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## Time scale separation in control of a single-link flexible-joint robot manipulator

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In this article, an adaptive tracking control is presented for a class of flexible-joint robotic manipulators in the presence of parametric uncertainties. For this purpose, the parameter separation technique and the idea of positive function of linearly connected parameters are coupled effectively with combination of backstepping and time scale separation. Subsequently, the virtual/actual control inputs are derived from the solutions of a series of fast dynamical equations. Furthermore, the adaptation law of unknown parameters can be derived based on Lyapunov theory in the backstepping technique. Stability proof of the overall closed-loop system is given via the theorem in singular perturbation theory. In this paper, the problem of ‘explosion of complexity’ existing in the conventional backstepping control method is avoided. The detailed simulation results are provided to demonstrate the effectiveness of the proposed controller.

**Keywords:** single-link flexible-joint robot; backstepping technique; time scale separation; parameter separation; positive function.

### 1. Introduction

During the past two decades, trajectory tracking control study of robotic manipulators with joint flexibilities has attracted considerable attention. Some approaches are proposed for the control problem of flexible-joint robots such as the feedback linearization method (Spong & Vidyasagar, 1989; Ozgoli & Taghirad, 2006; Rodriguez *et al.*, 2006; De Luca & Book, 2008), the passivity approach (Battilotti *et al.*, 1997; Brogliato *et al.*, 2007; Ott *et al.*, 2008), the singular perturbation approach (Ozgoli & Taghirad, 2009), the integral manifold control (Gandhi & Ghorbel, 2005), the adaptive sliding mode technique (Huang & Chen, 2004) and the proportional-derivative control approach (Wang & Liu, 1992; De Luca *et al.*, 2005).

Recently, the control of various flexible-joint robots has been considered such as flexible-joint robots with random noises (Liu & Wu, 2014), with transformation delay (Jiang *et al.*, 2014), with uncertain time-varying perturbations (Chang & Yen, 2012). Despite these efforts, control problem of linearly/non-linearly parameterized flexible-joint robots has remained largely open.

The backstepping technique (Krstic *et al.*, 1995) is a newly developed technique to control non-linear systems with parameter uncertainty, particularly those systems in which the uncertainties do not satisfy matching conditions. Conventional backstepping has been successfully applied to the control of flexible-joint robots recently (Lozano & Brogliato, 1992; Liu *et al.*, 2008; Bang *et al.*, 2010). However, the existing backstepping control methods (Lozano & Brogliato, 1992; Liu *et al.*, 2008; Bang *et al.*, 2010)

suffer from two major limitations. The first is that the controlled systems do not contain linear/non-linear parameterization. The second limitation is the so-called problem of ‘explosion of complexity’, which is caused by repeated differentiations of certain non-linear functions such as virtual controls, and thus inevitably leads to a complicated algorithm with heavy computation burden. In Ferrara & Giacomini (2000) by combination of backstepping procedure with a second order VSC algorithm, control of linearly parameterized flexible-joint robot is considered. However, one of the restricted assumptions is that the upper and lower bounds of functions in the last equation have to be known. In addition, the ‘explosion of complexity’ is the other problem.

It can be observed that a wide class of mechanical systems affected by uncertainties can be expressed in a triangular structure of parametric-pure feedback type. Single-link flexible-joint robot manipulator affected by uncertainty is an example of a system belonging to the class mentioned.

To overcome the ‘explosion of complexity’ problem of backstepping method for pure feedback systems, dynamic surface control (DSC) has been proposed in the controller design, by employing first-order filtering of the synthetic virtual control input at each step of backstepping approach (Li *et al.*, 2013; Sun *et al.*, 2013; Shaocheng *et al.*, 2015; Yongming *et al.*, 2015). Moreover, in Gao *et al.* (2013) and Yoo (2012), this problem is solved by combination of the backstepping and time scale separation. For the controller design, first by employing the time scale separation method, the time derivatives of the virtual/actual control inputs are defined as solutions of fast dynamic equations and then, their integrals are used as the virtual/actual control inputs. In Yoo (2012) a state predictor is designed to derive the adaptive laws for estimation of the unknown non-linearly connected parameters. However, designing state estimator is not straightforward and causes the procedure to be much more complicated. On the other hand, because of the state prediction errors, the control which is designed for the state estimator is not as accurate as when it is designed for the original system.

In this paper, an adaptive control scheme for single-link flexible-joint robot manipulator in the presence of parametric uncertainties, which include both linear and non-linear parameterization, is investigated. For non-linearly connected parameters term, using parameter separation technique, the bounding function is obtained which is linear in new unknown parameters. For linearly connected parameters term, the idea of positive function of parameter is applied. By employing the two mentioned techniques directly in combination of the backstepping and time scale separation procedures, the virtual/actual control inputs are defined as solutions of fast dynamic equations. Furthermore, designing state predictor to derive the adaptation law which results in more complexity is excluded and the adaptation law can be derived directly using Lyapunov theory in the backstepping procedure. For stability analysis, one of the theorems in singular perturbation theory is used.

The novelty of our work can be summarized as follows:

1. In this paper, an adaptive control method has been developed for a single-link flexible-joint robot manipulator in the presence of parametric uncertainties. These parameters appear linearly and non-linearly and belong to an unknown compact set, i.e., no prior knowledge is required on the bound of the unknown parameters.
2. The problem of ‘explosion of complexity’ existing in the conventional backstepping control methods due to the repeated differentiations of the virtual control inputs is avoided.
3. In Yoo (2012) which ‘the control of non-linearly parameterized pure feedback systems’ has been studied, first a state predictor is developed for deriving adaptive laws of unknown parameters and then combination of time scale separation and backstepping are applied to this state estimator for obtaining the virtual/actual controls. In our work, the adaptation law of unknown non-linearly

connected parameters, is derived without designing state predictor, which avoids complexity in Yoo (2012).

4. One of the most important aspect of the proposed method is its universality for a more general expanded class of pure-feedback systems.

The rest of this article is organized as follows. Problem formulation as well as preliminary results is presented in Section 2. In Section 3, we develop the controller structure. Finally the simulation results and some conclusion remarks are given in Sections 4 and 5.

## 2. Problem formulation and preliminaries

### 2.1. Mathematical model of single-link robotic manipulator

A single-link arm with a flexible-joint is illustrated in Fig. 1. The equations of motion for such a system can be derived as:

$$\begin{aligned} I\ddot{q}_1 + Mgl \sin(q_1) + k(q_1 - q_2) &= 0 \\ J\ddot{q}_2 + B\dot{q}_2 - k(q_1 - q_2) &= u, \end{aligned} \quad (2.1)$$

where  $q_1$  and  $q_2$  are the angular positions of the link and the motor shaft, and  $u$  is the input torque applied to the motor shaft. The load and motor inertias  $I$ ,  $J$ , the motor damping constant  $B$ , the joint stiffness  $k$ , the link mass  $M$ , the position of the link's center of gravity  $l$  can all be unknown.

