bar u, \bar y)(d) = Min[D_{low}Y(\bar u, \bar y)(d)]$ can be established without hypothesis on the asymptotic directions of Y.

COROLLARY 9. *Let $\bar y \in V(\bar u)$ be a uniformly efficient point of Y. Assume that*
(i) *Y is semi-differentiable at $(\bar u, \bar y)$, closed at $\bar u$, and has the domination property around $(\bar u, \bar y)$, and*
(ii) $DY(\bar u, \bar y)(0) \cap -\mathbb{R}^p_+ = \{0\}$.
Then $D_{low}V(\bar u, \bar y)(d) = Min[D_{low}Y(\bar u, \bar y)(d)]$ for every $d \in domD_{low}V(\bar u, \bar y)$.

Proof. By a close inspection of the proof of Proposition 8 we need only to show that the sequence $\{r_k\}_k$ is bounded. We proceed by contradiction. Assume that $\lim_{k\to\infty}\|r_k\| = +\infty$ and $\lim_{k\to\infty} r_k/\|r_k\| = \bar r$ for some nonzero vector $\bar r \in \mathbb{R}^p_+$. Choose any vector $\tilde v \in D_{low}V(\bar u, \bar y)(d)$. By definition, there exists $\tilde z_k \in V(\bar u + t_k d_k)$ such that $\lim_{k\to\infty}(\tilde z_k - \bar y)/t_k = \tilde v$. Since $\bar y$ is uniformly efficient, we may assume that $\tilde z_k$ are all K^+_ϵ-efficient for some $\epsilon > 0$. We have also

$$\lim_{k\to\infty} \frac{z_k - \tilde z_k}{\|r_k\|} = \lim_{k\to\infty} \frac{z_k}{\|r_k\|} = -\bar r.$$

Hence, for k sufficiently large we have $(z_k - \tilde z_k)/\|r_k\| \in -\operatorname{int}(K^+_\epsilon)$, implying $z_k - \tilde z_k \in -\operatorname{int}(K^+_\epsilon)$. The latter relation is in contradiction with the K^+_ϵ-efficiency of $\tilde z_k$. The proof is complete. ∎

We are now able to give a main result on the semi-differentiability of the marginal mapping when the feasible value mapping is semi-differentiable.

THEOREM 10. *Let $\bar y \in V(\bar u)$ be a uniformly efficient point of Y. Assume the following conditions:*
(i) *Y is semi-differentiable at $(\bar u, \bar y)$, closed at $\bar u$, and has the domination property around $(\bar u, \bar y)$;*
(ii) $DY(\bar u, \bar y)(0) \cap -\mathbb{R}^p_+ = \{0\}$;
(iii) $Y^\infty(\bar u) \cap -\mathbb{R}^p_+ = \{0\}$.
Then V is semi-differentiable at $(\bar u, \bar y)$ and its semi-derivative is given by the formula

$$D_{low}V(\bar u, \bar y)(d) = Min[D_{low}Y(\bar u, \bar y)(d)] \quad \textit{for every } d \in \mathbb{R}^q. \tag{11}$$

Proof. Let $(d, w) \in gphDV(\bar{u}, \bar{y})$ with $(d, w) = \lim(u_\nu - \bar{u}, y_\nu - \bar{y})/s_\nu$ for some $y_\nu \in V(u_\nu)$, $s_\nu > 0$, and $s_\nu \to 0$ as ν tends to ∞. Let $t_k \downarrow 0$ and $d_k \to d$ be arbitrary and given. We have to find $\bar{w}_k \in V(\bar{u} + t_k d_k)$ such that

$$\lim_{k\to\infty} \frac{\bar{w}_k - \bar{y}}{t_k} = w. \tag{12}$$

To this end, we observe that because Y is semi-differentiable, there exists $z_k \in Y(\bar{u} + t_k d_k)$ such that

$$\lim_{k\to\infty} \frac{z_k - \bar{y}}{t_k} = w. \tag{13}$$

Due to the domination property, we find some $w_k \in V(\bar{u} + t_k d_k)$ and $r_k \in \mathbb{R}^p_+$ such that $z_k = w_k + r_k$. We wish to show that $\lim_{k\to\infty} r_k/t_k = 0$, which completes the proof by setting $\bar{w}_k = w_k$ and using (12) and (13). We observe that the proof of Claims 1 and 2 of Proposition 8 hold, while Claim 3 needs some explanation because in the present case it is not known whether w is an efficient element of $D_{low}Y(\bar{u}, \bar{y})(d)$. By contradiction, we suppose that Claim 3 is not true, that is, there is a subsequence $\{r_{\nu_k}/t_{\nu_k}\}_k$ that converges to some nonzero vector $\bar{r} \in \mathbb{R}^p_+$. It follows from (13) that $(d, w - \bar{r}) \in gphDV(\bar{u}, \bar{y})$. Again, by the semi-differentiability of Y, for the subsequences $\{s_{\nu_k}\}_k$ and $\{d_{\nu_k}\}_k$ from $\{s_\nu\}_\nu$ and $\{d_\nu\}_\nu$ given at the beginning of the proof, we may find some $y'_{\nu_k} \in Y(\bar{u} + s_{\nu_k} d_{\nu_k})$ such that

$$\lim_{k\to\infty} \frac{y'_{\nu_k} - \bar{y}}{s_{\nu_k}} = w - \bar{r}.$$

We deduce that

$$\lim_{k\to\infty} \frac{y'_{\nu_k} - y_{\nu_k}}{s_{\nu_k}} = \lim_{k\to\infty} \frac{(y'_{\nu_k} - \bar{y}) + (\bar{y} - y_{\nu_k})}{s_{\nu_k}} = -\bar{r}.$$

In view of Lemma 2, the latter relation contradicts the uniform efficiency of $\bar{y}$. We conclude that V is semi-differentiable at $(\bar{u}, \bar{y})$. The formula for the semi-derivative of V at $(\bar{u}, \bar{y})$ is obtained from Proposition 8. ∎

Remark 2. The domination property assumption in Proposition 8 and Theorem 10 can be replaced by requesting that Y be closed-valued around $\bar{u}$. Indeed, condition $Y^\infty(\bar{u}) \cap -\mathbb{R}^p_+ = \{0\}$ implies that for u close to $\bar{u}$ and for every y, the set $Y(u) \cap (y - \mathbb{R}^p_+)$ is bounded, and hence compact. Therefore, for each $y \in Y(u)$, that set admits an efficient point dominating y.

Remark 3. In part (iii) of Proposition 8 (hence condition (i) of Theorem 10 and the subsequent results), the closedness of Y at $\bar{u}$ can be replaced by the closedness of $Y + \mathbb{R}^p_+$ around $(\bar{u}, \bar{y})$. Indeed, the closedness hypothesis was used to derive $\lim_{k\to\infty} z_k = \bar{y} - r \in Y(\bar{u})$. The limit $\bar{y} - r$ may lie outside of $Y(\bar{u})$ when Y is not closed at $\bar{u}$, but the segment joining $(\bar{u}, \bar{y} - r)$ and $(\bar{u}, \bar{y})$ does lie in the closure of the graph of $Y + \mathbb{R}^p_+$. Because $Y + \mathbb{R}^p_+$ is closed around $(\bar{u}, \bar{y})$, we find some point $(\bar{u}, \hat{y})$ in the interior of that segment, which belongs to the graph of $Y + \mathbb{R}^p_+$. This contradicts the hypothesis that $\bar{y}$ is an efficient point of $Y(\bar{u})$.

Remark 4. We note that the formula of the lower semi-derivative established in Corollary 9 and Theorem 10 was already known in [24, Theorem 4.1 and Corollary 4.1]. There are, however, some inadequacies in that paper, some of which were already pointed out in [28]. Unfortunately, there is still a problem with these works because the main results of [24] and [28], including the above mentioned Theorem 4.1 and Corollary 4.1, are based on Proposition 3.2 of [24], which is wrong in general. A suitable modification of an example given in [21, p. 199] illustrates this observation.

