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Adaptive control of pure-feedback systems in the presence of parametric uncertainties

Mehrnoosh ASADI*, **Heydar TOOSSIAN SHANDIZ**
Department of Electrical and Robotic Engineering, Shahrood University of Technology, Shahrood, Iran

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Abstract: The purpose of this study is to investigate an adaptive control approach for completely nonaffine pure-feedback systems with linear/nonlinear parameterization. In this approach, the parameter separation technique and the idea of the positive function of linearly connected parameters are coupled effectively with the combination of backstepping and time-scale separation. Subsequently, a fast dynamical equation is derived from the original subsystem, where the solution is sought to approximate the corresponding ideal virtual/actual control inputs. Furthermore, the adaptation law of unknown parameters can be derived based on Lyapunov theory in the backstepping technique and there is no need to design a state predictor for this purpose. Therefore, it results in higher accuracy and avoids complexity. The closed-loop stability and the state regulation of these systems are proved. Finally, the simulation results are provided to demonstrate the effectiveness of the proposed approach.

Key words: Pure-feedback systems, parametric uncertainties, positive function of linearly connected parameters, parameter separation, time-scale separation

1. Introduction

In recent years, many different control designs such as feedback linearization, adaptive control, inversion control, and backstepping have been developed for nonlinear dynamical systems [1,2]. The previous research concentrated mainly on affine systems and there are limited studies about nonaffine ones. Some of these studies have used neural networks (NNs) [3,4] or fuzzy logic systems [5,6] for approximating the ideal control input. According to some other researchers, the original system in nonaffine form is transformed into an affine one, and then the obtained control input for this affine system can be applied directly to the transformed system [7,8]. In [9,10], a dynamic inversion method was introduced for controlling nonaffine systems. In this method, first for derivation of adaptive laws, the state predictor is designed, and then, based on time-scale separation, the dynamic inversion controller defined as a solution of a fast dynamical equation is approximated.

Among different nonlinear systems, pure-feedback systems can represent more practical processes such as biochemical processes [11], aircraft flight control systems [2], or mechanical systems [12]. To design a control for the pure-feedback systems, the backstepping control technique provides a systematic framework [13]. However, there are two difficulties with the backstepping method for pure-feedback systems: 1) the problem of "explosion of complexity" due to the repeated differentiations of the virtual control inputs, and 2) the cascade and nonaffine properties of these systems, which make it difficult to find the explicit virtual/actual control inputs. In [14,15] affine pure-feedback systems were investigated by an adaptive NN-based control method. Adaptive neural

*Correspondence: asadi.mehr77@gmail.com

backstepping control of completely nonaffine pure-feedback systems was presented in [16] using input-to-state stability analysis and the small gain theorem. In [14–16] the time derivatives of the virtual control inputs were either approximated by the NNs, resulting in a complicated controller design, or ignored completely, leading to poor tracking performance.

In [17,18], the two mentioned problems of the backstepping method for these systems were simultaneously solved by a combination of backstepping and time-scale separation. In this proposed technique, first by employing the time-scale separation method, the time derivatives of the virtual/actual control inputs are defined as solutions of fast dynamic equations, and then their integrals are used as the virtual/actual control inputs.

In the past few years, the control of various pure-feedback systems was considered, such as uncertain nonaffine pure-feedback systems with unknown dead zone [19], with unknown time delay [20], with hysteresis input [21], and with output constraints [22]. Despite these efforts, the control problem of completely nonaffine pure-feedback systems with nonlinear parameterization has remained largely open. These systems were considered in [18]. For the controller design, first using the series parallel model, a state predictor was designed to derive the adaptive laws for estimation of the unknown parameters. Backstepping and the singular perturbation concept were then combined to develop the approximate version of virtual/actual control laws for this state estimator. Tikhonov's theorem from the singular perturbation theorem was used for stability analysis. However, designing the state estimator is not straightforward and it causes the procedure to be much more complicated. On the other hand, because of the state prediction errors, the control designed for the state estimator is not as accurate as when it is designed for the original system.

In this paper, an adaptive control scheme for completely nonaffine pure-feedback systems in the presence of parametric uncertainties, which include both linear and nonlinear parameterization, is investigated. Therefore, the class of systems considered here is much more general than the systems in [18]. For nonlinearly connected parameter terms, using a parameter separation technique, the bounding function is obtained, which is linear in new unknown parameters. For linearly connected parameter terms, a positive function of the parameter is applied instead of using the parameter itself. By employing the two mentioned techniques directly in a combination of the backstepping and time-scale separation procedures, the virtual/actual control inputs are defined as solutions of fast dynamic equations. In contrast to [18], designing a state predictor to derive the adaptation law, which results in more complexity, is excluded and the adaptation law can be derived directly using Lyapunov theory in the backstepping procedure. For stability analysis, one of the theorems in singular perturbation theory is used.

This paper is organized as follows. Preliminary results as well as problem formulation are presented in Section 2. In Section 3, we develop the controller structure. Finally, the simulation results and some conclusion remarks are given in Sections 4 and 5.