We first try choice of the state variables  $\zeta_1 = q_1$ ,  $\zeta_2 = \dot{q}_1$ ,  $\zeta_3 = q_2$ ,  $\zeta_4 = \dot{q}_2$ . The dynamic Equation (2.1) becomes

$$\begin{aligned} \dot{\zeta}_1 &= \zeta_2, \\ \dot{\zeta}_2 &= -\frac{Mgl}{I} \sin(\zeta_1) + \frac{k}{I}(\zeta_3 - \zeta_1) \\ \dot{\zeta}_3 &= \zeta_4 \\ \dot{\zeta}_4 &= -\frac{B}{J}\zeta_4 + \frac{k}{J}(\zeta_1 - \zeta_3) + \frac{1}{J}u \end{aligned} \quad (2.2)$$

The control objective is to design a control law  $u(t)$  for system (2.2) such that the output  $y = \theta_m \zeta_1$  tracks a reference signal  $y_r$ , where  $\theta_m$  is the unknown parameter.

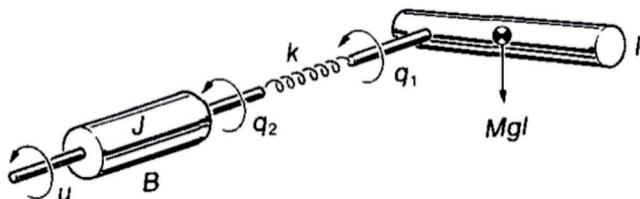


FIG. 1. Model of single-link flexible-joint arm.

By change of coordinates  $x = \theta_m \zeta$ , we obtain

$$\begin{aligned}\dot{x}_1 &= x_2, \\ \dot{x}_2 &= -\theta_m \frac{Mgl}{I} \sin\left(\frac{x_1}{\theta_m}\right) + \frac{k}{I} (x_3 - x_1), \\ \dot{x}_3 &= x_4, \\ \dot{x}_4 &= -\frac{B}{J} x_4 + \frac{k}{J} (x_1 - x_3) + \frac{\theta_m}{J} u, \\ y &= x_1.\end{aligned}\tag{2.3}$$

From (2.3) one has

$$\begin{aligned}\dot{x}_1 &= x_2, \\ \dot{x}_2 &= f(x_1, \theta_1) + \theta_2 (x_3 - x_1), \\ \dot{x}_3 &= x_4, \\ \dot{x}_4 &= \theta_3 x_4 + \theta_4 (x_1 - x_3) + \theta_5 u, \\ y &= x_1,\end{aligned}\tag{2.4}$$

where  $f(x_1, \theta_1) = -\theta_m \frac{Mgl}{I} \sin\left(\frac{x_1}{\theta_m}\right)$ ,  $\theta_2 = \frac{k}{I}$ ,  $\theta_3 = -\frac{B}{J}$ ,  $\theta_4 = \frac{k}{J}$ ,  $\theta_5 = \frac{\theta_m}{J}$

**ASSUMPTION 1** The reference signal  $y_r$  and its derivative  $\dot{y}_r$  are available and bounded.

## 2.2. Preliminaries on singular perturbation theory

Consider the problem of solving the state equation

$$\begin{aligned}\dot{x}(t) &= f(t, x(t), z(t), \varepsilon), \quad x(0) = \xi(\varepsilon), \\ \varepsilon \dot{z}(t) &= g(t, x(t), z(t), \varepsilon), \quad z(0) = \eta(\varepsilon),\end{aligned}\tag{2.5}$$

where  $\xi(\varepsilon)$  and  $\eta(\varepsilon)$  are smooth. It is assumed that the functions  $f$  and  $g$  are continuously differentiable in their arguments for  $(t, x, z, \varepsilon) \in [0, \infty) \times D_x \times D_z \times [0, \varepsilon_0]$  where  $D_x \subset R^n$  and  $D_z \subset R^m$  are open connected sets,  $\varepsilon_0 \gg 0$ . If  $g(t, x, z, 0) = 0$  has  $l \geq 1$  isolated real roots  $z = h_a(t, x)$ ,  $a = 1, 2, \dots, l$ , for each  $(t, x) \in [0, \infty) \times D_x$  when  $\varepsilon = 0$ , we say that the model (2.5) is in ‘standard form’. Let us choose one fixed parameter  $a \in \{1, 2, \dots, l\}$ , and drop the subscript  $a$  from  $h$  from now on. Let  $v = z - h(tx)$  where  $h(tx)$  denotes a chosen root of  $l$  roots satisfying  $g(t, x, z, 0) = 0$ . From singular perturbation theory, the ‘reduced system’ is represented by

$$\dot{x}(t) = f(t, x(t), h(t, x(t)), 0), \quad x(0) = \xi(0)\tag{2.6}$$

and the ‘boundary layer system’ with the new time scale  $\tau = t/\varepsilon$  is defined as

$$\frac{dv}{d\tau} = g(t, x, v + h(t, x(t)), 0), \quad v(0) = \eta_0 - h(0, \xi_0),\tag{2.7}$$

where  $\eta_0 = \eta(0)$  and  $\xi_0 = \xi(0)$  are treated as fixed parameters. The following theorem is introduced (Theorem 11.4 in Khalil, 1996).

**THEOREM 1** Consider the singular perturbation system (2.5) and assume that the following assumptions are satisfied for all  $(t, x, \varepsilon) \in [0, \infty) \times B_r \times [0, \varepsilon_0]$ ,

- (A1)  $f(t, 0, 0, \varepsilon) = 0$  and  $g(t, 0, 0, \varepsilon) = 0$ .
- (A2) The equation  $0 = g(t, x, z, 0)$  has an isolated root  $z = h(t, x)$  such that  $h(t, 0) = 0$ .
- (A3) The functions  $f, g, h$  and their partial derivatives up to order 2 are bounded for  $z - h(t, x) \in B_\rho$ .
- (A4) The origin of the boundary layer system  $\frac{dv}{d\tau} = g(t, x, v + h(t, x), 0)$  is exponentially stable, uniformly in  $(t, x)$ .
- (A5) The origin of the reduced system  $\dot{x} = f(t, x, h(t, x), 0)$  is exponentially stable. There is a Lyapunov function  $V(t, x)$  for the reduced system, which satisfies

$$\begin{aligned} c_1 \|x\|^2 &\leq V(t, x) \leq c_2 \|x\|^2 \\ \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x, h(t, x), 0) &\leq -c_3 \|x\|^2. \end{aligned}$$

For some positive constants  $c_i, i = 1, 2, 3$ .