Example 11. Let $Y : \mathbb{R} \rightrightarrows \mathbb{R}^2$ be given by

$$Y(u) := \begin{cases} \big\{(y_1, 0) : u\sin(\ln(u)) \leqq y_1 \leqq u\big\} \cup \big\{(2u, -u)\big\} & \text{for } u > 0, \\ \{(0,0)\} & \text{for } u = 0, \\ \big\{(u\sin(\ln(|u|)), 0), (2|u|, -|u|)\big\} & \text{for } u < 0. \end{cases}$$

Then at $(\bar{u}, \bar{y}) = (0, (0,0))$, we have

$$D_{low}(Y + \mathbb{R}^2_+)(\bar{u}, \bar{y})(d) = \big\{(l_1, l_2) : l_1 \geqq |d|, l_2 \in \mathbb{R}^2_+\big\} \cup \big\{(l_1, l_2) : l_1 \geqq 2|d|, l_2 \geqq -|d|\big\},$$
$$D_{low}Y(\bar{u}, \bar{y})(d) = \begin{cases} \{(d, 0), (2d, -d)\} & \text{for } d > 0, \\ \{(2|d|, -|d|)\} & \text{for } d \leq 0. \end{cases}$$

Hence $\min D_{low}(Y + \mathbb{R}^2_+)(\bar{u}, \bar{y})(-1) \nsubseteq D_{low}Y(\bar{u}, \bar{y})(-1)$, which refutes Proposition 3.2 of [24]. Moreover, since

$$V(u) = \begin{cases} \{(u\sin(\ln(|u|)), 0), (2|u|, -|u|)\} & \text{for } u \neq 0, \\ \{(0,0)\} & \text{for } u = 0, \end{cases}$$

we have $D_{low}V(\bar{u}, \bar{y})(1) = \{(2, -1)\}$, by which $\min D_{low}Y(\bar{u}, \bar{y})(1) \nsubseteq D_{low}V(\bar{u}, \bar{y})(1)$. Notice that for this mapping Y all assumptions of [24, Theorem 4.1 and Corollary 4.1] and [28, Theorems 2.1 and 2.2] are satisfied; nevertheless, equality between $\min D_{low}Y(\bar{u}, \bar{y})$ and $D_{low}V(\bar{u}, \bar{y})$ does not hold.

COROLLARY 12. *Let $\bar{y} \in V(\bar{u})$ be a uniformly efficient point of Y. Assume the following conditions:*

(i) *Y is semi-differentiable at $(\bar{u}, \bar{y})$, closed at $\bar{u}$, and has the domination property around $(\bar{u}, \bar{y})$;*

(ii) *either Y is calm at $\bar{u}$, or*
   (iia) *Y is calm at $(\bar{u}, \bar{y})$ and*
   (iib) *$Y^\infty(\bar{u}) \cap -\mathbb{R}^p_+ = \{0\}$.*

Then V is semi-differentiable at $(\bar{u}, \bar{y})$ and its derivative is given by (11).

Proof. To prove this corollary it suffices to show that conditions (ii) and (iii) of Theorem 10 hold. First, we prove that (iia) implies condition (ii) of Theorem 10. In fact, let $w \in DY(\bar{u}, \bar{y})(0), w \neq 0$. There are $(u_k, y_k) \in gphY$ converging to $(\bar{u}, \bar{y})$ and $t_k > 0$ converging to 0 such that $w = \lim_{k\to\infty}(y_k - \bar{y})/t_k$ and $0 = \lim_{k\to\infty}(u_k - \bar{u})/t_k$. Because Y is calm at $(\bar{u}, \bar{y})$, we may find some $\tilde{y}_k \in Y(\bar{u})$ and $\ell > 0$ such that $\|\tilde{y}_k - y_k\| \leqq \ell\|u_k - \bar{u}\|$ for every k sufficiently large. We deduce that

$$\lim_{k\to\infty} \frac{\tilde{y}_k - \bar{y}}{t_k} = \lim_{k\to\infty} \left(\frac{\tilde{y}_k - y_k}{t_k} + \frac{y_k - \bar{y}}{t_k} \right) = w.$$

Since $\bar{y}$ is a properly efficient point of $Y(\bar{u})$, due to (R3), we conclude that $w \notin -\mathbb{R}^p_+$. Next, we prove that if Y is calm at $\bar{u}$, then condition (iib) is satisfied. Indeed, because Y is calm at $\bar{u}$, we have

$$Y^\infty(\bar{u}) \subseteq [Y(\bar{u})]_\infty. \tag{14}$$

Moreover, being uniformly efficient, the point $\bar{y}$ is a properly efficient point of $Y(\bar{u})$. Therefore, we deduce from (R3) that

$$[Y(\bar{u})]_\infty \cap -\mathbb{R}^p_+ \subseteq \mathrm{cl}(cone(Y(\bar{u}) - \bar{y})) \cap -K^+_\epsilon = \{0\}$$

for some $\epsilon > 0$. This latter relation and (14) yield (iib). The proof is complete. □

COROLLARY 13. *Let $\bar{y} \in V(\bar{u})$ be a uniformly efficient point of Y. Assume the following conditions:*

(i) *Y is proto-differentiable, pseudo-Lipschitz at $(\bar{u}, \bar{y})$, closed at $\bar{u}$, and has the domination property around $(\bar{u}, \bar{y})$;*

(ii) $Y^\infty(\bar{u}) \cap -\mathbb{R}^p_+ = \{0\}$.

Then V is semi-differentiable at $(\bar{u}, \bar{y})$ and its derivative is given by (11).

Proof. We know from (R2) that a proto-differentiable and pseudo-Lipschitz mapping is semi-differentiable. Therefore, due to (i), the first condition of Corollary 12 holds true. Furthermore, the pseudo-Lipschitz assumption on Y at $(\bar{u}, \bar{y})$ implies that it is calm at this point. It remains to apply Corollary 12 to complete the proof. □

In the what remains of this section we wish to find conditions on the feasible mapping X and the objective function f to ensure the semi-differentiability of V and to compute its semi-derivative. For $(u, y) \in gphY$ we define

$$\hat{X}(u, y) := (f(u, .))^{-1}(y) \cap X(u) = \{x \in X(u) : f(u, x) = y\}.$$

In the next result "*dist*" denotes distance.

LEMMA 14. *Let $(\bar{u}, \bar{y}) \in gphY$. Assume the following conditions:*

(i) *f is locally Lipschitz at $(\bar{u}, \bar{x}), \bar{x} \in \hat{X}(\bar{u}, \bar{y})$;*

(ii) *X is closed around $\bar{u}$ and pseudo-Lipschitz at $(\bar{u}, \bar{x}), \bar{x} \in \hat{X}(\bar{u}, \bar{y})$;*

(iii)
$$\limsup_{(u,y) \xrightarrow{gphY} (\bar{u},\bar{y})} dist[0, \hat{X}(u, y)] < +\infty.$$

Then Y is pseudo-Lipschitz at $(\bar{u}, \bar{y})$.

Proof. Our aim is to apply (R1). We first prove that $gphY$ is locally closed at $(\bar{u}, \bar{y})$. To this end we claim that there exist $\alpha > 0$ and a neighborhood Q of $(\bar{u}, \bar{y})$ such that

$$\hat{X}(u, y) \cap (\alpha B_n) \neq \emptyset \quad \forall (u, y) \in gphY \cap Q. \tag{15}$$

Indeed, if this is not the case, there exist $\{(u_k, y_k)\}_k \subseteq gphY$ with $(u_k, y_k) \to (\bar{u}, \bar{y})$ and $\hat{X}(u_k, y_k) \cap (kB_n) = \emptyset$. Then for the sequence $\{(u_k, y_k)\}_k$, one has $\lim_{k\to\infty} dist[0, \hat{X}(u_k, y_k)] = +\infty$, which contradicts (iii). Further, we choose a closed neighborhood $Q' \subseteq Q$ of $(\bar{u}, \bar{y})$ and show that $gphY \cap Q'$ is closed. To this end, let $\{(u_k, y_k)\}_k \subseteq gphY \cap Q'$ converge to (u, y). By (15), there are $x_k \in X(u_k) \cap (\alpha B_n)$ with $f(u_k, x_k) = y_k$. We may assume that $x_k \to \bar{x}$. Due to the continuity of f and due to condition (ii), we have $(u, y) \in gphY \cap Q'$ as requested. Thus, $gphY$ is locally closed at $(\bar{u}, \bar{y})$.