2. Preliminaries and problem formulation

2.1. Preliminaries on the singular perturbation theory [23]

Consider the problem of solving the state equation

$$\dot{x}\left(\mathbf{t}\right)=\mathbf{f}\left(\mathbf{t},\mathbf{x}\left(\mathbf{t}\right),\mathbf{z}\left(\mathbf{t}\right),\varepsilon\right),\qquad \mathbf{x}\left(\mathbf{0}\right)=\xi(\varepsilon)$$

$$\varepsilon\dot{z}\left(\mathbf{t}\right)=\mathbf{g}\left(\mathbf{t},\mathbf{x}\left(\mathbf{t}\right),\mathbf{z}\left(\mathbf{t}\right),\varepsilon\right),\qquad \mathbf{z}\left(\mathbf{0}\right)=\eta(\varepsilon), \tag{1}$$

where $\xi(\varepsilon)$ and $\eta(\varepsilon)$ are smooth. It is assumed that the functions $\mathbf{f}$ and $\mathbf{g}$ are continuously differentiable in their arguments for $(\mathbf{t},\mathbf{x},\mathbf{z},\varepsilon)\in[\mathbf{0},\infty)\times\mathbf{D_x}\times\mathbf{D_z}\times[\mathbf{0},\varepsilon_\mathbf{0}]$ where $\mathbf{D_x}\subset\mathbf{R^n}$ and $\mathbf{D_z}\subset\mathbf{R^m}$ are open connected sets, $\varepsilon_\mathbf{0}\gg\mathbf{0}$. If $\mathbf{g}(\mathbf{t},\mathbf{x},\mathbf{z},\mathbf{0})=\mathbf{0}$ has $\mathbf{l}\geq\mathbf{1}$ isolated real roots $\mathbf{z}=\mathbf{h_a}(\mathbf{t},\mathbf{x})$, $\mathbf{a}=\mathbf{1},\mathbf{2},\ldots,\mathbf{l}$, for each $(\mathbf{t},\mathbf{x})\in[\mathbf{0},\infty)\times\mathbf{D_x}$ when $\varepsilon=\mathbf{0}$, we say that the model of Eq. (1) is in 'standard form'. Let us choose one fixed parameter $\mathbf{a}\in\{\mathbf{1},\mathbf{2},\ldots,\mathbf{l}\}$, and drop the subscript a from h from now on. Let $\mathbf{v}=\mathbf{z}-\mathbf{h}(\mathbf{t},\mathbf{x})$ where $\mathbf{h}(\mathbf{t},\mathbf{x})$ denotes a chosen root of l roots satisfying $\mathbf{g}(\mathbf{t},\mathbf{x},\mathbf{z},\mathbf{0})=\mathbf{0}$. From singular perturbation theory, the 'reduced system' is represented by

$$\dot{x}(\mathbf{t})=\mathbf{f}(\mathbf{t},\mathbf{x}(\mathbf{t}),\mathbf{h}(\mathbf{t},\mathbf{x}(\mathbf{t})),\mathbf{0}),\qquad \mathbf{x}(\mathbf{0})=\xi(\mathbf{0}) \tag{2}$$

and the 'boundary layer system' with the new time scale $\tau=\mathbf{t}/\varepsilon$ is defined as

$$\frac{\mathbf{dv}}{\mathbf{d}\tau}=\mathbf{g}(\mathbf{t},\mathbf{x},\mathbf{v}+\mathbf{h}(\mathbf{t},\mathbf{x}(\mathbf{t})),\mathbf{0}),\qquad \mathbf{v}(\mathbf{0})=\eta_\mathbf{0}-\mathbf{h}(\mathbf{0},\xi_\mathbf{0}), \tag{3}$$

where $\eta_\mathbf{0}=\eta(\mathbf{0})$ and $\xi_\mathbf{0}=\xi(\mathbf{0})$ are treated as fixed parameters. The following theorem is introduced (Theorem 11.2 in [23]).

Theorem 1 *Consider the singular perturbation system of Eq. (1) and assume that the following assumptions are satisfied for all* $(\mathbf{t},\mathbf{x},\varepsilon)\in[\mathbf{0},\infty)\times\mathbf{B_r}$[1] $\times[\mathbf{0},\varepsilon_\mathbf{0}]$,

(A1) $f(t,0,0,\varepsilon)=0$ *and* $g(t,0,0,\varepsilon)=0$.

(A2) The equation $0=g(t,x,z,0)$ *has an isolated root* $z=h(t,x)$ *such that* $h(t,0)=0$.

(A3) The functions f,g,h *and their partial derivatives up to order 2 are bounded for* $z-h(t,x)\in B_\rho$[2].

(A4) The origin of the boundary layer system $\frac{\mathbf{dv}}{\mathbf{d}\tau}=\mathbf{g}(\mathbf{t},\mathbf{x},\mathbf{v}+\mathbf{h}(\mathbf{t},\mathbf{x}),\mathbf{0})$ *is uniformly exponentially stable in* $(\mathbf{t},\mathbf{x})$.

(A5) The origin of the reduced system $\dot{x}=f(t,x,h(t,x),0)$ *is exponentially stable. Namely, there is a Lyapunov function* $V(t,x)$ *for the reduced system that satisfies*

$$c_1\|x\|^2\leq V(t,x)\leq c_2\|x\|^2$$

$$\frac{\partial V}{\partial t}+\frac{\partial V}{\partial x}f(t,x,h(t,x),0)\leq -c_3\|x\|^2$$

for some positive constants c_i, $i=1,\ldots,3$.