Then, there exists a positive constant  $\varepsilon^*$  such that for all  $\varepsilon < \varepsilon^*$ , the origin of (2.5) is exponentially stable.

**REMARK 1** Assumption (A4) in Theorem 1 can be verified locally via linearization (Khalil, 1996). It can be shown that if there exists  $\varphi_0 > 0$  such that the Jacobian matrix  $(\partial g / \partial v)$  satisfies the eigenvalue condition  $\text{Re}[\lambda \{ \partial g(t, x, v + h(t, x), 0) / \partial v \}] \leq -\varphi_0 < 0$  for all  $x \in D_x$ , then Assumption (A4) is satisfied.

**REMARK 2** According to the proof procedure of Theorem 1 (Theorem 11.4 in Khalil, 1996), if the assumption (A5) is replaced by (A5')

(A5)' The origin of the reduced system  $\dot{x} = f(t, x, h(t, x), 0)$  is asymptotically stable. There is a Lyapunov function  $V(t, x)$  for the reduced system, which satisfies

$$\begin{aligned} V(t, x) &> 0 \\ \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x, h(t, x), 0) &\leq -c_3 \|x\|^2. \end{aligned}$$

Then, in Theorem 1, we can conclude that there exists a positive constant  $\varepsilon^*$  such that for all  $\varepsilon < \varepsilon^*$ , the origin of (2.5) is asymptotically stable.

### 2.3. Parameter separation

**LEMMA 1** For any real-valued continuous function  $f(x, y)$ , where  $x \in R^m, y \in R^n$ , there are smooth scalar functions  $a(x) \geq 0, b(y) \geq 0, c(x) \geq 1$  and  $d(y) \geq 1$ , such that

$$|f(x, y)| \leq a(x) + b(y) \quad (2.8)$$

$$|f(x,y)| \leq c(x) d(y) \quad (2.9)$$

a constructive proof is given in Lin & Qian (2002).

**REMARK 3** According to Lemma 1, there exist two smooth functions  $\gamma(x_1) \geq 1$  and  $\Lambda(\theta_1) \geq 1$  satisfying

$$|f(x_1, \theta_1)| \leq \gamma(x_1) \Lambda(\theta_1). \quad (2.10)$$

Let  $\Theta = \Lambda(\theta_1)$  be a new unknown constant. Using Remark 3, it is deduced that

$$|f(x_1, \theta_1)| \leq \gamma(x_1) \Theta. \quad (2.11)$$

To overcome the non-linear parameterization problem, using the above parameter separation method, instead of estimating the unknown parameter  $\theta_1$ , the unknown constant  $\Theta$  which appears linearly is estimated. Moreover, for designing the adaptive controller, in lieu off (.), the bounding functions  $\gamma(.)$  can be used.

### 3. Controller design and stability analysis

In this section, the adaptive control is developed by combination of backstepping, singular perturbation theory, parameter separation method and the idea of positive function of linearly connected parameters (PFLP). The design procedure is presented in the following. Introduce the change of coordinates  $z_1 = x_1 - y_r$ , and  $z_i = x_i - \alpha_{i-1}$  where  $i = 2, 3, 4$

*Step 1.* We start with the first equation of (2.4) by considering  $x_2$  as the control variable. The derivative of  $z_1$  is given as

$$\dot{z}_1(t) = z_2 + \alpha_1 - \dot{y}_r. \quad (3.1)$$

Then,  $\alpha_1$  as the first virtual controller can be specified as the solution of

$$z_2 + \alpha_1 - \dot{y}_r = -k_1 z_1 \quad (3.2)$$

resulting in the asymptotically stable closed-loop dynamics  $\dot{z}_1 = -k_1 z_1$  for the first subsystem.  $k_1 > 0$  is the first control gain. According to the following fast dynamics based on time-scale separation concept, an approximate virtual controller is designed

$$\varepsilon_1 \dot{\alpha}_1 = -\text{sign} \left( \frac{\partial Q_1}{\partial \alpha_1} \right) Q_1(t, \bar{z}_2, \alpha_1) \quad (3.3)$$

with the initial condition  $\alpha_1(0) = \alpha_{1,0}$ ,  $\varepsilon_1 \ll 1$ ,  $\bar{z}_2 = [z_1, z_2]^T$

$$Q_1(t, \bar{z}_2, \alpha_1) = k_1 z_1 + z_2 + \alpha_1 - \dot{y}_r \quad (3.4)$$

Let  $\alpha_1 = h_1(t, \bar{z}_2)$  be an isolated root of  $Q_1(t, \bar{z}_2, \alpha_1) = 0$ . Then the reduced system is defined as

$$\dot{z}_1 = -k_1 z_1, \quad z_1(0) = z_{1,0} \quad (3.5)$$

and the boundary layer system can be represented by

$$\frac{dy_1}{d\tau_1} = -\text{sign}\left(\frac{\partial Q_1}{\partial \alpha_1}\right) Q_1(t, \bar{z}_2, y_1 + h_1(t, \bar{z}_2)), \quad (3.6)$$

where  $y_1 = \alpha_1 - h_1(t, \bar{z}_2)$  and  $\tau_1 = t/\varepsilon_1$

Considering the control Lyapunov function  $V_1 = \frac{1}{2}z_1^2$  and using the reduced system (3.5), it is deduced that

$$\dot{V}_1 = -k_1 z_1^2. \quad (3.7)$$

*Step 2* :The derivative of  $z_2$  is expressed as

$$\dot{z}_2 = f(x_1, \theta_1) + \theta_2(z_3 + \alpha_2 - x_1) - \dot{\alpha}_1 \quad (3.8)$$

similar to step 1, we should find  $\alpha_2$  such that

$$f(x_1, \theta_1) + \theta_2(z_3 + \alpha_2 - x_1) - \dot{\alpha}_1 = -k_2 z_2, \quad (3.9)$$

where  $k_2 > 0$  is the positive control gain. In this step, the time derivative of the virtual control input  $\dot{\alpha}_1$  is appeared which has been designed in the previous step  $\dot{\alpha}_1 = -\text{sign}\left(\frac{\partial Q_1}{\partial \alpha_1}\right) Q_1(t, \bar{z}_2, \alpha_1)/\varepsilon_1$