Now we suppose to the contrary that Y is not pseudo-Lipschitz at $(\bar{u}, \bar{y})$. In view of (R1), we have $(\sigma, 0) \in N_{gphY}(\bar{u}, \bar{y})$ for some $\sigma \neq 0$. By definition there are positive numbers ϵ_k converging to 0, a sequence $\{(u_k, y_k)\}_k \in gphY$ converging to $(\bar{u}, \bar{y})$, and $\{(\sigma_k, \mu_k)\}_k$ with $(\sigma_k, \mu_k) \in \hat{N}^{\epsilon_k}_{gphY}(u_k, y_k)$ such that $\lim_{k\to\infty}(\sigma_k, \mu_k) = (\sigma, 0)$. In view of (ii), we may find a sequence $\{x_k\}_k$ such that $y_k = f(u_k, x_k)$ and a subsequence $\{x_{\nu_k}\}$ converging to some $\bar{x} \in \hat{X}(\bar{u}, \bar{y})$. Then for every k we have

$$\begin{aligned}\epsilon_{\nu_k} &\geqq \limsup_{(u,y)\xrightarrow{gphY}(u_{\nu_k},y_{\nu_k})} \frac{\langle \sigma_{\nu_k}, u - u_{\nu_k}\rangle + \langle \mu_{\nu_k}, y - y_{\nu_k}\rangle}{\|u - u_{\nu_k}\| + \|y - y_{\nu_k}\|} \\ &\geqq \limsup_{(u,x)\xrightarrow{gphX}(u_{\nu_k},x_{\nu_k})} \frac{\langle \sigma_{\nu_k}, u - u_{\nu_k}\rangle + \langle \mu_{\nu_k}, f(u,x) - f(u_{\nu_k}, x_{\nu_k})\rangle}{\|u - u_{\nu_k}\| + \|f(u,x) - f(u_{\nu_k}, x_{\nu_k})\|}.\end{aligned}$$

Let $\ell > 0$ be a Lipschitz constant of f at $(\bar{u}, \bar{x})$. For k sufficiently large we deduce that

$$(1+\ell)(\epsilon_{\nu_k} + \|\mu_{\nu_k}\|) \geqq \limsup_{(u,x)\xrightarrow{gphX}(u_{\nu_k},x_{\nu_k})} \frac{\langle \sigma_{\nu_k}, u - u_{\nu_k}\rangle}{\|u - u_{\nu_k}\| + \|x - x_{\nu_k}\|},$$

which implies $(\sigma_{\nu_k}, 0) \in \hat{N}_{gphX}^{(1+\ell)(\epsilon_{\nu_k}+\|\mu_{\nu_k}\|)}(u_{\nu_k}, x_{\nu_k})$. By passing to the limit in the latter relation when k tends to ∞, we obtain $(\sigma, 0) \in N_{gphX}(\bar{u}, \bar{x})$. This contradicts the pseudo-Lipschitz continuity of X. The proof is complete. ∎

To go further, we recall the concept of an asymptotic function of a real function $\phi : \mathbb{R}^q \times \mathbb{R}^n \to \mathbb{R}$ with respect to the mapping X at $\bar{u}$ (see [15]):

$$\phi_X^\infty(\bar{u}; v) := \inf\Big\{ \lim_{k\to\infty} t_k \phi(u_k, x_k) : t_k \downarrow 0, x_k \in X(u_k), u_k \to \bar{u}, t_k x_k \to v \Big\} \quad \text{for } v \in \mathbb{R}^n. \tag{16}$$

Note that ϕ_X^∞ may take the value $-\infty$ or $+\infty$.

COROLLARY 15. *Assume f is locally Lipschitz at $(\bar{u}, \bar{x})$ and X is locally closed and pseudo-Lipschitz at $(\bar{u}, \bar{x})$ for every $\bar{x} \in \hat{X}(\bar{u}, \bar{y})$. The mapping Y is pseudo-Lipschitz at $(\bar{u}, \bar{y})$ if any of the following conditions holds:*
- (i) *for every sequence $\{(u_k, y_k)\}_k \subseteq gphY$ converging to $(\bar{u}, \bar{y})$, there exists a sequence $\{x_k\}_k$ with $x_k \in \hat{X}(u_k, y_k)$ admitting a cluster point $\bar{x} \in \hat{X}(\bar{u}, \bar{y})$;*
- (ii) *the mapping $\hat{X}$ is locally bounded at $(\bar{u}, \bar{y})$;*
- (iii) $\{v \in \mathbb{R}^n : (f_i)_X^\infty(\bar{u}; v) \leqq 0, i = 1, \dots, p\} = \{0\}$.

Proof. We observe that (i) is equivalent to condition (ii) of Lemma 14 and (ii) clearly implies (i). Therefore, to complete the proof it suffices to show that (iii) implies (ii). Indeed, suppose to the contrary that $\hat{X}$ is not locally bounded at $(\bar{u}, \bar{y})$. Then we may find a sequence $\{(u_k, y_k)\}_k \subseteq gphY$ converging to $(\bar{u}, \bar{y})$ and $x_k \in \hat{X}(u_k, y_k)$ such that $\lim_{k\to\infty} \|x_k\| = \infty$. Without loss of generality we may assume $\{x_k/\|x_k\|\}_k$ converges to some nonzero vector v. We have $\lim_{k\to\infty} f(u_k, x_k)/\|x_k\| = \lim_{k\to\infty} y_k/\|x_k\| = 0$, which shows that $(f_i)_X^\infty(\bar{u}; v) \leqq 0$ for all $i = 1, \dots, p$. This contradicts (iii) and completes the proof. ∎

Let $h : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function. If the directional derivative $h'(\bar{x}; d)$ of h at $\bar{x}$ in any direction $d \in \mathbb{R}^n$ exists and coincides with the Clarke directional derivative, then one says that h is regular at $\bar{x}$. A vector-valued locally Lipschitz function is said to be regular at $\bar{x}$ if its components are regular at $\bar{x}$ (see [7] for more information).

LEMMA 16. *Let $(\bar{u}, \bar{y}) \in gphY$. Assume f is locally Lipschitz and regular at $(\bar{u}, \bar{x})$, $\bar{x} \in \hat{X}(\bar{u}, \bar{y})$. Then*

$$\{f'((\bar{u}, \bar{x}); (d, v)) : v \in D_{low}X(\bar{u}, \bar{x})(d)\} \subseteq D_{low}Y(\bar{u}, \bar{y})(d) \quad \textit{for } d \in \mathbb{R}^q. \tag{17}$$

If, in addition,

(i) X *is semi-differentiable at* $(\bar{u}, \bar{x}), \bar{x} \in \hat{X}(\bar{u}, \bar{y})$;

(ii)
$$\limsup_{(u,y) \xrightarrow{gphY} (\bar{u},\bar{y})} dist[0, \hat{X}(u, y)] < +\infty;$$

(iii) $DX(\bar{u}, \bar{x})(0) \cap \{v \in \mathbb{R}^n : f'((\bar{u}, \bar{x}); (0, v)) = 0\} = \{0\}$, $\bar{x} \in \hat{X}(\bar{u}, \bar{y})$,

then the mapping Y *is semi-differentiable at* $(\bar{u}, \bar{y})$ *and the semi-derivative of* Y *at* $(\bar{u}, \bar{y})$ *is given by*

$$D_{low}Y(\bar{u}, \bar{y})(d) = \bigcup_{\bar{x} \in \hat{X}(\bar{u},\bar{y})} \{f'((\bar{u}, \bar{x}); (d, v)) : v \in D_{low}X(\bar{u}, \bar{x})(d)\}. \tag{18}$$

Proof. If $D_{low}X(\bar{u}, \bar{x})(d)$ is empty, then there is nothing to prove. Let $v \in D_{low}X(\bar{u}, \bar{x})(d)$. We show that the vector $f'((\bar{u}, \bar{x}); (d, v))$ belongs to $D_{low}Y(\bar{u}, \bar{y})(d)$. In fact, let $t_k > 0$ converging to 0 and d_k converging to d be given. By definition, there are some $x_k \in X(\bar{u} + t_k d_k)$ such that $\lim_{k\to\infty}(x_k - \bar{x})/t_k = v$. Because f is regular, we deduce that $\lim_{k\to\infty}(f(\bar{u} + t_k d_k, x_k) - f(\bar{u}, \bar{x}))/t_k = f'((\bar{u}, \bar{x}); (d, v))$. Hence $f'((\bar{u}, \bar{x}); (d, v)) \in D_{low}Y(\bar{u}, \bar{y})(d)$.