Then there exists a positive constant ε^* *such that for all* $\varepsilon<\varepsilon^*$, *the origin of Eq. (1) is exponentially stable.*

Remark 1 *According to the proof of Theorem 1 (Theorem 11.4 in [23]), if assumption (A5) is replaced by* $(\mathbf{A5})'$, *which is stated below:*

$(A5)'$ *The origin of the reduced system* $\dot{x}=f(t,x,h(t,x),0)$ *is asymptotically stable. Namely, there is a Lyapunov function* $V(t,x)$ *for the reduced system that satisfies*

$$V(t,x)>0$$

[1] $B_r=\{x\in R^n\mid\|x\|\leq r\}$
[2] $B_\rho=\{y\in R^m\mid\|y\|\leq\rho\}$

$$\frac{\partial V}{\partial t}+\frac{\partial V}{\partial x}f(t,x,h\left(t,x\right),0)\leq -c_3\left\|x\right\|^2$$

then we can conclude that there exists a positive constant ε^* such that for all $\varepsilon<\varepsilon^*$, the origin of Eq. (1) is asymptotically stable in Theorem 1.

2.2. Problem statement

Consider the following pure-feedback system with both linear and nonlinear parameterization:

$$\dot{x}_{\mathbf{i}}\left(\mathbf{t}\right)=\mathbf{f_{i1}}\left(\bar{x}_{\mathbf{i}}\left(\mathbf{t}\right),\mathbf{x_{i+1}}\left(\mathbf{t}\right)\right)+\mathbf{f_{i2}}\left(\bar{x}_{\mathbf{i}}\left(\mathbf{t}\right),\theta\right)+\mathbf{f_{i3}^{T}}\left(\bar{x}_{\mathbf{i}}\left(\mathbf{t}\right),\mathbf{x_{i+1}}\left(\mathbf{t}\right)\right)\theta,\quad \mathbf{i=1,\ldots,n-1}$$

$$\dot{x}_{\mathbf{n}}\left(\mathbf{t}\right)=\mathbf{f_{n1}}\left(\bar{x}_{\mathbf{n}}(\mathbf{t}),\mathbf{u}(\mathbf{t})\right)+\mathbf{f_{n2}}\left(\bar{x}_{\mathbf{n}}\left(\mathbf{t}\right),\theta\right)+\mathbf{f_{n3}^{T}}\left(\bar{x}_{\mathbf{n}}\left(\mathbf{t}\right),\mathbf{u}\left(\mathbf{t}\right)\right)\theta, \tag{4}$$

where $\bar{x}_{\mathbf{i}}=\left[\mathbf{x_1},\mathbf{x_2},\ldots,\mathbf{x_i}\right]^{\mathbf{T}}\in\mathbf{R^i}$ and $\mathbf{u}\in\mathbf{R}$ are the system states and control input, respectively. $\theta\in\mathbf{R^s}$ denotes an unknown constant vector. $(\bar{x}_{\mathbf{i}},\theta)\in\mathbf{D}_{\bar{x}_{\mathbf{i}}}\times\mathbf{D}_{\theta}$, $(\bar{x}_{\mathbf{i}},\mathbf{x_{i+1}})\in\mathbf{D}_{\bar{x}_{\mathbf{i+1}}}$, $(\bar{x}_{\mathbf{n}},\mathbf{u})\in\mathbf{D}_{\bar{x}_{\mathbf{n}}}\times\mathbf{D_u}$, $\mathbf{D}_{\bar{x}_{\mathbf{i}}}\in\mathbf{R^i}$, $\mathbf{D}_{\bar{x}_{\mathbf{i+1}}}\in\mathbf{R^{i+1}}$, $\mathbf{D}_{\bar{x}_{\mathbf{n}}}\in\mathbf{R^n}$, $\mathbf{D}_{\theta}\in\mathbf{R^s}$, and $\mathbf{D_u}\in\mathbf{R}$ are domains containing their respective origins. $\mathbf{f_{i1}}:\mathbf{D}_{\bar{x}_{\mathbf{i+1}}}\rightarrow\mathbf{R}$, $\mathbf{f_{i2}}:\mathbf{D}_{\bar{x}_{\mathbf{i}}}\times\mathbf{D}_{\theta}\rightarrow\mathbf{R}$, $\mathbf{f_{i3}}:\mathbf{D}_{\bar{x}_{\mathbf{i+1}}}\rightarrow\mathbf{R^s}$ $\mathbf{f_{n1}}:\mathbf{D}_{\bar{x}_{\mathbf{n}}}\times\mathbf{D_u}\rightarrow\mathbf{R}$, $\mathbf{f_{n2}}:\mathbf{D}_{\bar{x}_{\mathbf{n}}}\times\mathbf{D}_{\theta}\rightarrow\mathbf{R}$, $\mathbf{f_{n3}}:\mathbf{D}_{\bar{x}_{\mathbf{n}}}\rightarrow\mathbf{R^s}$ are continuously differentiable nonlinear functions in their arguments.