The second approximate virtual controller can be designed as the following fast dynamics

$$\varepsilon_2 \dot{\alpha}_2 = -\text{sign}\left(\frac{\partial Q_2}{\partial \alpha_2}\right) Q_2(t, \bar{z}_3, \alpha_2, \hat{\Theta}, \hat{\theta}_2), \quad (3.10)$$

where  $\alpha_2(0) = \alpha_{2,0}$ ,  $\varepsilon_2 \ll 1$ ,  $\bar{z}_3 = [z_1, \dots, z_3]^T$  and

$$Q_2(t, \bar{z}_3, \alpha_2, \hat{\Theta}, \hat{\theta}_2) = k_2 z_2 + \text{sign}(z_2) \gamma(x_1) \hat{\Theta} + \Xi(\hat{\theta}_2)(z_3 + \alpha_2 - x_1) - \dot{\alpha}_1, \quad (3.11)$$

where according to Remark 3,  $\gamma(x_1)$  is the scalar function,  $\hat{\Theta}$  is an estimate of  $\Theta$ ,  $\hat{\theta}_2$  is an estimate of  $\theta_2$  and  $\Xi(\hat{\theta}_2)$  is a positive function of  $\hat{\theta}_2$  whose derivative is bounded away from zero; that is  $(\partial \Xi(\hat{\theta}_2)/\partial \hat{\theta}_2)$  is either positive or negative.

Let  $\alpha_2 = h_2(t, \bar{z}_3, \hat{\Theta}, \hat{\theta}_2)$  be an isolated root of  $Q_2(t, \bar{z}_3, \alpha_2, \hat{\Theta}, \hat{\theta}_2) = 0$ . Then the reduced system is defined as

$$\dot{z}_2 = -k_2 z_2 + f(x_1, \theta_1) - \text{sign}(z_2) \gamma(x_1) \hat{\Theta} + (x_3 - x_1) (\theta_2 - \Xi(\hat{\theta}_2)), \quad z_2(0) = z_{2,0} \quad (3.12)$$

and the boundary layer system can be represented by

$$\frac{dy_2}{d\tau_2} = -\text{sign}\left(\frac{\partial Q_2}{\partial \alpha_2}\right) Q_2(t, \bar{z}_3, y_2 + h_2(t, \bar{z}_3, \hat{\Theta}, \hat{\theta}_2), \hat{\Theta}, \hat{\theta}_2) \quad (3.13)$$

where  $y_2 = \alpha_2 - h_2(t, \bar{z}_3, \hat{\Theta}, \hat{\theta}_2)$  and  $\tau_2 = t/\varepsilon_2$ . Considering the control Lyapunov function  $V_2 = V_1 + \frac{1}{2}z_2^2$  and using the reduced system (3.12), it is deduced that

$$\begin{aligned}\dot{V}_2 &\leqslant -k_1 z_1^2 - k_2 z_2^2 + |z_2| |f(x_1, \theta_1)| - z_2 \text{sign}(z_2) \gamma(x_1) \hat{\Theta} + z_2(x_3 - x_1) (\theta_2 - \Xi(\hat{\theta}_2)) \\ &\leqslant -k_1 z_1^2 - k_2 z_2^2 + |z_2| \gamma(x_1) \tilde{\Theta} + z_2(x_3 - x_1) \tilde{\theta}_2\end{aligned}\quad (3.14)$$

which  $\tilde{\Theta} = \Theta - \hat{\Theta}$ ,  $\tilde{\theta}_2 = \theta_2 - \Xi(\hat{\theta}_2)$ .

*Step 3.* Similar to steps 1 and 2

$$\dot{z}_3 = z_4 + \alpha_3 - \dot{\alpha}_2 \quad (3.15)$$

$$\varepsilon_3 \dot{\alpha}_3 = -\text{sign}\left(\frac{\partial Q_3}{\partial \alpha_3}\right) Q_3(t, \bar{z}_4, \alpha_3) \quad (3.16)$$

with the initial condition  $\alpha_3(0) = \alpha_{3,0}$ ,  $\varepsilon_3 \ll 1$ ,  $\bar{z}_4 = [z_1, z_2, z_3, z_4]^T$ ,  $Q_3(t, \bar{z}_4, \alpha_3) = k_3 z_3 + z_4 + \alpha_3 - \dot{\alpha}_2$ , where  $k_3 > 0$  is the positive control gain and  $\dot{\alpha}_2 = -\text{sign}\left(\frac{\partial Q_2}{\partial \alpha_2}\right) Q_2(t, \bar{z}_3, \alpha_2, \hat{\Theta}, \hat{\theta}_2)/\varepsilon_2$ . Let  $\alpha_3 - h_3(t, \bar{z}_4)$  be an isolated root of  $Q_3(t, \bar{z}_4, \alpha_3) = 0$ . Then the reduced system is defined as

$$\dot{z}_3 = -k_3 z_3, \quad z_3(0) = z_{3,0} \quad (3.17)$$

and the boundary layer system can be represented by

$$\frac{dy_3}{d\tau_3} = -\text{sign}\left(\frac{\partial Q_3}{\partial \alpha_3}\right) Q_3(t, \bar{z}_4, y_3 + h_3(t, \bar{z}_4)), \quad (3.18)$$

where  $y_3 = \alpha_3 - h_3(t, \bar{z}_4)$  and  $\tau_3 = t/\varepsilon_3$

Considering the control Lyapunov function  $V_3 = V_2 + \frac{1}{2}z_3^2$ , it is deduced that

$$\dot{V}_3 \leqslant -k_1 z_1^2 - k_2 z_2^2 - k_3 z_3^2 + \tilde{\Theta} |z_2| \gamma(x_1) + z_2(x_3 - x_1) \tilde{\theta}_2 \quad (3.19)$$

*Step 4.* In the last step, the actual control input  $u$  appears and is at our disposal. We derive the  $z_4$  dynamics

$$\dot{z}_4 = \theta_3 x_4 + \theta_4(x_1 - x_3) + \theta_5 u - \dot{\alpha}_3. \quad (3.20)$$

We now obtain an approximate actual control input via time-scale separation to satisfy

$$\theta_3 x_4 + \theta_4(x_1 - x_3) + \theta_5 u - \dot{\alpha}_3 = -k_4 z_4 \quad (3.21)$$

as

$$\varepsilon_4 \dot{u} = -\text{sign}\left(\frac{\partial Q_4}{\partial u}\right) Q_4(t, \bar{z}_4, u, \hat{\theta}_3, \hat{\theta}_4, \hat{\theta}_5) \quad (3.22)$$

with the initial condition  $u(0) = u_0$ ,  $\varepsilon_4 \ll 1$  and

$$Q_4(t, \bar{z}_4, u, \hat{\theta}_3, \hat{\theta}_4, \hat{\theta}_5) = k_4 z_4 + \hat{\theta}_3 x_4 + \hat{\theta}_4(x_1 - x_3) + \Xi(\hat{\theta}_5) u - \dot{\alpha}_3 \quad (3.23)$$

$\bar{z}_4 = [z_1, z_2, z_3, z_4]^T$ .  $k_4 > 0$  is the positive control gain and  $\dot{\alpha}_3 = -\text{sign}\left(\frac{\partial Q_3}{\partial \alpha_3}\right) Q_3(t, \bar{z}_4, \alpha_3)/\varepsilon_3$ .

