Optim Lett
DOI 10.1007/s11590-016-0999-3

ORIGINAL PAPER

On Benson's scalarization in multiobjective optimization

Majid Soleimani-damaneh[1,2] · Moslem Zamani[1,2]

Received: 10 February 2014 / Accepted: 5 January 2016

Abstract In this paper, a popular scalarization problem in multiobjective optimization, introduced by Benson, is considered. In the literature it was proved that, under convexity assumption, the set of properly efficient points is empty when the Benson's problem is unbounded. In this paper, it is shown that this result is still valid in general case without convexity assumption.

Keywords Multiobjective programming · Scalarization · Benson's problem

1 Preliminaries

In this paper, we consider the following general multiobjective optimization problem (MOP):

$$\min_{\mathbf{x}\in X} \mathbf{f}(\mathbf{x}) = \Big(f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_m(\mathbf{x})\Big). \qquad \textbf{(MOP)}$$

The set of feasible solutions of this problem is $X \subseteq \mathbb{R}^n$ and $\mathbf{f} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a vector-valued objective function.

✉ Majid Soleimani-damaneh
soleimani@khayam.ut.ac.ir

Moslem Zamani
moslemzamani@ut.ac.ir

1 School of Mathematics, Statistics and Computer Science, College of Science, University of Tehran, Tehran, Iran

2 School of Mathematics, Institute for Research in Fundamental Sciences (IPM), 19395-5746 Tehran, Iran

Published online: 19 January 2016

The solution notion for (MOP) is defined with respect to an ordering cone which is used for ordering the criterion space $\mathbb{R}^m$. We use the natural ordering cone defined as

$$\mathbb{R}^m_+ = \{\mathbf{x} \in \mathbb{R}^m : x_j \geq 0, \ \forall j = 1, 2, \ldots, m\}.$$

Utilizing this ordering cone, a feasible solution $\hat{\mathbf{x}} \in X$ is called an efficient solution to (MOP) if

$$\left(\mathbf{f}(\hat{\mathbf{x}}) - \mathbb{R}^m_+\right) \bigcap \mathbf{f}(X) = \{\mathbf{f}(\hat{\mathbf{x}})\}.$$

One of the solution concepts which plays an important role in multiobjective optimization, from both theoretical and practical points of view, is the proper efficiency notion [5,11,12]. This concept has been introduced to eliminate the efficient solutions with unbounded tradeoffs. There are different definitions for proper efficiency in the literature [2,3,6,9,10]. We use the following ones.

Definition 1 [6] A feasible solution $\hat{\mathbf{x}} \in X$ is called a properly efficient solution to (MOP) in the *Geoffrion*'s sense, if it is efficient and there is a real number $M > 0$ such that for all $i \in \{1, 2, \ldots, m\}$ and $\mathbf{x} \in X$ satisfying $f_i(\mathbf{x}) < f_i(\hat{\mathbf{x}})$ there exists an index $j \in \{1, 2, \ldots, m\}$ such that $f_j(\mathbf{x}) > f_j(\hat{\mathbf{x}})$ and

$$\frac{f_i(\hat{\mathbf{x}}) - f_i(\mathbf{x})}{f_j(\mathbf{x}) - f_j(\hat{\mathbf{x}})} \leq M.$$

Definition 2 [2] A feasible solution $\hat{\mathbf{x}} \in X$ is called a properly efficient solution to (MOP) in the *Benson*'s sense, if

$$cl\Big(cone\big(\mathbf{f}(X) + \mathbb{R}^m_+ - \mathbf{f}(\hat{\mathbf{x}})\big)\Big) \bigcap (-\mathbb{R}^m_+) = \{\mathbf{0}\},$$

where $cone(A) = \{\alpha\mathbf{a} : \alpha \geq 0, \ \mathbf{a} \in A\} = \bigcup_{\alpha \geq 0} \alpha A$, and $cl(A)$ is the closure of A.

Definition 3 [9] A feasible solution $\hat{\mathbf{x}} \in X$ is called a properly efficient solution to (MOP) in the *Henig*'s sense if $(\mathbf{f}(\hat{\mathbf{x}}) - C) \bigcap \mathbf{f}(X) = \{\mathbf{f}(\hat{\mathbf{x}})\}$, for some convex pointed cone C satisfying $\mathbb{R}^m_+ \backslash \{\mathbf{0}\} \subseteq int(C)$.