Remark 2 *As illustrated above, f_{i3} and f_{n3} can be developed as the function of x_{i+1} and u respectively, which finally leads to a more general expanded class of systems.*

The control objective is to design a control law $\mathbf{u}(\mathbf{t})$ for the system of Eq. (4) such that the origin of the system is asymptotically stable.

Assumption 1 $(\partial\mathbf{f_{i1}}/\partial\mathbf{x_{i+1}})$, $(\partial\mathbf{f_{i3}}/\partial\mathbf{x_{i+1}})$, $(\partial\mathbf{f_{n1}}/\partial\mathbf{u})$, *and* $(\partial\mathbf{f_{n3}}/\partial\mathbf{u})$ *are bounded away from zero for* $\bar{x}_{\mathbf{i+1}}\in\mathbf{\Omega}_{\bar{x}_{\mathbf{i+1}}}\subset\mathbf{D}_{\bar{x}_{\mathbf{i+1}}}$ *and* $(\bar{x}_{\mathbf{n}},\mathbf{u})\in\mathbf{\Omega}_{\bar{x}_{\mathbf{n}},\mathbf{u}}\subset\mathbf{D}_{\bar{x}_{\mathbf{n}}}\times\mathbf{D_u}$, *where* $\mathbf{\Omega}_{\bar{x}_{\mathbf{i+1}}}$ *and* $\mathbf{\Omega}_{\bar{x}_{\mathbf{n}},\mathbf{u}}$ *are compact sets; that is,* $(\partial\mathbf{f_{i1}}/\partial\mathbf{x_{i+1}})$, $(\partial\mathbf{f_{i3}}/\partial\mathbf{x_{i+1}})$, $(\partial\mathbf{f_{n1}}/\partial\mathbf{u})$, *and* $(\partial\mathbf{f_{n3}}/\partial\mathbf{u})$ *are either positive or negative. Without losing the generality, we assume* $\left(\frac{\partial\mathbf{f_{i1}}}{\partial\mathbf{x_{i+1}}}\right)>\mathbf{0}$, $\left(\frac{\partial\mathbf{f_{i3}}}{\partial\mathbf{x_{i+1}}}\right)>\mathbf{0}$, $\left(\frac{\partial\mathbf{f_{n1}}}{\partial\mathbf{u}}\right)>\mathbf{0}$, *and* $\left(\frac{\partial\mathbf{f_{n3}}}{\partial\mathbf{u}}\right)>\mathbf{0}$.

Lemma 1 *For any real-valued continuous function* $\mathbf{f}\left(\mathbf{x},\mathbf{y}\right)$, *where* $\mathbf{x}\in\mathbf{R^m}$, $\mathbf{y}\in\mathbf{R^n}$, *there are smooth scalar functions* $\mathbf{a}(\mathbf{x})\geq\mathbf{0},\mathbf{b}(\mathbf{y})\geq\mathbf{0}$, $\mathbf{c}(\mathbf{x})\geq\mathbf{1}$, *and* $\mathbf{d}(\mathbf{y})\geq\mathbf{1}$, *such that*

$$\left|\mathbf{f}(\mathbf{x},\mathbf{y})\right|\leq\mathbf{a}\left(\mathbf{x}\right)+\mathbf{b}\left(\mathbf{y}\right) \tag{5}$$

$$\left|\mathbf{f}(\mathbf{x},\mathbf{y})\right|\leq\mathbf{c}\left(\mathbf{x}\right)\mathbf{d}\left(\mathbf{y}\right) \tag{6}$$

and a constructive proof is given in [24].

Remark 3 *According to Lemma 1, there exist two smooth functions* $\gamma_{\mathbf{i}}(\bar{x}_{\mathbf{i}})\geq\mathbf{1}$ *and* $\mathbf{\Lambda_i}\left(\theta\right)\geq\mathbf{1}$ *satisfying*

$$\left|\mathbf{f_{i2}}\left(\bar{x}_{\mathbf{i}},\theta\right)\right|\leq\gamma_{\mathbf{i}}\left(\bar{x}_{\mathbf{i}}\right)\mathbf{\Lambda_i}\left(\theta\right),\qquad \mathbf{i=1,\ldots,n}. \tag{7}$$

Let $\mathbf{\Theta}=\sum_{\mathbf{i=1}}^{\mathbf{n}}\mathbf{\Lambda_i}\left(\theta\right)$ *be a new unknown constant. Using Remark 3, it is deduced that*

$$\left|\mathbf{f_{i2}}\left(\bar{x}_{\mathbf{i}},\theta\right)\right|\leq\gamma_{\mathbf{i}}\left(\bar{x}_{\mathbf{i}}\right)\mathbf{\Theta},\qquad \mathbf{i=1,\ldots,n}. \tag{8}$$

In order to overcome the nonlinear parameterization problem, using the above parameter separation method, instead of estimating the unknown parameter $\theta \in \mathbf{R^s}$*, the unknown constant* $\mathbf{\Theta}$*, which appears linearly, is estimated. Moreover, for designing the adaptive controller, in lieu of* $\mathbf{f_{i2}}(.)$ *and* $\mathbf{f_{n2}}(.)$*, the bounding functions* $\gamma_\mathbf{i}(.)$ *and* $\gamma_\mathbf{n}(.)$ *can be used.*