Let  $u = h_4(t, \bar{z}_4, \hat{\theta}_3, \hat{\theta}_4, \hat{\theta}_5)$  be an isolated root of  $Q_4(t, \bar{z}_4, u, \hat{\theta}_3, \hat{\theta}_4, \hat{\theta}_5) = 0$ . Then the reduced system is defined as

$$\dot{z}_4 = -k_4 z_4 + \tilde{\theta}_3 x_4 + \tilde{\theta}_4 (x_1 - x_3) + (\theta_5 - \Xi(\hat{\theta}_5)) u z_4(0) = z_{4,0} \quad (3.24)$$

and the boundary layer system can be represented by

$$\frac{dy_4}{d\tau_4} = -\text{sign}\left(\frac{\partial Q_4}{\partial u}\right) Q_4\left(t, \bar{z}_4, y_4 + h_4\left(t, \bar{z}_4, \hat{\theta}_3, \hat{\theta}_4, \hat{\theta}_5\right), \hat{\theta}_3, \hat{\theta}_4, \hat{\theta}_5\right), \quad (3.25)$$

where  $y_4 = u - h_4(t, \bar{z}_4, \hat{\theta}_3, \hat{\theta}_4, \hat{\theta}_5)$  and  $\tau_4 = t/\varepsilon_4$ . We choose the Lyapunov function  $V_4 = V_3 + \frac{1}{2}z_4^2 + \frac{1}{2}\tilde{\theta}^2 + \frac{1}{2}\tilde{\theta}^T \Gamma^{-1} \tilde{\theta}$  where  $\Gamma$  is a positive definite matrix and  $\tilde{\theta} = [\tilde{\theta}_2, \tilde{\theta}_3, \tilde{\theta}_4, \tilde{\theta}_5]^T$ .

The resulting derivatives of  $V_4$  is given as

$$\begin{aligned} \dot{V}_4 &\leq \sum_{i=1}^4 -k_i z_i^2 + \tilde{\Theta} |z_2| \gamma(x_1) - \tilde{\Theta} \dot{\tilde{\Theta}} + \tilde{\theta}^T \phi - \tilde{\theta}^T \Gamma^{-1} \Delta^T \\ &\leq \sum_{i=1}^4 -k_i z_i^2 + \tilde{\Theta} (|z_2| \gamma(x_1) - \dot{\tilde{\Theta}}) + \tilde{\theta}^T (\phi - \Gamma^{-1} \Delta^T), \end{aligned} \quad (3.26)$$

where  $\phi = [z_2(x_3 - x_1), z_4 x_4, z_4(x_1 - x_3), z_4 u]^T$  and  $\Delta = \left[ \dot{\tilde{\theta}}_2 \frac{\partial \Xi(\hat{\theta}_2)}{\partial \hat{\theta}_2}, \dot{\tilde{\theta}}_3, \dot{\tilde{\theta}}_4, \dot{\tilde{\theta}}_5 \frac{\partial \Xi(\hat{\theta}_5)}{\partial \hat{\theta}_5} \right]^T$ .

Finally, we can eliminate the  $\tilde{\Theta}$  and  $\tilde{\theta}^T$  terms from (3.26) by designing the adaptation law as

$$\dot{\tilde{\Theta}} = |z_2| \gamma(x_1), \quad (3.27)$$

$$\dot{\tilde{\theta}} = \Gamma \phi * \eta, \quad \eta = \left[ 1 / \frac{\partial \Xi(\hat{\theta}_2)}{\partial \hat{\theta}_2}, 1, 1, 1 / \frac{\partial \Xi(\hat{\theta}_5)}{\partial \hat{\theta}_5} \right]^T, \quad (3.28)$$

where  $\Gamma \phi * \eta$  is defined as the element-by-element multiplication of array  $\Gamma \phi$  by  $\eta$ .

Therefore, the derivative of  $V_n$  is

$$\dot{V}_4 \leq \sum_{i=1}^4 -k_i z_i^2 \quad (3.29)$$

By using the Lasalle's Theorem, this Lyapunov function guarantees the asymptotic stability of the origin of reduced system (3.5), (3.12), (3.17) and (3.24).

**REMARK 4** In each step  $i$ , the time derivative of the virtual control input  $\dot{\alpha}_{i-1}$  is appeared which has been designed in the previous step  $\dot{\alpha}_{i-1} = -\text{sign}\left(\frac{\partial Q_{i-1}}{\partial \alpha_{i-1}}\right) Q_{i-1}/\varepsilon_{i-1}$ . Therefore, the ‘explosion of complexity’ arising from the calculation of this term is avoided.

**THEOREM 2** Consider the closed-loop system (2.4). Under Assumption 1, the adaptive controller (3.22) with virtual controllers (3.3), (3.10) and (3.16), and parameter adaptive laws (3.27) and (3.28) guarantees that output system tracks the desired signal.

*Proof.* Using Theorem 1, it is verified that the Assumptions (A1)–(A3) are satisfied with (3.1), (3.3), (3.8), (3.10), (3.15), (3.16), (3.20) and (3.22). Note that because of the tracking problem, Assumptions (A1) and (A2) can be fulfilled by applying change of variable.

The exponential stability of the boundary layer system (3.6), (3.13), (3.18) and (3.25) can be easily obtained locally by linearization with respect to  $y_i, i = 1, 2, 3, 4$ . Using Remark 1 yields

$$\text{sign} \left( \frac{\partial Q_i}{\partial \alpha_i} \right) > 0 \quad i = 1, 2, 3, \quad (3.30)$$

$$\text{sign} \left( \frac{\partial Q_4}{\partial u} \right) > 0. \quad (3.31)$$

This confirms that the boundary layer system has a locally exponentially stable origin.