Since we are using the natural cone, the above three definitions are equivalent [9,11]. Hereafter, the set of efficient solutions and the set of properly efficient solutions are denoted by X_E and X_{PE}, respectively. Also, setting $Y = \mathbf{f}(X)$, the set of nondominated points, denoted by Y_N, is defined by $Y_N = \mathbf{f}(X_E)$; and the set of properly nondominated points, denoted by Y_{PN}, is defined by $Y_{PN} = \mathbf{f}(X_{PE})$.

The set Y is called $\mathbb{R}^m_+-$closed if $Y + \mathbb{R}^m_+$ is closed. Similarly, Y is called $\mathbb{R}^m_+-$convex if $Y + \mathbb{R}^m_+$ is convex. If X is a convex set and $f_1, f_2, \ldots, f_m$ are convex functions, then Y is $\mathbb{R}^m_+-$convex.

2 Scalarization

In this section, Benson's method [1], which is a popular scalarization tool, is dealt with. One important area of multiobjective programming research concerns the existence of properly and improperly efficient solutions. Benson's method [1] gives an examination of the existence of efficient and properly efficient solutions for multiobjective programming problems. Given a feasible solution, $\mathbf{x}^0 \in X$, the efficiency of $\mathbf{x}^0$ is checked utilizing l_1-norm, by solving the following single-objective optimization problem:

$$\begin{aligned} \max \quad & \sum_{k=1}^{m} l_k \\ \text{s.t.} \quad & f_k(\mathbf{x}^0) - l_k - f_k(\mathbf{x}) = 0, \quad k = 1, 2, \ldots, m, \\ & l_k \geq 0, \quad k = 1, 2, \ldots, m, \\ & \mathbf{x} \in X. \end{aligned} \tag{1}$$

A linear version of the above model was studied by Ecker and Kouada [4]. Some useful examples clarifying Model (1) can be found in Benson [1] and Giannessi et al. [7].

The vector $\mathbf{x}^0 \in X$ is efficient if and only if the optimal value of Problem (1) is zero; see [2,5]. If $(\hat{\mathbf{x}}, \hat{l})$ is an optimal solution to Model (1), then $\hat{\mathbf{x}}$ is an efficient solution; see [2,5]. One of the main questions is that, what happens when Problem (1) is unbounded. Benson [1] answered this question under convexity assumptions. He proved the following result. See also Theorem 4.16 in [5].

Theorem A *Assume that f_k, $k = 1, 2, \ldots, m$ are convex functions and X is a convex set. If Problem (1) is unbounded, then $X_{PE} = \emptyset$.*

As can be seen from the above theorem and discussion, any optimal solution of Problem (1) yields an efficient solution. Furthermore, in many cases the unboundedness of Problem (1) shows that no properly efficient solutions exist. In the following, we prove that this important result holds in all cases (without any convexity assumption).

Theorem 1 *If Problem (1) is unbounded, then $X_{PE} = \emptyset$.*

Proof If $\mathbf{x}^0 \in X$ is efficient, then Problem (1) has a finite optimal value equal to zero (see [1,5]). Therefore, due to the assumption, we have $\mathbf{x}^0 \in X \backslash X_E$.

By contradiction, assume that there exists $\hat{\mathbf{x}} \in X_{PE}$. Since $\hat{\mathbf{x}} \in X_{PE}$, due to the Henig proper efficiency, there exists a convex and pointed cone C, such that $\mathbb{R}^m_+ \backslash \{\mathbf{0}\} \subseteq intC$ and

$$(\mathbf{f}(\hat{\mathbf{x}}) - C \backslash \{\mathbf{0}\}) \cap \mathbf{f}(X) = \emptyset.$$

According to $\mathbb{R}^m_+ \backslash \{\mathbf{0}\} \subseteq intC$, we have $\mathbf{e}_i \in intC$ for each $i = 1, 2, \ldots, m$ (vector $\mathbf{e}_i$ denotes the $i-$th unit vector in $\mathbb{R}^m$). Thus there exists $r_i > 0$ such that $B(\mathbf{e}_i; r_i) = \{\mathbf{y} \in \mathbb{R}^m \ : \ \|\mathbf{y} - \mathbf{e}_i\| < r_i\} \subseteq C$.