3. Controller design

The adaptive control is developed by the combination of backstepping, singular perturbation theory, the parameter separation method, and the idea of the positive function of linearly connected parameter terms. Similar to the backstepping method, this design procedure contains n steps. Employing the time-scale separation concept, the virtual control laws $\alpha_\mathbf{i}$, $i = 1, \ldots, n-1$ are obtained. At each step, the parameter separation method and the idea of the positive function of linearly connected parameters are applied. Finally, the actual control law u is designed at step n and in addition the update law of unknown parameters is derived. The design procedure is presented in the following. Let us introduce the change of coordinates $\mathbf{z_1}=\mathbf{x_1}$ and $\mathbf{z_i}=\mathbf{x_i}-\alpha_{\mathbf{i-1}}$ where $\mathbf{i} = \mathbf{2}, \ldots, \mathbf{n}$

Step 1. We start with the first equation of Eq. (4) by considering $\mathbf{x_2}$ as the control variable. The derivative of $\mathbf{z_1}$ is given as

$$\dot{z}_\mathbf{1}(\mathbf{t}) = \mathbf{f_{11}}(\mathbf{x_1}, \mathbf{z_2}+\alpha_\mathbf{1}) + \mathbf{f_{12}}(\mathbf{x_1}, \theta) + \mathbf{f_{13}^T}(\mathbf{x_1}, \mathbf{z_2}+\alpha_\mathbf{1})\,\theta \tag{9}$$

Then $\alpha_\mathbf{1}$ as the first virtual controller can be specified as the solution of

$$\mathbf{f_{11}}(\mathbf{x_1}, \mathbf{z_2}+\alpha_\mathbf{1}) + \mathbf{f_{12}}(\mathbf{x_1}, \theta) + \mathbf{f_{13}^T}(\mathbf{x_1}, \mathbf{z_2}+\alpha_\mathbf{1})\,\theta = -\mathbf{k_1 z_1} \tag{10}$$

resulting in the asymptotically stable closed-loop dynamics $\dot{z}_\mathbf{1} = -\mathbf{k_1 z_1}$ for the first subsystem. $\mathbf{k_1} > 0$ is the first control gain. However, in Eq. (10), because of the nonaffine property of the nonlinear functions, $\alpha_\mathbf{1}$ cannot be explicitly computed. According to the following fast dynamics based on the time-scale separation concept, an approximate virtual controller is designed:

$$\varepsilon_\mathbf{1}\dot{\alpha}_\mathbf{1} = -\mathbf{sign}\left(\frac{\partial \mathbf{Q_1}}{\partial \alpha_\mathbf{1}}\right)\mathbf{Q_1}(\bar{z}_\mathbf{2}, \alpha_\mathbf{1}, \hat{\Theta}, \hat{\theta}), \tag{11}$$

with the initial condition $\alpha_\mathbf{1}(\mathbf{0}) = \alpha_{\mathbf{1,0}}$, $\varepsilon_\mathbf{1} \ll \mathbf{1}$, $\bar{z}_\mathbf{2} = [\mathbf{z_1}, \mathbf{z_2}]^\mathbf{T}$

$$\mathbf{Q_1}\left(\bar{z}_\mathbf{2}, \alpha_\mathbf{1}, \hat{\Theta}, \hat{\theta}\right) = \mathbf{k_1 z_1} + \mathbf{f_{11}}(\mathbf{x_1}, \mathbf{z_2}+\alpha_\mathbf{1}) + \mathbf{sat}(\mathbf{z_1}/\mu)\,\gamma_\mathbf{1}(\mathbf{x_1})\,\hat{\Theta} + \mathbf{f_{13}^T}(\mathbf{x_1}, \mathbf{z_2}+\alpha_\mathbf{1})\,\mathbf{\Xi}(\hat{\theta}) \tag{12}$$

where according to Remark 3, $\gamma_\mathbf{1}(\mathbf{z_1})$ is the scalar function, $\hat{\Theta}$ is an estimate of $\mathbf{\Theta}$, $\hat{\theta}$ is an estimate of θ, $\mu \ll \mathbf{1}$ is the constant, and $\mathbf{\Xi}(\hat{\theta}) \in \mathbf{R^s}$ is a positive function of $\hat{\theta}$ whose derivative is bounded away from zero; that is, $(\partial \mathbf{\Xi}(\hat{\theta})/\partial \hat{\theta})$ is either positive or negative.