Let $(d, w) \in gphDY(\bar{u}, \bar{y})$, say $w = \lim_{k\to\infty}(y_k - \bar{y})/t_k$, $d = \lim_{k\to\infty}(u_k - \bar{u})/t_k$, where $(u_k, y_k) \in gphY$ and $t_k > 0$ converges to 0. In view of (ii) we may find $x_k \in X(u_k)$ such that $y_k = f(u_k, x_k)$ and the sequence $\{x_k\}_k$ is bounded, which admits a cluster point $\bar{x} \in \hat{X}(\bar{u}, \bar{y})$. By working with subsequences if necessary one may assume that $\bar{x}$ is the limit of that sequence. We consider the sequence $\{(x_k - \bar{x})/t_k\}_k$. If it is unbounded, then we may suppose that the sequence $\{(x_k - \bar{x})/\|x_k - \bar{x}\|\}_k$ converges to some nonzero vector $\bar{v}$ and $\{t_k/\|x_k - \bar{x}\|\}_k$ converges to 0. The latter limit implies $\lim_{k\to\infty}(u_k - \bar{u})/\|x_k - \bar{x}\| = 0$ and $\lim_{k\to\infty}(y_k - \bar{y})/\|x_k - \bar{x}\| = 0$. This yields $\bar{v} \in DX(\bar{u}, \bar{x})(0)$ and $f'((\bar{u}, \bar{x}); (0, \bar{v})) = 0$, which is in contradiction with (iii). By this we may suppose the sequence $\{(x_k - \bar{x})/t_k\}_k$ converges to some vector v and deduce that $(d, v) \in gphDX(\bar{u}, \bar{x})$. Moreover,

$$w = \lim_{k\to\infty} \frac{f(u_k, x_k) - f(\bar{u}, \bar{x})}{t_k} = f'((\bar{u}, \bar{x}); (d, v)).$$

Let $\tilde{t}_k > 0$ converge to 0 and d_k converge to d. Because X is semi-differentiable at $(\bar{u}, \bar{y})$, there exists $\tilde{x}_k \in X(\bar{u} + \tilde{t}_k d_k)$ converging to $\bar{x}$ such that $\lim_{k\to\infty}(\tilde{x}_k - \bar{x})/\tilde{t}_k = v$. Setting $\tilde{y}_k = f(\bar{u} + \tilde{t}_k d_k, \tilde{x}_k)$ we have

$$\lim_{k\to\infty}(\tilde{y}_k - \bar{y})/\tilde{t}_k = \lim_{k\to\infty}(f(\bar{u} + \tilde{t}_k d_k, \tilde{x}_k) - f(\bar{u}, \bar{x}))/\tilde{t}_k = f'((\bar{u}, \bar{x}), (d, v)) = w.$$

This proves that Y is semi-differentiable at $(\bar{u}, \bar{x})$. The formula for the semi-derivative of Y is immediate. ∎

We note that when X is graph convex and $\bar{u} \in \text{int}(domX)$, the mapping X is semi-differentiable and $D_{low}X(\bar{u}, \bar{x})(d)$ is nonempty for every direction d (see [10, Theorem 11.1.36]). In the general case, a certain regularity condition is needed to ensure the nonemptiness of $D_{low}X(\bar{u}, \bar{x})(d)$ (see [4, section 4.2], in which first order feasible directions at $\bar{x}$ relative to d are exactly those of the lower Dini derivative $D_{low}X(\bar{u}, \bar{x})(d)$).

COROLLARY 17. *Let* $(\bar{u}, \bar{y}) \in gphY$. *Assume the following conditions:*

(i) $\hat{X}$ *is calm at* $(\bar{u}, \bar{y})$ *with* $\hat{X}(\bar{u}, \bar{y}) = \{\bar{x}\}$ *a singleton;*

(ii) *f is locally Lipschitz and regular, and X is semi-differentiable at $(\bar{u}, \bar{x})$.*
Then the mapping Y is semi-differentiable at $(\bar{u}, \bar{y})$ and the semi-derivative of Y at $(\bar{u}, \bar{y})$ is given by

$$D_{low}Y(\bar{u}, \bar{y})(d) = \{f'((\bar{u}, \bar{x}); (d, v)) : v \in D_{low}X(\bar{u}, \bar{x})(d)\}\,.$$

Proof. This corollary is obtained from Lemma 16 by observing that when $\hat{X}(\bar{u}, \bar{y})$ is a singleton, the calmness of $\hat{X}$ at $(\bar{u}, \bar{y})$ is equivalent to (iii) of that lemma. □

We say that Y has *locally bounded sections* at $(\bar{u}, \bar{y})$ if there is a neighborhood Q of $(\bar{u}, \bar{y})$ and a constant $\alpha > 0$ such that

$$Y(u) \cap (y - \mathbb{R}^p_+) \subseteq \alpha B_p \quad \text{for all } (u, y) \in Q, \tag{19}$$

and f has *locally bounded level sets* at $(\bar{u}, \bar{y})$ if there is a neighborhood Q of $(\bar{u}, \bar{y})$ and a constant $\alpha > 0$ such that

$$\{x \in X(u) : f(u, x) \in y - \mathbb{R}^p_+\} \subseteq \alpha B_n \quad \text{for all } (u, y) \in Q. \tag{20}$$

It is clear that with f continuous, (20) implies (19). Moreover, a sufficient condition for (20) can be given by $\{v \in \mathbb{R}^n : (f_i)^\infty_X(\bar{u}; v) \leqq 0, i = 1, \ldots, p\} = \{0\}$. In particular, this is true when X is locally bounded at $\bar{u}$. We are now able to present a main result on the semi-differentiability of the mapping V under the semi-differentiability of the feasible solution mapping.

THEOREM 18. *Let $\bar{y} \in V(\bar{u})$ be a uniformly efficient point. Assume the following conditions:*

(i) *f is locally Lipschitz, regular at $(\bar{u}, \bar{x})$ for every $\bar{x} \in \hat{X}(\bar{u}, \bar{y})$, and has locally bounded level sets at $(\bar{u}, \bar{y})$;*

(ii) *X is closed around $\bar{u}$, pseudo-Lipschitz, and semi-differentiable at $(\bar{u}, \bar{x}), \bar{x} \in \hat{X}(\bar{u}, \bar{y})$;*

(iii) *$DX(\bar{u}, \bar{x})(0) \cap \{v \in \mathbb{R}^n : f'((\bar{u}, \bar{x}); (0, v)) = 0\} = \{0\}, \bar{x} \in \hat{X}(\bar{u}, \bar{y})$.*

Then V is semi-differentiable at $(\bar{u}, \bar{y})$ and its semi-derivative is given by the formula

$$D_{low}V(\bar{u}, \bar{y})(d) = Min\left\{\bigcup_{\bar{x} \in \hat{X}(\bar{u}, \bar{y})} \{f'((\bar{u}, \bar{y}); (d, v)) : v \in D_{low}X(\bar{u}, \bar{x})(d)\}\right\}.$$

Proof. We wish to apply Theorem 10 combined with Remark 3. Checking for three conditions of that theorem will be done by five claims.

Claim 1. Y is semi-differentiable at $(\bar{u}, \bar{y})$ and its semi-derivative is given by (18).

Indeed, in view of Lemma 16 we have only to prove that

$$\limsup_{(u,y) \xrightarrow{gphY} (\bar{u}, \bar{y})} dist[0, \hat{X}(u, y)] < +\infty. \tag{21}$$

Let Q be a neighborhood of $(\bar{u}, \bar{y} + e)$ and $\alpha > 0$ be such that (20) is satisfied. Choose a small neighborhood U of $\bar{u}$ such that $U \times \{\bar{y} + e\} \subset Q$. We deduce from (20) that

$$\begin{aligned} \hat{X}(u, y) &\subseteq \{x \in X(u) : f(u, x) \leqq y\} \\ &\subseteq \{x \in X(u) : f(u, x) \leqq \bar{y} + te\} \\ &\subseteq \alpha B_n \end{aligned}$$

for all (u, y) from the neighborhood $U \times \{\bar{y} + e - \text{int}(\mathbb{R}^p_+)\}$ of $(\bar{u}, \bar{y})$. This implies (21) and so Claim 1 is obtained from Lemma 16.

Claim 2. $Y+\mathbb{R}^p_+$ is closed around $(\bar u,\bar y)$.

Since f has closed bounded level sets at $(\bar u,\bar y)$, we can have (20) for a closed neighborhood Q of $(\bar u,\bar y)$ and X is closed on the projection of Q on $\mathbb{R}^q$. We prove that $gph(Y+\mathbb{R}^p_+)\cap Q$ is closed. Indeed, let $\{(u_k,y_k+r_k)\}_k\subseteq gph(Y+\mathbb{R}^p_+)\cap Q$ converge to (u,y) with $y_k\in Y(u_k)$ and $r_k\in\mathbb{R}^p_+$. Due to (20) and (ii), we may choose $x_k\in X(u_k)$ such that $f(u_k,x_k)\leqq y_k+r_k$ and $x_k\to x\in X(u)$. It follows from the continuity of f that $f(u,x)\in Y(u)$ and $f(u,x)\leqq y$. Hence $y\in Y(u)+\mathbb{R}^p_+$, which shows $Y+\mathbb{R}^p_+$ is closed around $(\bar u,\bar y)$.