Now, consider the following system, in which α, θ, and $\mathbf{y}$ are variables:

$$\begin{cases} \mathbf{f}(\mathbf{x}^0) - \alpha\mathbf{e}_i = \mathbf{f}(\hat{\mathbf{x}}) - \theta(\mathbf{e}_i + \mathbf{y}), \\ \|\mathbf{y}\| < r_i, \\ \alpha > 0, \ \theta > 0. \end{cases} \tag{2}$$

Set $\bar{\alpha}_i := 2r_i^{-1}\|\mathbf{f}(\hat{\mathbf{x}}) - \mathbf{f}(\mathbf{x}^0)\|$ and $\bar{\mathbf{y}} := \frac{r_i}{2} \times \frac{\mathbf{f}(\hat{\mathbf{x}})-\mathbf{f}(\mathbf{x}^0)}{\|\mathbf{f}(\hat{\mathbf{x}})-\mathbf{f}(\mathbf{x}^0)\|}$. The vector

$$(\alpha, \theta, \mathbf{y}) = (\bar{\alpha}_i, \bar{\alpha}_i, \bar{\mathbf{y}})$$

is a solution to system (2), and $\mathbf{e}_i + \mathbf{y} \in B(\mathbf{e}_i; r_i)$. Thus

$$\mathbf{f}(\mathbf{x}^0) - \bar{\alpha}_i\mathbf{e}_i - \mathbf{f}(\hat{\mathbf{x}}) \in -C.$$

Also, clearly we have

$$\forall \alpha > \bar{\alpha}_i, -(\alpha - \bar{\alpha}_i)\mathbf{e}_i \in -C\backslash\{\mathbf{0}\}.$$

Therefore, for each $\alpha > \bar{\alpha}_i$, we have $\mathbf{f}(\mathbf{x}^0) - \alpha\mathbf{e}_i - \mathbf{f}(\hat{\mathbf{x}}) \in -C\backslash\{\mathbf{0}\}$, because C is a convex cone. This implies $\mathbf{f}(\mathbf{x}^0) - \alpha\mathbf{e}_i \in \mathbf{f}(\hat{\mathbf{x}}) - C\backslash\{\mathbf{0}\}$. Hence, $\mathbf{f}(\mathbf{x}^0) - \alpha\mathbf{e}_i \notin \mathbf{f}(X)$ due to Henig proper efficiency.

So far, we proved that for each $\mathbf{e}_i$, $i = 1, 2, \ldots, m$, there exists some positive scalar $\bar{\alpha}_i > 0$, such that

$$\forall \alpha > \bar{\alpha}_i, \mathbf{f}(\mathbf{x}^0) - \alpha\mathbf{e}_i \notin \mathbf{f}(X). \tag{3}$$

Now defining $\bar{\alpha} := \max_{1\leq i\leq m} \bar{\alpha}_i$, we show that for any $\mathbf{d} \in \mathbb{R}^m_+$ satisfying $\sum_{i=1}^m d_i = 1$ and any $\alpha > \bar{\alpha}$, we have $\mathbf{f}(\mathbf{x}^0) - \alpha\mathbf{d} \notin \mathbf{f}(X)$.

The cone $-C$ is convex and also, for each i, we have $\mathbf{f}(\mathbf{x}^0) - \bar{\alpha}\mathbf{e}_i - \mathbf{f}(\hat{\mathbf{x}}) \in -C$. Therefore,

$$\sum_{i=1}^m d_i \Big(\mathbf{f}(\mathbf{x}^0) - \bar{\alpha}\mathbf{e}_i - \mathbf{f}(\hat{\mathbf{x}})\Big) \in -C.$$

This implies

$$\mathbf{f}(\mathbf{x}^0) - \bar{\alpha}\mathbf{d} - \mathbf{f}(\hat{\mathbf{x}}) \in -C.$$

Furthermore, we have $\mathbf{d} \in C$, which implies

$$-(\alpha - \bar{\alpha})\mathbf{d} \in -C\backslash\{0\}, \quad \forall \alpha > \bar{\alpha}.$$

Thus

$$\mathbf{f}(\mathbf{x}^0) - \alpha\mathbf{d} - \mathbf{f}(\hat{\mathbf{x}}) \in -C\backslash\{\mathbf{0}\}, \quad \forall \alpha > \bar{\alpha}.$$

Therefore, due to the proper efficiency of $\hat{\mathbf{x}}$, we get

$$\mathbf{f}(\mathbf{x}^0) - \alpha \mathbf{d} \notin \mathbf{f}(X), \quad \forall \alpha > \bar{\alpha}.$$

Thus, $\bar{\alpha}$ provides an upper bound for the objective function of Problem (1). This contradicts the unboundedness assumption on Problem (1), and completes the proof. □

Now we consider the following single-objective problem which has been studied by Guddat et al. [8] as hybrid scalarization method; see also [5].

$$\begin{aligned} \min \quad & \sum_{k=1}^{m} \lambda_k f_k(\mathbf{x}) \\ s.t. \quad & f_k(\mathbf{x}) \le f_k(\mathbf{x}^0), \quad k = 1, 2, \ldots, m, \\ & \mathbf{x} \in X. \end{aligned} \tag{4}$$

In this problem, $\lambda_1, \lambda_2, \ldots, \lambda_m$ are positive fixed scalars. This problem is an extension of Problem (1). Setting $\lambda_i = 1$ for each $i = 1, 2, \ldots, m$, Model (4) leads to Model (1).