Let $\alpha_\mathbf{1} = \mathbf{h_1}(\bar{z}_\mathbf{2}, \hat{\Theta}, \hat{\theta})$ be an isolated root of $\mathbf{Q_1}\left(\bar{z}_\mathbf{2}, \alpha_\mathbf{1}, \hat{\Theta}, \hat{\theta}\right) = \mathbf{0}$. Then the reduced system is defined as

$$\dot{z}_\mathbf{1} = -\mathbf{k_1 z_1} + \mathbf{f_{12}}(\mathbf{x_1}, \theta) - \mathbf{sat}(\mathbf{z_1}/\mu)\,\gamma_\mathbf{1}(\mathbf{x_1})\,\hat{\Theta} + \mathbf{f_{13}^T}(\mathbf{x_1}, \mathbf{x_2})\left(\theta - \mathbf{\Xi}\left(\hat{\theta}\right)\right)$$

$$\mathbf{z_1}(\mathbf{0}) = \mathbf{z_{1,0}} \tag{13}$$

and the boundary layer system can be represented by

$$\frac{\mathbf{dy_1}}{\mathbf{d}\tau_1}=-\mathbf{sign}\left(\frac{\partial \mathbf{Q_1}}{\partial \alpha_1}\right)\mathbf{Q_1}\left(\bar{z}_2,\mathbf{y_1}+\mathbf{h_1}\left(\bar{z}_2,\hat{\Theta},\hat{\theta}\right),\hat{\Theta},\hat{\theta}\right) \tag{14}$$

where $\mathbf{y_1}=\alpha_1-\mathbf{h_1}(\bar{z}_2,\hat{\Theta},\hat{\theta})$ and $\tau_1=\mathbf{t}/\varepsilon_1$

Considering the control Lyapunov function $\mathbf{V_1}=\frac{1}{2}\mathbf{z_1^2}$ and using the reduced system of Eq. (13), it is deduced that

$$\dot{V}_1\leq -\mathbf{k_1z_1^2}+|\mathbf{z_1}|\,|\mathbf{f_{12}}\left(\mathbf{x_1},\theta\right)|-\mathbf{z_1sat}\left(\mathbf{z_1}/\mu\right)\gamma_1\left(\mathbf{x_1}\right)\hat{\Theta}+\mathbf{z_1f_{13}^T}\left(\mathbf{x_1},\mathbf{x_2}\right)\left(\theta-\Xi\left(\hat{\theta}\right)\right)$$

$$\leq -\mathbf{k_1z_1^2}+|\mathbf{z_1}|\,\gamma_1\left(\mathbf{x_1}\right)\tilde{\Theta}\;+\;\mathbf{z_1f_{13}^T}\left(\mathbf{x_1},\mathbf{x_2}\right)\tilde{\theta}, \tag{15}$$

where $\tilde{\Theta}=\mathbf{\Theta}-\hat{\Theta}$, $\tilde{\theta}=\theta-\Xi\left(\hat{\theta}\right)$. In the above equation, $\mathbf{sat}(\mathbf{z_1}/\mu)$ for $\mu\ll\mathbf{1}$ is approximated to $\mathbf{sign}(\mathbf{z_1})$.

Step i $(\mathbf{i}=\mathbf{2},\ldots,\mathbf{n}-\mathbf{1})$**.** The derivative of $\mathbf{z_i}$ is expressed as

$$\dot{z}_\mathbf{i}=\mathbf{f_{i1}}\left(\bar{x}_\mathbf{i}(\mathbf{t}),\mathbf{x_{i+1}}(\mathbf{t})\right)+\mathbf{f_{i2}}\left(\bar{x}_\mathbf{i}\left(\mathbf{t}\right),\theta\right)+\mathbf{f_{i3}^T}\left(\bar{x}_\mathbf{i}(\mathbf{t}),\mathbf{x_{i+1}}(\mathbf{t})\right)\theta-\dot{\alpha}_{\mathbf{i-1}} \tag{16}$$

Similar to Step 1, we should find $\alpha_\mathbf{i}$ such that

$$\mathbf{f_{i1}}\left(\bar{x}_\mathbf{i},\mathbf{z_{i+1}}+\alpha_\mathbf{i}\right)+\mathbf{f_{i2}}\left(\bar{x}_\mathbf{i},\theta\right)+\mathbf{f_{i3}^T}\left(\bar{x}_\mathbf{i},\mathbf{z_{i+1}}+\alpha_\mathbf{i}\right)\theta-\dot{\alpha}_{\mathbf{i-1}}=-\mathbf{k_iz_i} \tag{17}$$

where $\mathbf{k_i}>\mathbf{0}$ is the $\mathbf{i}$th positive control gain. In this step, the time derivative of the virtual control input $\dot{\alpha}_{\mathbf{i-1}}$ appears, which has been designed in the previous step, $\dot{\alpha}_{\mathbf{i-1}}=-\mathbf{sign}\left(\frac{\partial \mathbf{Q_{i-1}}}{\partial \alpha_{\mathbf{i-1}}}\right)\mathbf{Q_{i-1}}(\bar{z}_\mathbf{i},,\alpha_{\mathbf{i-1}},\hat{\Theta},\hat{\theta})/\varepsilon_{\mathbf{i-1}}$. Therefore, the "explosion of complexity" arising from the calculation of this term is avoided.