Claim 3. Y has the domination property around $(\bar u,\bar y)$.

In fact, we know that Y has locally bounded sections at $(\bar u,\bar y)$. Therefore, when $(u,y)\in gphY$ is sufficiently close to $(\bar u,\bar y)$, the set $Y(u)\cap(y-\mathbb{R}^p_+)$ is nonempty and compact. Consequently, it admits efficient points. Any efficient point of this section is also an efficient point of $Y(u)$ and dominates y.

Claim 4. $DY(\bar u,\bar y)(0)\cap -\mathbb{R}^p_+=\{0\}$.

Let $w\in DY(\bar u,\bar y)(0)$ be a nonzero vector, say $w=\lim_{k\to\infty}(y_k-\bar y)/t_k$ for some $(u_k,y_k)\in gphY$ converging to $(\bar u,\bar y)$ and $t_k>0$ converging to 0 such that $\lim_{k\to\infty}(u_k-\bar u)/t_k=0$. We may find $x_k\in\hat X(u_k,y_k)$ that converge to some $\bar x\in\hat X(\bar u,\bar y)$. Since X is pseudo-Lipschitz at $(\bar u,\bar x)$, there exists $\tilde x_k\in X(\bar u)$ such that $\|x_k-\tilde x_k\|\leqq \ell\|u_k-\bar u\|$ for some $\ell>0$ and for all $k\geqq 0$. Set $\tilde y_k=f(\bar u,\tilde x_k)$. Then

$$\lim_{k\to\infty}\frac{\tilde y_k-\bar y}{t_k}=\lim_{k\to\infty}\left(\frac{f(\bar u,\tilde x_k)-f(u_k,x_k)}{t_k}+\frac{y_k-\bar y}{t_k}\right)=w,$$

because $\|f(\bar u,\tilde x_k)-f(u_k,x_k)\|\leqq\kappa(\|u_k-\bar u\|+\|\tilde x_k-x_k\|)\leqq\kappa(1+\ell)\|u_k-\bar u\|$, where κ is a Lipschitz constant of f near $(\bar u,\bar x)$, which yields

$$\lim_{k\to\infty}\frac{f(\bar u,\tilde x_k)-f(u_k,x_k)}{t_k}=0.$$

We deduce that $w\in T_{Y(\bar u)}(\bar y)$. Since $\bar y$ is a properly efficient point of $Y(\bar u)$ we conclude that $w\notin -\mathbb{R}^p_+$.

Claim 5. $Y^\infty(\bar u)\cap -\mathbb{R}^p_+=\{0\}$.

Let $w\in Y^\infty(\bar u)$ be a nonzero vector, say $w=\lim_{k\to\infty}t_ky_k$, with $t_k>0$ converging to 0 and $y_k\in Y(u_k)$ with $u_k\to\bar u$ and $\|y_k\|\to+\infty$. According to the proof of Claim 3, we find some $\tilde y_k\in V(u_k)$ such that $\{\tilde y_k\}_k$ is a bounded sequence. Then

$$w=\lim_{k\to\infty}t_ky_k=\lim_{k\to\infty}t_k(y_k-\tilde y_k)\in\limsup_{(u,y)\xrightarrow{gphV}(\bar u,\bar y)} cone(Y(u)-y).$$

In view of Lemma 2, $w\notin -\mathbb{R}^p_+$ and Claim 5 is established.

To complete the proof it remains to apply Theorem 10 and Lemma 16. ∎

Remark 5. In a number of papers devoted to sensitivity analysis of vector optimization problems (see [5, 11, 12, 23] for instance), the derivative (called the S-derivative)

$$D_SF(\bar u,\bar x)(d):=\Big\{v\in\mathbb{R}^n : v=\lim_{k\to\infty}t_k(x_k-\bar x)\text{ with }t_k>0,(u_k,x_k)\in gphF,$$
$$u_k\to\bar u, t_k(u_k-\bar u,x_k-\bar x)\to(d,v)\Big\}$$

is frequently considered for a mapping $F : \mathbb{R}^q \rightrightarrows \mathbb{R}^n$ at $(\bar{u}, \bar{x}) \in gphF$. This derivative is very large, even larger than the contingent derivative in many nonconvex cases. Therefore, conditions involving it in computation of derivatives of the efficient value mapping are sometimes exceedingly restrictive. For instance, the condition

$$D_S F(\bar{u}, \bar{x})(0) = \{0\}$$

implies existence of a number $\gamma > 0$ and of a neighborhood U of $\bar{u}$ such that

$$F(u) \subseteq \{\bar{x}\} + \gamma\|u - \bar{u}\|B_n \quad \text{for all } u \in U.$$

In particular, the condition $D_S X(\bar{u}, \bar{x})(0) = \{0\}$ used in [5, Proposition 4.1] leads to the fact that $\limsup_{u \to \bar{u}} X(u) = \{\bar{x}\}$, which, in particular, tells us that the feasible solution set of $\mathrm{P}(\bar{u})$ must be a singleton whenever X is lower semi-continuous at $\bar{u}$. Note further that condition $D_S Y(\bar{u}, \bar{y})(0) \cap -\mathbb{R}^p_+ = \{0\}$ used in [11, 23] to establish estimates for contingent derivatives implies both conditions (ii) and (iii) of Theorem 10. The converse is not true except for the convex case.

Remark 6. An important result of [12, Theorem 4.2] for the proto-derivative was expressed in terms of the S-derivative. It states that, under the following conditions, $D_{adj} V(\bar{u}, \bar{y})(d) \subseteq WMin[D_{adj} Y(\bar{u}, \bar{y})(d)]$:

(i) f is continuously differentiable at $(\bar{u}, \bar{x})$, where $\bar{x} \in \hat{X}(\bar{u}, \bar{y})$;

(ii) X is semi-differentiable at $(\bar{u}, \bar{x})$;

(iii) $D_S \hat{X}((\bar{u}, \bar{y}), \bar{x})(0, 0) = \{0\}$.

Actually, this result is true when f is locally Lipschitz and regular at $(\bar{u}, \bar{x})$. In fact, according to Remark 3, condition (iii) implies existence of a constant $\gamma > 0$ and neighborhoods U of $\bar{u}$ and W of $\bar{y}$ such that

$$\hat{X}(u, y) \subseteq \{\bar{x}\} + \gamma(\|u - \bar{u}\| + \|y - \bar{y}\|)B_n \quad \text{for all } u \in U, y \in W.$$

This, in its turn, implies both conditions (ii) and (iii) of Lemma 16, by which the mapping Y is semi-differentiable at $(\bar{u}, \bar{y})$. In particular, Y is proto-differentiable and its proto-derivative coincides with the lower semi-derivative at $(\bar{u}, \bar{y})$. It remains to apply (10) to conclude.

We now give an example for which Theorem 4.2 of [12] is not applicable because condition (iii) discussed in Remark 4 is not satisfied, while our approach works and gives

$$D_{adj} V(\bar{u}, \bar{y})(d) = D_{low} V(\bar{u}, \bar{y})(d) = Min[D_{low} Y(\bar{u}, \bar{y})(d)] \subseteq WMin[D_{low} Y(\bar{u}, \bar{y})(d)],$$

the last inclusion being evident.

Example 19. Let $X : \mathbb{R} \rightrightarrows \mathbb{R}$ be given by $X(u) =: \{x \in \mathbb{R} : 0 \leqq x \leqq 1\}$ and let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be given by $f(u, x) = 1$. Then V is proto-differentiable (strictly semi-differentiable) at $(\bar{u}, \bar{y}) = (0, 1)$ while $D_S \hat{X}((\bar{u}, \bar{y}), \bar{x})(0, 0) \neq \{0\}$ for each $\bar{x} \in \hat{X}(\bar{u}, \bar{y})$.

Remark 7. An interesting case is when the feasible set is unperturbed (see e.g., [4, 17]). Conditions for semi-differentiability of V in such a case are much simplified. Recall that a set $K \subseteq \mathbb{R}^n$ is *derivable* at x if $T_K(x) = T_K^{adj}(x)$.