The vector $\mathbf{x}^0 \in X$ is efficient if and only if $\mathbf{x}^0 \in X$ is an optimal solution to Problem (4); see [5,8]. It is not difficult to see that the unboundedness of Problem (4) implies the unboundedness of Problem (1). Therefore, by Theorem 1, we get the following corollary about the problem of Guddat et al. [8].

Corollary 1 *If Problem (4) is unbounded, then* $X_{PE} = \emptyset$.

Under the $\mathbb{R}^m_+-$closedness and $\mathbb{R}^m_+-$convexity assumptions, Benson [1] proved that, the unboundedness of Problem (1) implies $X_E = \emptyset$. Due to the above discussion, it can be seen that, under $\mathbb{R}^m_+-$closedness and $\mathbb{R}^m_+-$convexity assumptions, the unboundedness of Problem (4) implies $X_E = \emptyset$ as well. Although, in Theorem 1, we omitted the $\mathbb{R}^m_+-$convexity assumption for investigating X_{PE}, the following example shows that one can not omit this assumption for X_E.

Example 1 Let $X = \{(x_1, x_2) : x_1 < 0, \;\; x_2 \le 0, \;\; x_2 \ge \frac{1}{x_1}\} \cup \{(0, 0)\}$ and $f_1(\mathbf{x}) = x_1, \;\; f_2(x) = x_2$. Hence $Y = X$ is $\mathbb{R}^2_+-$closed, while it is not $\mathbb{R}^2_+-$convex. Considering $\mathbf{x}^0 = (0, 0)$, Problem (1) is unbounded while $X_E = \{(x_1, x_2) \in X : \; x_2 = \frac{1}{x_1}\}$.

Acknowledgments The authors would like to express their gratitude to anonymous referee and handling editor for their helpful comments on the first version of this paper. This research was in part supported by a Grant from IPM (No. 94260124).

References

1. Benson, H.: Existence of efficient solutions for vector maximization problems. J. Optim. Theory Appl. **26**(4), 569–580 (1978)
2. Benson, H.: An improved definition of proper efficiency for vector maximization with respect to cones. J. Optim. Theory Appl. **71**, 232–241 (1979)

3. Borwein, J.M.: Proper efficient points for maximization with respect to cones. SIAM J. Control Optim. **15**, 57–63 (1977)
4. Ecker, J.G., Kouada, I.A.: Finding efficient points for linear multiple objective programs. Math. Programm. **8**, 375–377 (1975)
5. Ehrgott, M.: Multicriteria Optim. Springer, Berlin (2005)
6. Geoffrion, A.: Proper efficiency and the theory of vector maximization. J. Math. Anal. Appl. **22**, 618–630 (1968)
7. Giannessi, F., Matroeni, G., Pellegrini, L.: On the theory of vector optimization and variational inequalities; image space analysis and separation. In: Giannessi, F. (ed.) Vector Variational Inequalities and Vector Equilibria. Mathematical Theories, vol. 38, pp. 141–215, Kluwer, Dordrecht (2000)
8. Guddat, J., Guerra Vasquez, F., Tammer, K., Wendler, K.: Multiobjective and Stochastic Optimization Based on Parametric Optimization, Akademie, Berlin (1985)
9. Henig, M.: Proper efficiency with respect to cones. J. Optim. Theory Appl. **36**(3), 387–407 (1982)
10. Kuhn, H., Tucker, A.: Nonlinear programming. In: Neyman, J. (ed.) Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, pp. 481–492. University of California Press, Berkeley (1951)
11. Sawaragi, Y., Nakayama, H., Tanino, T.: Theory of Multiobjective Optimization. Academic Press, Orlando (1985)
12. Soleimani-damaneh, M.: Nonsmooth optimization using Mordukhovichs subdifferential. SIAM J. Control Optim. **48**, 3403–3432 (2010)