To overcome the nonaffine property, the $\mathbf{i}$th approximate virtual controller can be designed as the following $\mathbf{i}$th fast dynamics:

$$\varepsilon_\mathbf{i}\dot{\alpha}_\mathbf{i}=-\mathbf{sign}\left(\frac{\partial \mathbf{Q_i}}{\partial \alpha_\mathbf{i}}\right)\mathbf{Q_i}\left(\bar{z}_{\mathbf{i+1}},\alpha_\mathbf{i},\hat{\Theta},\hat{\theta}\right) \tag{18}$$

where $\alpha_\mathbf{i}\left(\mathbf{0}\right)=\alpha_{\mathbf{i,0}}$, $\varepsilon_\mathbf{i}\ll\mathbf{1}$, $\bar{z}_{\mathbf{i+1}}=[\mathbf{z_1},\ldots,\mathbf{z_{i+1}}]^\mathbf{T}$, and

$$\mathbf{Q_i}\left(\bar{z}_{\mathbf{i+1}},\alpha_\mathbf{i},\hat{\Theta},\hat{\theta}\right)=\mathbf{k_iz_i}\;+\mathbf{f_{i1}}\left(\bar{x}_\mathbf{i},\mathbf{z_{i+1}}+\alpha_\mathbf{i}\right)+\mathbf{sat}\left(\mathbf{z_i}/\mu\right)\gamma_\mathbf{i}\left(\bar{x}_\mathbf{i}\right)\hat{\Theta}+\left(\bar{x}_\mathbf{i},\mathbf{z_{i+1}}+\alpha_\mathbf{i}\right)\Xi(\hat{\theta})-\dot{\alpha}_{\mathbf{i-1}}. \tag{19}$$

Let $\alpha_\mathbf{i}=\mathbf{h_i}(\bar{z}_{\mathbf{i+1}}\hat{\Theta},\hat{\theta},)$ be an isolated root of $\mathbf{Q_i}\left(\bar{z}_{\mathbf{i+1}},\alpha_\mathbf{i},\hat{\Theta},\hat{\theta}\right)=\mathbf{0}$. Then the reduced system is defined as

$$\dot{z}_\mathbf{i}=-\mathbf{k_iz_i}\;+\mathbf{f_{i2}}\left(\bar{x}_\mathbf{i},\theta\right)-\mathbf{sat}\left(\mathbf{z_i}/\mu\right)\gamma_\mathbf{i}\left(\bar{x}_\mathbf{i}\right)\hat{\Theta}+\mathbf{f_{i3}^T}\left(\bar{x}_\mathbf{i},\mathbf{x_{i+1}}\right)\left(\theta-\Xi\left(\hat{\theta}\right)\right)$$

$$\mathbf{z_i}\left(\mathbf{0}\right)=\mathbf{z_{i,0}} \tag{20}$$

and the boundary layer system can be represented by

$$\frac{\mathbf{dy_i}}{\mathbf{d}\tau_\mathbf{i}}=-\mathbf{sign}\left(\frac{\partial \mathbf{Q_i}}{\partial \alpha_\mathbf{i}}\right)\mathbf{Q_i}\left(\bar{z}_{\mathbf{i+1}},\mathbf{y_i}+\mathbf{h_i}\left(\bar{z}_{\mathbf{i+1}},\hat{\Theta},\hat{\theta}\right),\hat{\Theta},\hat{\theta}\right) \tag{21}$$

where $\mathbf{y_i}=\alpha_\mathbf{i}-\mathbf{h_i}(\bar{z}_{\mathbf{i+1}},\hat{\Theta},\hat{\theta})$ and $\tau_\mathbf{i}=\mathbf{t}/\varepsilon_\mathbf{i}$. Considering the control Lyapunov function $\mathbf{V_i}=\mathbf{V_{i-1}}+\frac{\mathbf{1}}{\mathbf{2}}\mathbf{z_i^2}$ and using the reduced system of Eq. (20), it is deduced that

$$\dot{V}_\mathbf{i}\leq\left(\sum_{\mathbf{j=1}}^{\mathbf{i-1}}-\mathbf{k_j z_j^2}+|\mathbf{z_j}|\,\gamma_\mathbf{j}\left(\bar{x}_\mathbf{j}\right)\tilde{\Theta}+\mathbf{z_j f_{j3}^T}\left(\bar{x}_\mathbf{j},\mathbf{x_{j+1}}\right)\tilde{\theta}\right)-\mathbf{k_i z_i^2}+|\mathbf{z_i}|\,|\mathbf{f_{i2}}\left(\bar{x}_\mathbf{i},\theta\right)|$$

$$-\mathbf{z_i sat}\left(\mathbf{z_i}/\mu\right)\gamma_\mathbf{i}\left(\bar{x}_\mathbf{i}\right)\hat{\Theta}+\mathbf{z_i f_{i3}^T}\left(\bar{x}_\mathbf{i},\mathbf{x_{i+1}}\right)\left(\theta-\mathbf{\Xi}\left(\hat{\theta}\right)\right)$$

$$\leq\sum_{\mathbf{j=1}}^{\mathbf{i}}-\mathbf{k_j z_j^2}+|\mathbf{z_j}|\,\gamma_\mathbf{j}\left(\bar{x}_\mathbf{j}\right)\tilde{\Theta}+\mathbf{z_j f_{j3}^T}\left(\bar{x}_\mathbf{j},\mathbf{x_{j+1}}\right)\tilde{\theta} \tag{22}$$

Step n. In the last step, the actual control input $\mathbf{u}$ appears and is at our disposal. We derive the $\mathbf{z_n}$ dynamics:

$$\dot{z}_\mathbf{n}=\mathbf{f_{n1}}\left(\bar{x}_\mathbf{n},\mathbf{u}\right)+\mathbf{f_{n2}}\left(\bar{x}_\mathbf{n},\theta\right)+\mathbf{f_{n3}^T}\left(\bar{x}_\mathbf{n},\mathbf{u}\right)\theta-\dot{\alpha}_\mathbf{n-1}, \tag{23}$$