Corollary 20. *Let $\bar{y} \in V(\bar{u})$ be a uniformly efficient point of Y at $\bar{u}$. Assume the following conditions:*

(i) *f is locally Lipschitz, regular at $(\bar{u}, \bar{x})$ for every $\bar{x} \in \hat{X}(\bar{u}, \bar{y})$;*

(ii) *$X(u) = K$ for all u in some neighborhood of $\bar{u}$, where $K \subseteq \mathbb{R}^n$ is compact, derivable at every point $\bar{x} \in \hat{X}(\bar{u}, \bar{y})$ and satisfies*

$$T_K(\bar{x}) \cap \{v \in \mathbb{R}^n : f'((\bar{u}, \bar{x}); (0, v)) = 0\} = \{0\}.$$

Then V is semi-differentiable at $(\bar{u}, \bar{y})$ and its semi-derivative is given by the formula

$$D_{low}V(\bar{u}, \bar{y})(d) = Min\,\{\nabla_u f(\bar{u}, \bar{y})d + \nabla_x f(\bar{u}, \bar{y})v : v \in T_K(\bar{x})\}\,.$$

Proof. Because K is derivable, the constant mapping X is semi-differentiable and its derivative is given by $DX(\bar{u}, \bar{x})(d) = D_{low}X(\bar{u}, \bar{x})(d) = T_K(\bar{x})$. The corollary is now immediate from Theorem 18. ∎

Remark 8. In the convex case, in which X is graph convex and f is convex, the following corollary strengthens the main result of [26, Theorem 5.3], in which the author gives the formula for the contingent derivative under the same conditions except for (ii), where it is requested that $\bar{y}$ be normally efficient (hence uniformly efficient in this case).

COROLLARY 21. *Assume that X is graph-convex, f is convex and continuously differentiable, and the following conditions hold:*

(i) *$\bar{u} \in \operatorname{int}(domY)$;*
(ii) *$\bar{y}$ is a uniformly efficient point of Y at $\bar{u}$;*
(iii) *$X(u)$ is closed for every u in a neighborhood of $\bar{u}$;*
(iv) *$\hat{X}$ is calm at $((\bar{u}, \bar{y}), \bar{x})$, where $\hat{X}(\bar{u}, \bar{y}) = \{\bar{x}\}$.*

Then V is semi-differentiable at $(\bar{u}, \bar{y})$ and its semi-derivative is given by the formula

$$D_{low}V(\bar{u}, \bar{y})(d) = Min\,\{\nabla_u f(\bar{u}, \bar{y})d + \nabla_x f(\bar{u}, \bar{y})v : v \in D_{low}X(\bar{u}, \bar{x})(d)\}\,.$$

Proof. Since X is graph convex and closed-valued and $\bar{u} \in \operatorname{int}(domX)$, the mapping X is continuous (closed and lower semi-continuous) at $\bar{u}$ and pseudo-Lipschitz at $(\bar{u}, \bar{x})$ (see [21, Theorem 9.34 and Example 9.52]). In order to apply Theorem 18 we have only to prove that f has locally bounded level sets at $(\bar{u}, \bar{y})$ because condition $DX(\bar{u}, \bar{x})(0) \cap \operatorname{Ker} \nabla_x f(\bar{u}, \bar{x}) = \{0\}$ is equivalent to (iv), as we have already mentioned in the proof of Corollary 17. Let us consider the mapping

$$\tilde{X}(u, y) := X(u) \cap \{x \in \mathbb{R}^n : f(u, x) \in y - \mathbb{R}^p_+\} \quad \text{for } (u, y) \in \mathbb{R}^q \times \mathbb{R}^p.$$

Direct verification shows that this mapping is graph convex and closed at $(\bar{u}, \bar{y})$. Moreover, because $\bar{y}$ is an efficient point of $Y(\bar{u})$, we have $\tilde{X}(\bar{u}, \bar{y}) = \hat{X}(\bar{u}, \bar{y}) = \{\bar{x}\}$. We show that $\tilde{X}$ is locally bounded at $(\bar{u}, \bar{y})$. Indeed, suppose to the contrary that for some sequence $\{(u_k, y_k)\}_k$ converging to $(\bar{u}, \bar{y})$, there exists $x_k \in \tilde{X}(u_k, y_k)$ such that $\lim_{k\to\infty} \|x_k\| = \infty$. We may assume without loss of generality that $\lim_{k\to\infty} x_k/\|x_k\| = v$ for some $v \neq 0$. Since X is lower semi-continuous at $\bar{u}$, there exists $w_k \in X(u_k)$ with $w_k \to \bar{x}$. Since $X(u_k)$ is convex, when k is sufficiently large, we have $w_k + \frac{x_k - w_k}{\|x_k - w_k\|} \in X(u_k)$ and the closedness of X implies

$$\lim_{k\to\infty} w_k + \frac{x_k - w_k}{\|x_k - w_k\|} = \bar{x} + v \in X(\bar{u}).$$

By convexity of f, one has also

$$f\left(u_k, w_k + \frac{x_k - w_k}{\|x_k - w_k\|}\right) \leqq \left(1 - \frac{1}{\|x_k - w_k\|}\right) f(u_k, w_k) + \frac{1}{\|x_k - w_k\|} f(u_k, x_k).$$

By passing to the limit when k tends to ∞ in the latter inequality, we deduce that

$$f(\bar{u}, \bar{x} + v) \leqq f(\bar{u}, \bar{x}).$$

Because $\bar{y} = f(\bar{u}, \bar{x})$ is an efficient point of $Y(\bar{y})$ we deduce that $\bar{y} = f(\bar{u}, \bar{x} + v)$. This contradicts the fact that $\tilde{X}(\bar{u}, \bar{y}) = \{\bar{x}\}$. It is then clear that f has closed and locally bounded level sets at $(\bar{u}, \bar{y})$. The proof is complete. ∎

By using Corollary 20 it is easy to construct examples for which the results of [26] cannot apply, while Theorem 21 is suitable. For instance, if f is the identity function on the set $X(u) := [0, 1] \times [0, 1]$ in $\mathbb{R}^2$ for every $u \in \mathbb{R}$, then the point $\bar{y} = (0, 0)$ is uniformly efficient. In view of Corollary 20 the marginal mapping V is semi-differentiable (it is a constant singleton). Since $\bar{y}$ is not normally efficient, we cannot apply [26, Theorem 5.3] to obtain its contingent derivative.

We recall that an efficient solution $\bar{x}$ of P(u) is *strict* if there is no other efficient solution x such that $f(u, x) = f(u, \bar{x})$. In other words $\bar{x}$ is a strict efficient solution if $\hat{X}(\bar{u}, f(\bar{u}, \bar{x}))$ is a singleton.

COROLLARY 22. *Let $\bar{x} \in S(\bar{u})$ be a strict efficient solution with $\bar{y} = f(\bar{u}, \bar{x})$ uniformly efficient. Assume the following conditions:*

(i) *f is continuously differentiable at $(\bar{u}, \bar{x})$ and $\nabla_x f(\bar{u}, \bar{x})$ is an injection on $DX(\bar{u}, \bar{x})(d)$ for every $d \in \mathbb{R}^q$ and has locally bounded level sets at $(\bar{u}, \bar{y})$;*

(ii) *X is closed around $\bar{u}$, pseudo-Lipschitz and semi-differentiable at $(\bar{u}, \bar{x})$.*

Then S is semi-differentiable at $(\bar{u}, \bar{y})$ and its semi-derivative is given by the formula

$$D_{low}S(\bar{u}, \bar{x})(d) \\ := \{v \in D_{low}X(\bar{u}, \bar{x})(d) : \nabla_u f(\bar{u}, \bar{x})d + \nabla_x f(\bar{u}, \bar{x})v \in D_{low}V(\bar{u}, \bar{y})(d)\}.$$

Proof. First, we observe that due to the injection hypothesis of $\nabla_x f(\bar{u}, \bar{x})$ on $DX(\bar{u}, \bar{x})(0)$, condition (iii) of Theorem 18 is satisfied. The other conditions being already stated in the corollary, we conclude that V is semi-differentiable at $(\bar{u}, \bar{x})$. To prove semi-differentiability of S, let $(d, v) \in gphDS(\bar{u}, \bar{x})$, say $v = \lim_{k\to\infty}(x_k - \bar{x})/t_k$, $d = \lim_{k\to\infty}(u_k - \bar{u})/t_k$, with $(u_k, x_k) \in gphS$ and $t_k > 0$ converging to 0. By setting $\bar{y} = f(\bar{u}, \bar{x})$, we have

$$w := \nabla_u f(\bar{u}, \bar{x})d + \nabla_x f(\bar{u}, \bar{x})v \in gphDV(\bar{u}, \bar{y}) = gphD_{low}V(\bar{u}, \bar{y}).$$