Finally, since in the previous part, we showed the asymptotic stability of the origin of reduced system (3.5), (3.12), (3.17) and (3.24), considering Remark 2, assumption(A5)' is satisfied.

According to Theorem 1 and Remark 2, there exist constant  $\varepsilon_i^* > 0$  such that for  $0 < \varepsilon_i < \varepsilon_i^*, i = 1, 2, 3, 4$ , the origins of the systems (3.1), (3.8), (3.15) and (3.20) are asymptotically stable. It follows that  $z_i \rightarrow 0, i = 1, 2, 3, 4$  as  $t \rightarrow \infty$ . Since  $z_1 = y - y_r$ , the error between system output and reference signal converges to zero asymptotically.  $\square$

**REMARK 5** The idea of PFLP or using positive function  $\Xi(\hat{\theta})$  instead of  $\hat{\theta}$  is just for satisfying (3.30) and (3.31).

**REMARK 6** A single-link flexible-joint robot manipulator affected by uncertainty is an example of a system belonging to the parametric-pure feedback type in the form

$$\begin{aligned} \dot{x}_i(t) &= f_{i1}(\bar{x}_i(t), x_{i+1}(t)) + f_{i2}(\bar{x}_i(t), \theta) + f_{i3}^T(\bar{x}_i(t), x_{i+1}(t))\theta \quad i = 1, \dots, n-1, \\ \dot{x}_n(t) &= f_{n1}(\bar{x}_n(t), u(t)) + f_{n2}(\bar{x}_n(t), \theta) + f_{n3}^T(\bar{x}_n(t), u(t))\theta, \end{aligned} \quad (3.32)$$

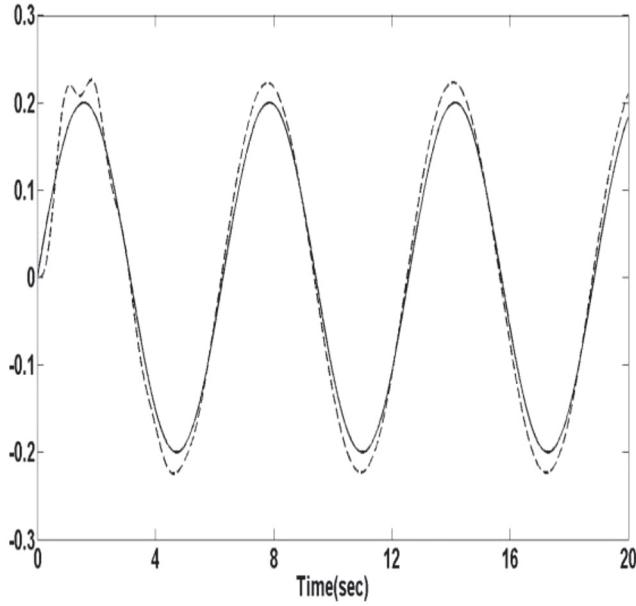
where  $\bar{x}_i = [x_1, x_2, \dots, x_i]^T \in R^i$  and  $u \in R$  are the system states and control input, respectively.  $\theta \in R^s$  denotes an unknown constant vector. As illustrated above,  $f_{i3}$  and  $f_{n3}$  can be developed as the function of  $x_{i+1}$  and  $u$ , respectively which finally leads to more general expanded class of pure feedback systems. The control design procedure in this paper can be used to control the system (3.32).

#### 4. Simulation results

Consider the system (2.4), where  $Mgl = 5, I = 1, J = 0.3, k = 100, B = .1, \theta_n = 1$ . To obtain the bounding function of the non-linearly connected parameter term, a direct calculation gives  $\left| -\theta_m \frac{Mgl}{I} \sin \left( \frac{x_1}{\theta_m} \right) \right| \leqslant |\theta_m \frac{Mgl}{I}| \times 1$ . We define  $\gamma(x_1) = 1$  and  $\Theta = |\theta_m \frac{Mgl}{I}|$ .

In order to show the effectiveness of the proposed approach, the simulation results are evaluated for different parameter values in two cases:

**Case 1:** Here the reference signal is defined as  $y_r = 0.2 \sin(t)$  and the design parameters in the control scheme are chosen as  $k_1 = 8, k_2 = 5, k_3 = 3, k_4 = 7, \varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = 0.01$ . Furthermore,

FIG. 2. Trajectories of  $y$  (dashed line) and  $y_r$  (solid line).

$\mathcal{E}(\hat{\theta}_2) = \exp(\hat{\theta}_2)$ ,  $\mathcal{E}(\hat{\theta}_5) = 1 + \tanh(\hat{\theta}_5)$  and  $\Gamma$  is the  $4 \times 4$  identity matrix. The initial conditions are chosen as  $x_1(0) = x_2(0) = x_3(0) = x_4(0) = 0$ ,  $a_1(0) = a_2(0) = a_3(0) = u(0) = 0$ ,  $\hat{\theta}(0) = [1 \ 1 \ 1 \ 1]^T \hat{\Theta}(0) = 1$ .

The fast dynamic for determining the solution of control signal is designed as

$$\varepsilon_1 \dot{\alpha}_1 = -[k_1 z_1 + z_2 + \alpha_1 - \dot{y}_r], \quad (4.1)$$