Let $d_k \in \mathbb{R}^q$ converging to d and $s_k > 0$ converging to 0 be arbitrary and given. We wish to find $\bar{x}_k \in S(\bar{u} + s_k d_k)$ such that $\lim_{k\to\infty}(\bar{x}_k - \bar{x})/s_k = v$. Indeed, due to the semi-differentiability of V, we find $\bar{y}_k \in V(\bar{u} + s_k d_k)$ such that $\lim_{k\to\infty}(\bar{y}_k - \bar{y})/s_k = w$. Choose any $\bar{x}_k \in S(\bar{u} + s_k d_k)$ such that $\bar{y}_k = f(\bar{u} + s_k d_k, \bar{x}_k)$ and $\{\bar{x}_k\}_k$ is bounded (which is possible given the local bounded level sets of f). We show that the sequence $\{\bar{x}_k\}_k$ is the one we are looking for. In fact, condition (iii) of Theorem 18 implies that the sequence $\{(\bar{x}_k - \bar{x})/s_k\}_k$ is bounded. Let $\{(\bar{x}_{\nu_k} - \bar{x})/s_{\nu_k}\}_k$ be a subsequence converging to some limit $\bar{v} \in DX(\bar{u}, \bar{x})(d)$. Then $w = \nabla_u f(\bar{u}, \bar{x})d + \nabla_x f(\bar{u}, \bar{x})\bar{v}$. Hence, $\nabla_x f(\bar{u}, \bar{x})v = \nabla_x f(\bar{u}, \bar{x})\bar{v}$. In view of (i), $\bar{v} = v$ proving the semi-differentiability of S. The formula of the semi-derivative of S is clear. ∎

Remark 9. As far as we know, the only result that exists on differentiability of S was given in [12, Theorem 3.1] for proto-differentiability under the following hypotheses: (a) f is continuously differentiable at $(\bar{u}, \bar{x})$ and $\nabla_x f(\bar{u}, \bar{x})$ is strictly monotone on $\mathbb{R}^n$ in the sense that $\langle \nabla_x f(\bar{u}, \bar{x})(v_1 - v_2), v_1 - v_2 \rangle > 0$ for all $v_1 \neq v_1$; (b) $D_S\hat{X}(\bar{u}, \bar{x})(0, 0) = \{0\}$; and (c) V is proto-differentiable at $(\bar{u}, \bar{x})$. Notice that the strict monotonicity of $\nabla_x f(\bar{u},\bar{x})$ implies that $p = n$ and $\nabla_x f(\bar{u},\bar{x})$ is positive definite. It is rather restrictive, especially in applications, where most of practical problems have two or three objective components while the number of decision variable components may be very large. Condition (b), as we already discussed in Remark 2, implies that $\bar{x}$ is a strict efficient solution. Actually the argument of the proof of Corollary 22 can also be applied to establish the formula for the proto-derivative under the condition that $\bar{x}$ be strictly efficient instead of (b), which is a much stronger result than [12, Theorem 3.1].

Example 23. Let $X : \mathbb{R} \rightrightarrows \mathbb{R}^3$ be given by $X(u) =: \{x \in B_3 : x_1 - x_2 = 2u, -x_2 \leqq 0\}$ and let

$$f(u,x) = \begin{pmatrix} u^2+1 & u+1 & 1 \\ 1 & u^2 & u-1 \end{pmatrix} x.$$

At $\bar{u} = 0$, $\bar{x}^T = (0 \quad 0 \quad 0)$ is strictly efficient. On account of Lemma 3, $f(\bar{u},\bar{x}) = 0$ is uniformly efficient. In view of (R4) (see section 5), X is pseudo-Lipschitz and semi-differentiable at the given point and $D_{low}X(\bar{u},\bar{x})(d) = \{v : v_1 - v_2 = 2d, -v_2 \leqq 0\}$. We show that $\nabla_x f(\bar{u},\bar{x})$ is injective on $DX(\bar{u},\bar{x})(d)$ for every $d \in \mathbb{R}$. Indeed, let $v^1, v^2 \in DX(\bar{u},\bar{x})(d)$ with $\nabla_x f(\bar{u},\bar{x})v^1 = \nabla_x f(\bar{u},\bar{x})v^2$. Then,

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix} (v^1 - v^2) = 0,$$
$$\begin{pmatrix} 1 & -1 & 0 \end{pmatrix} (v^1 - v^2) = 0.$$

We deduce that $v^1 = v^2$. All hypotheses of Corollary 22 are fulfilled at $(\bar{u},\bar{x})$ and so S is semi-differentiable at the given point. Moreover, due to this corollary and Theorem 18, $D_{low}S(\bar{u},\bar{x})(d)$ is equal to the set of feasible points of the following multiobjective problem for each $d \in \mathbb{R}$:

$$\begin{aligned} \min \quad & \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix} v \\ \text{s.t.} \quad & v_1 - v_2 = 2d, \\ & -v_2 \leqq 0. \end{aligned}$$

In this example $\nabla_x f(\bar{u},\bar{x})$ is not a square matrix, and so [12, Theorem 3.1] cannot be applied.

5. Problem with general constraint.

In this section we consider the problem $\mathrm{P}(u)$, in which the constraint set $X(u)$ is given by

$$X(u) = \{x \in C : g(u,x) \in K\}, \tag{22}$$

where the sets $C \subset \mathbb{R}^n$ and $K \subset \mathbb{R}^m$ are closed and $g : \mathbb{R}^q \times \mathbb{R}^n \to \mathbb{R}^m$ is a continuously differentiable function. The constraint qualification (CQ) at $(\bar{u},\bar{x})$ is the following:

$$N_K(g(\bar{u},\bar{x})) \cap \{\nabla_x g(\bar{u},\bar{x})(T_C(\bar{x}))\}^+ = \{0\}.$$

We will need the following result ([21, Example 9.51]).

(R4) Let C and K be regular at $\bar{x}$ and $g(\bar{u},\bar{x})$, respectively, that is, $T_C(\bar{x}) = T_C^{cl}(\bar{x})$ and $T_K(g(\bar{u},\bar{x})) = T_K^{cl}(g(\bar{u},\bar{x}))$. If (CQ) holds at $(\bar{u},\bar{x})$, then X is both strictly semi-differentiable and pseudo-Lipschitz there and its strict semi-derivative is given by

$$D_{low}X(\bar{u},\bar{x})(d) = \{v \in T_C(\bar{x}) : \nabla_u g(\bar{u},\bar{x})d + \nabla_x g(\bar{u},\bar{x})v \in T_K(g(\bar{u},\bar{x}))\}.$$

THEOREM 24. *Let $\bar{y} \in V(\bar{u})$ be a uniformly efficient point. Assume the following conditions:*

(i) *$C \subset \mathbb{R}^n$ is nonempty compact and regular at $\bar{x}$ with $\bar{x} \in \hat{X}(\bar{u}, \bar{y})$, and $K \subseteq \mathbb{R}^m$ is nonempty closed and regular at $g(\bar{u}, \bar{x})$;*

(ii) *f and g are continuously differentiable and g satisfies* (CQ) *at every $(\bar{u}, \bar{x})$ with $\bar{x} \in \hat{X}(\bar{u}, \bar{y})$;*

(iii) *$\{v \in T_C(\bar{x}) : \nabla_x g(\bar{u}, \bar{x})v \in T_K(g(\bar{u}, \bar{x}))\} \cap \operatorname{Ker} \nabla_x f(\bar{u}, \bar{x}) = \{0\}$ for $\bar{x} \in \hat{X}(\bar{u}, \bar{y})$.*

Then V is semi-differentiable at $(\bar{u}, \bar{y})$ and

$$D_{low}V(\bar{u}, \bar{y})(d) = \bigcup_{\bar{x} \in \hat{X}(\bar{u}, \bar{y})} \{\nabla_u f(\bar{u}, \bar{x})d + \nabla_x f(\bar{u}, \bar{x})v : v \in T_C(\bar{x}), \nabla_x g(\bar{u}, \bar{x})v \in T_K(g(\bar{u}, \bar{x}))\}.$$

Proof. We wish to apply Theorem 18. First, we observe that in view of (R4), the mapping X is pseudo-Lipschitz and semi-differentiable at $(\bar{u}, \bar{x})$ with $\bar{x} \in \hat{X}(\bar{x}, \bar{y})$ because g is continuously differentiable and satisfies (CQ) at $(\bar{u}, \bar{x})$. Moreover, since C is compact, f has locally bounded and closed level sets, and X is closed around $\bar{u}$. It remains to verify (iii) of Theorem 18. According to (R4) and as X is semi-differentiable, we have $DX(\bar{u}, \bar{x})(0) = \{v \in T_C(\bar{x}) : \nabla_x g(\bar{u}, \bar{x})v \in T_K(g(\bar{u}, \bar{x}))\}$. Therefore, (iii) of Theorem 18 is immediate from (iii). The proof is complete. ◻

Let us consider a particular case when $K = -\mathbb{R}^p_+$ and C is compact and regular. The constraint $g(u, x) \in K$ becomes the system of inequalities $g_i(u, x) \leqq 0, i = 1, \ldots, m$, where $g_1, \ldots, g_m$ are the components of g. The feasible set mapping is given by

$$X(u) = \{x \in C : g_i(u, x) \leqq 0, i = 1, \ldots, m\}. \tag{23}$$

The active index set at $(\bar{u}, \bar{x}) \in gphX$ is given by

$$I(\bar{u}, \bar{x}) := \{i \in \{1, \ldots, m\} : g_i(\bar{u}, \bar{x}) = 0\}.$$

The constraint qualification (CQ) at $(\bar{u}, \bar{x}) \in gphX$ is the known Mangasarian–Fromovitz constraint qualification.

COROLLARY 25. *Let $\bar{y} \in V(\bar{u})$ be a uniformly efficient point. Assume the following conditions:*

(i) *f and g are continuously differentiable and g satisfies* (CQ) *at every $(\bar{u}, \bar{x})$ with $\bar{x} \in \hat{X}(\bar{x}, \bar{y})$;*

(ii) $\{v \in T_C(\bar{x}) : \nabla_x g_i(\bar{u}, \bar{x})v \leqq 0, i \in I(\bar{u}, \bar{x})\} \cap \operatorname{Ker} \nabla_x f(\bar{u}, \bar{x}) = \{0\}$.

Then V is semi-differentiable at $(\bar{u}, \bar{y})$ and

$$D_{low}V(\bar{u}, \bar{y})(d) = \bigcup_{\bar{x} \in \hat{X}(\bar{u}, \bar{x})} \{\nabla_u f(\bar{u}, \bar{x})d + \nabla_x f(\bar{u}, \bar{x})v : v \in T_C(\bar{x}), \nabla_x g_i(\bar{u}, \bar{x})v \leqq 0,$$
$$i \in I(\bar{u}, \bar{x})\}.$$

Proof. An application of Theorem 24 yields the result. ◻

6. Problem over a union of convex polyhedral sets. We consider the problem of minimizing a linear function over a union of polyhedral set described at the end of section 3:

$$\begin{aligned} \min \quad & C(u)x \\ \text{s.t. } & x \in X(u), \end{aligned}$$

where $C(u)$ is a $p \times n$-matrix, $X(u) = \bigcup_{i=1}^{m} X_i(u)$ with $X_i(u) = \{x \in \mathbb{R}^p : A_i(u)x \leqq b_i(u)\}$, in which $A_i(u)$ is an $m_i \times n$-matrix and $b_i(u)$ is an m_i-vector for each $u \in \mathbb{R}^q$ and $i = 1, \dots, m$. Here, it is assumed that all functions $C(.)$, $A_i(.)$, and $b_i(.)$ are continuously differentiable. We set $Y_i(u) := C(u)X_i(u)$ and $V_i(u) := MinY_i(u)$. The next theorem provides sufficient conditions for semi-differentiability of the marginal mapping V.

THEOREM 26. *Let $\bar{y} \in V(\bar{u})$. Assume the following conditions:*

(i) *for each $i \in \{1, \dots, m\}$ there exists x_i with $A_i(\bar{u})x_i < b(\bar{u})$;*

(ii) *for each $i \in \{1, \dots, m\}$ with $\bar{y} \in Y_i(\bar{u})$, there is a unique efficient solution $\bar{x}_i \in X_i(\bar{u})$ such that $\bar{y} = C(\bar{u})\bar{x}_i$ and*

$$\{\lambda \in -\mathbb{R}^p_+ : C^T(\bar{u})\lambda \in N^{co}_{X_i(\bar{u})}(\bar{x}_i)\} \subseteq -\operatorname{int}(\mathbb{R}^p_+) \cup \{0\}; \tag{24}$$

(iii) *for each $i \in \{1, \dots, m\}$ with $\bar{y} \notin Y_i(\bar{u})$ one has*

$$\{v : C(\bar{u})v \leqq 0\} \cap \{v : A_i(\bar{u})v \leqq 0\} = \{0\}. \tag{25}$$

Then V is semi-differentiable at $(\bar{u}, \bar{y})$ and

$$D_{low}V(\bar{u}, \bar{y})(d) = Min \bigcup_{\substack{i \in \{1,\dots,m\}: \\ \bar{y} = C(\bar{u})\bar{x}_i}} \{C'(\bar{u}; d)\bar{x}_i + C(\bar{u})v : v \in D_{low}X_i(\bar{u}, \bar{x}_i)(d)\}.$$

Proof. In order to apply Theorem 18, we show that $\bar{y}$ is a uniformly efficient point. To this end we check the hypotheses of Corollary 6. Clearly condition (i) of Corollary 6 is the same as (i). Condition (ii) of Corollary 6 follows from (25) for those i with $\bar{y} \notin Y_i(\bar{u})$, and from the uniqueness of $\bar{x}_i$ in (ii) for those i with $\bar{y} \in Y_i(\bar{u})$. Condition (iii) of Corollary 6 is obtained from (25), and finally (24) is exactly (iv) of Corollary 6. It remains to check the hypotheses (i)–(iii) of Theorem 18. First, C has locally bounded level sets because, otherwise, we may find some $y \in \mathbb{R}^p$, $(u_k, x_k) \in gphX$ such that $\lim_{k\to\infty} u_k = \bar{u}$, $\lim_{k\to\infty} \|x_k\| = +\infty$ and $C(u_k)x_k \leqq y$ for all $k \geqq 1$. Without loss of generality we may assume that $x_k \in X_i(u_k)$ for all k and some $i \in \{1, \dots, m\}$, and $\lim_{k\to\infty} x_k/\|x_k\| = v$ for some $v \neq 0$. Then we obtain $C(\bar{u})v \leqq 0$ and $A_i(\bar{u})v \leqq 0$. The two latter inequalities contradict (ii) if i is such that $\bar{y} \in Y_i(\bar{u})$, and (25) if i is such that $\bar{y} \notin Y_i(\bar{u})$. Second, it follows from (ii) that $\hat{X}(\bar{u}, \bar{y}) = \{\bar{x}_i : i \in \{1, \dots, m\}$ such that $\bar{y} = C(\bar{u})\bar{x}_i\}$. Moreover, since A_i and b_i are continuous, the mappings X_i are closed around $\bar{u}$. In view of (i), the constraint qualification (CQ) is satisfied, and hence X_i are pseudo-Lipschitz and semi-differentiable at $\bar{x}_i$. We deduce that X is closed around $\bar{u}$, pseudo-Lipshitz, and semi-differentiable at $(\bar{u}, x)$, $x \in \hat{X}(\bar{u}, \bar{y})$. Finally, the uniqueness of $\bar{x}_i$ in (ii) is equivalent to the condition that

$$\operatorname{Ker}(C(\bar{u})) \cap \{v : A_i^j(\bar{u})v \leqq 0, j \in I_i(\bar{u}, \bar{x})\} = \{0\},$$

where $A_i^j(\bar{u})$ stands for the jth row of $A_i(\bar{u})$. This, in its turn, implies condition (iii) of Theorem 18 because $DX_i(\bar{u}, \bar{x})(0) \subseteq \{v : A_i^j(\bar{u})v \leqq 0, j \in I_i(\bar{u}, \bar{x})\}$. It remains to apply Theorem 18 to complete the proof. □

We end this section with an example to illustrate Theorem 26.

Example 27. Let $X_1, X_2, X_3 : \mathbb{R} \rightrightarrows \mathbb{R}^2$ be given by $X_1(u) =: \{x : -ux_1 - 2x_2 \leqq (u-1)^2\}$, $X_2(u) =: \{x : -2x_1 - ux_2 \leqq 0\}$, and $X_3(u) =: \{x : 3x_1 - ux_2 \leqq -u\}$ and let

$$C(u) = \begin{pmatrix} u^2 & u-1 \\ 1-u & 2u \end{pmatrix}.$$

At $\bar{u} = 1$ and $\bar{y}^T = (0 \quad 0)$, we have $\bar{y} \in C(\bar{u})X_1(\bar{u}) \cap C(\bar{u})X_2(\bar{u})$ while $\bar{y} \notin C(\bar{u})X_3(\bar{u})$. It is verified that all conditions of Theorem 26 hold at the given point. Thus, V is semi-differentiable at the given point and, for each $d \in \mathbb{R}$,

$$D_{low}V(\bar{u}, \bar{y})(d) = cone\left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -4 \end{pmatrix} \right\}.$$

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