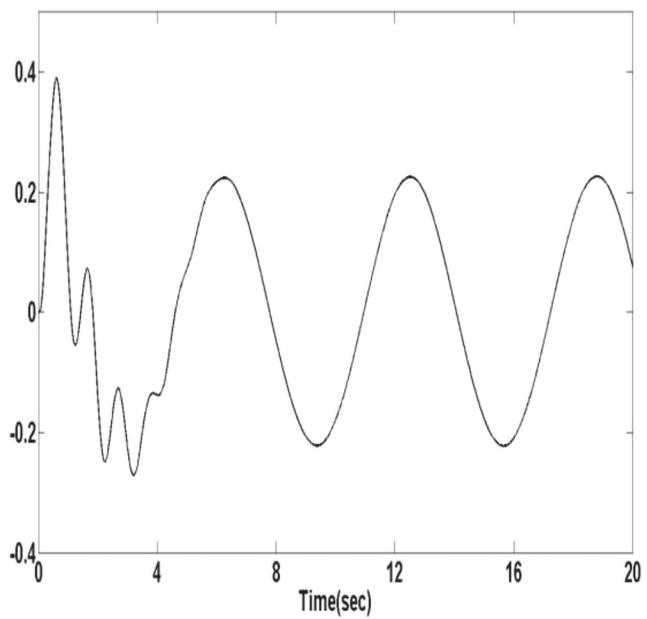
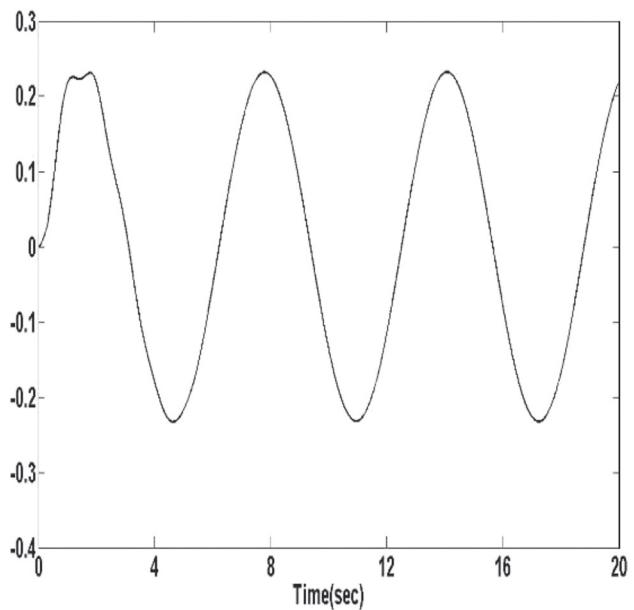
$$\varepsilon_2 \dot{\alpha}_2 = -[k_2 z_2 + \text{sign}(z_2) \gamma(x_1) \hat{\Theta} + \mathcal{E}(\hat{\theta}_2)(z_3 + \alpha_2 - x_1) - \dot{\alpha}_1], \quad (4.2)$$

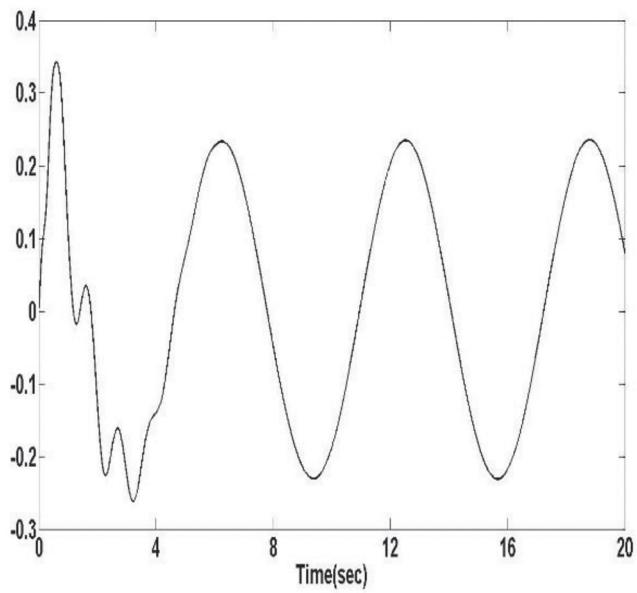
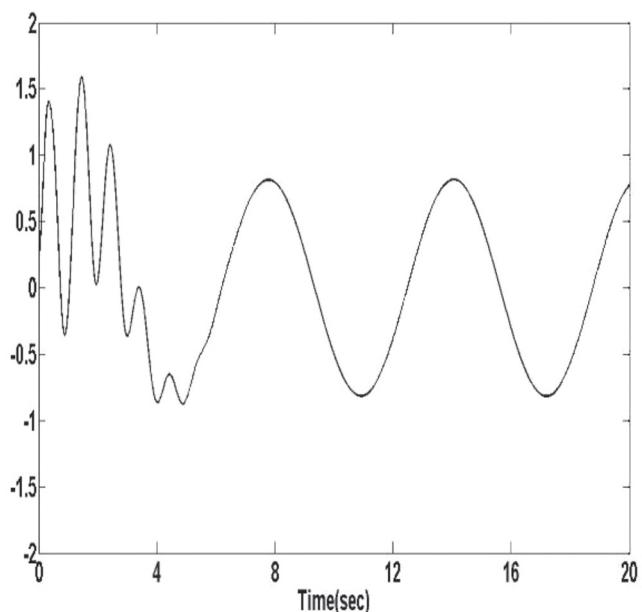
$$\varepsilon_3 \dot{\alpha}_3 = -[k_3 z_3 + z_4 + \alpha_3 - \dot{\alpha}_2], \quad (4.3)$$

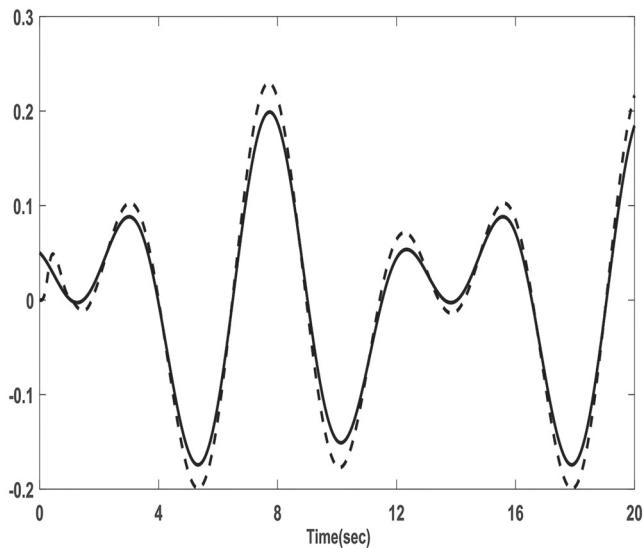
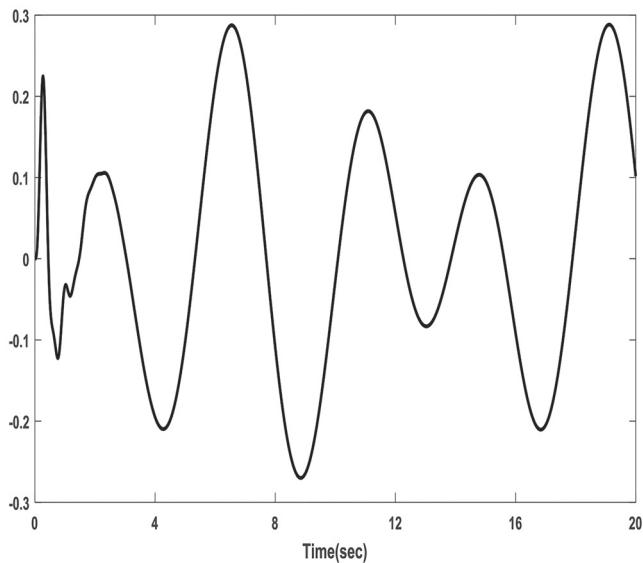
$$\varepsilon_4 \dot{u} = -[k_4 z_4 + \hat{\theta}_3 x_4 + \hat{\theta}_4(x_1 - x_3) + \mathcal{E}(\hat{\theta}_5) u - \dot{\alpha}_3]. \quad (4.4)$$

*Case 2:* In this case,  $y_r = 0.1 \cos(1.5t + \pi/3) + 0.1 \sin(t)$  is the reference signal and the design parameters are chosen as  $k_1 = k_2 = k_3 = 4$ ,  $\varepsilon_1 = \varepsilon_2 = 0.1$ ,  $\varepsilon_3 = \varepsilon_4 = 0.01$ ,  $\mathcal{E}(\hat{\theta}_2) = \exp(\hat{\theta}_2)$  and  $\mathcal{E}(\hat{\theta}_5) = \exp(\hat{\theta}_5)$ . The initial conditions are chosen as  $\hat{\theta}(0) = [0.1 \ 0.1 \ 1 \ 1]^T \hat{\Theta}(0) = 1.5$  and the other design parameters are similar to case 1.

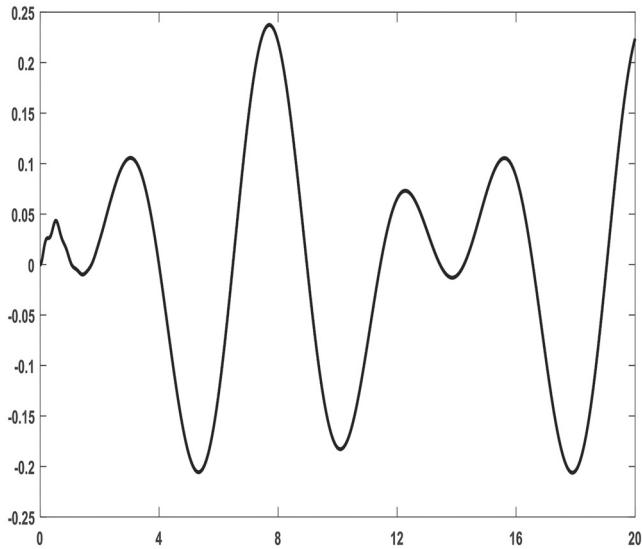
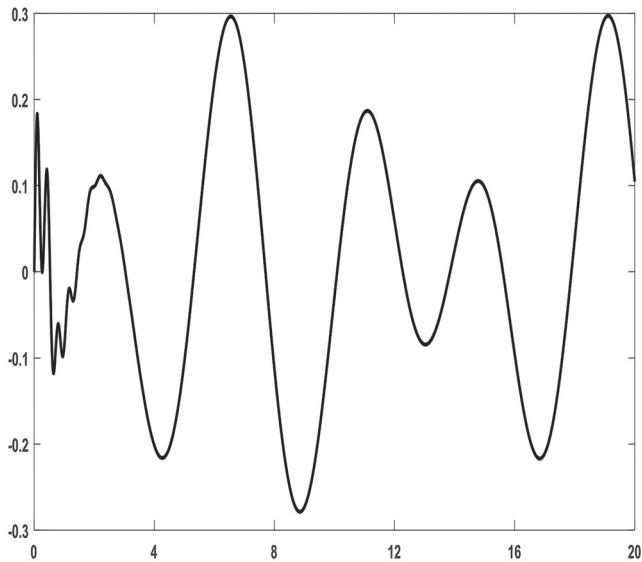
$x_1$  or the angular position  $q_1$  on the link side is depicted in Figs 2 and 3.  $x_2$  or the angular velocity  $\dot{q}_1$  on the link side is shown in Figs 4 and 5.  $x_3$  or the angular position  $q_2$  on the motor side is depicted in Figs 6 and 7.  $x_4$  or the angular velocity  $\dot{q}_2$  on the motor side is shown in Figs 8 and 9. The input voltage  $u$  is represented in Figs 10 and 11.

FIG. 3. Trajectory of  $x_2$ FIG. 4. Trajectory of  $x_3$ .

FIG. 5. Trajectory of  $x_4$ .FIG. 6. Trajectory of  $u$ .

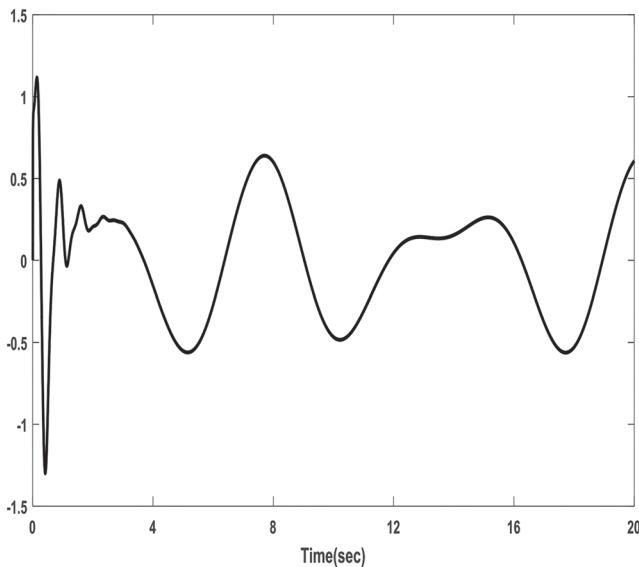
FIG. 7. Trajectories of  $y$  (dashed line) and  $y_r$  (solid line).FIG. 8. Trajectory of  $x_2$ .

These figures reveal that the proposed approach has the fairly good tracking performance and control regardless of parametric uncertainties. In addition, note that the states and the control input in the controlled closed-loop system are bounded.

FIG. 9. Trajectory of  $x_3$ .FIG. 10. Trajectory of  $x_4$ .

## 5. Conclusion

In this paper, an adaptive control has been developed for a single-link robotic manipulator in the presence of parametric uncertainties. By combination of backstepping and singular perturbation concept and coupling it effectively with the parameter separation technique and taking in to account of the idea

FIG. 11. Trajectory of  $u$ .

of (PFLP), virtual/actual control inputs as well as adaptation law of unknown parameters have been derived. The proposed control approach can overcome the uncertainties and the problem of ‘explosion of complexity’. Simulation results have been presented to illustrate the satisfactory tracking and good transient performance of the proposed algorithm. Future research will concentrate on an adaptive output-feedback control design of single-link robotic manipulator with immeasurable states in the presence of linear and non-linear parameterization.

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