And we now obtain an approximate actual control input via time-scale separation to satisfy

$$\mathbf{f_{n1}}\left(\bar{x}_\mathbf{n},\mathbf{u}\right)+\mathbf{f_{n2}}\left(\bar{x}_\mathbf{n},\theta\right)+\mathbf{f_{n3}^T}\left(\bar{x}_\mathbf{n},\mathbf{u}\right)\theta-\dot{\alpha}_\mathbf{n-1}=-\mathbf{k_n z_n}, \tag{24}$$

as

$$\varepsilon_\mathbf{n}\dot{u}=-\mathbf{sign}\left(\frac{\partial\mathbf{Q_n}}{\partial\mathbf{u}}\right)\mathbf{Q_n}(\bar{z}_\mathbf{n},\mathbf{u},\hat{\Theta},\hat{\theta}), \tag{25}$$

with the initial condition $\mathbf{u}\left(\mathbf{0}\right)=\mathbf{u_0}$, $\varepsilon_\mathbf{n}\ll\mathbf{1}$ and

$$\mathbf{Q_n}\left(\bar{z}_\mathbf{n},\mathbf{u},\hat{\Theta},\hat{\theta}\right)=\mathbf{k_n z_n}+\mathbf{f_{n1}}\left(\bar{x}_\mathbf{n},\mathbf{u}\right)+\mathbf{sat}\left(\mathbf{z_n}/\mu\right)\gamma_\mathbf{n}\left(\bar{x}_\mathbf{n}\right)\hat{\Theta}+\mathbf{f_{n3}^T}\left(\bar{x}_\mathbf{n},\mathbf{u}\right)\mathbf{\Xi}(\hat{\theta})-\dot{\alpha}_\mathbf{n-1} \tag{26}$$

$\bar{z}_\mathbf{n}=[\mathbf{z_1},\ldots,\mathbf{z_n}]^\mathbf{T}$. Here, $\mathbf{k_n}$ is the nth control gain.

Let $\mathbf{u}=\mathbf{h_n}(\bar{z}_\mathbf{n},\hat{\Theta},\hat{\theta})$ be an isolated root of $\mathbf{Q_n}(\bar{z}_\mathbf{n},\mathbf{u},\hat{\Theta},\hat{\theta})=\mathbf{0}$. Then the reduced system is defined as

$$\dot{z}_\mathbf{n}=-\mathbf{k_n z_n}+\mathbf{f_{n2}}\left(\bar{x}_\mathbf{n},\theta\right)-\mathbf{sat}\left(\mathbf{z_n}/\mu\right)\gamma_\mathbf{n}\left(\bar{x}_\mathbf{n}\right)\hat{\Theta}+\mathbf{f_{n3}^T}\left(\bar{x}_\mathbf{n},\mathbf{u}\right)\left(\theta-\mathbf{\Xi}\left(\hat{\theta}\right)\right)$$

$$\mathbf{z_n}\left(\mathbf{0}\right)=\mathbf{z_{n,0}}, \tag{27}$$

and the boundary layer system can be represented by

$$\frac{\mathbf{dy_n}}{\mathbf{d}\tau_\mathbf{n}}=-\mathbf{sign}\left(\frac{\partial\mathbf{Q_n}}{\partial\mathbf{u}}\right)\mathbf{Q_n}\left(\bar{z}_\mathbf{n},\mathbf{y_n}+\mathbf{h_n}\left(\bar{z}_\mathbf{n},\hat{\Theta},\hat{\theta}\right),\hat{\Theta},\hat{\theta}\right) \tag{28}$$

where $\mathbf{y_n}=\mathbf{u}-\mathbf{h_n}(\bar{z}_\mathbf{n},\hat{\Theta},\hat{\theta})$ and $\tau_\mathbf{n}=\mathbf{t}/\varepsilon_\mathbf{n}$. We choose the Lyapunov function $\mathbf{V_n}=\mathbf{V_{n-1}}+\frac{\mathbf{1}}{\mathbf{2}}\mathbf{z_n^2}+\frac{\mathbf{1}}{\mathbf{2}}\tilde{\Theta}^\mathbf{2}+\frac{\mathbf{1}}{\mathbf{2}}\tilde{\theta}^\mathbf{T}\mathbf{\Gamma}^{-\mathbf{1}}\tilde{\theta}$, where $\mathbf{\Gamma}$ is a positive definite matrix. The resulting derivatives of $\mathbf{V_n}$ is given as

$$\dot{V}_\mathbf{n}\leq\left(\sum_{\mathbf{j=1}}^{\mathbf{n-1}}-\mathbf{k_j z_j^2}+|\mathbf{z_j}|\,\gamma_\mathbf{j}\left(\bar{x}_\mathbf{j}\right)\tilde{\Theta}+\mathbf{z_j f_{j3}^T}(\bar{x}_\mathbf{j},\mathbf{x_{j+1}})\tilde{\theta}\right)-\mathbf{k_n z_n^2}+|\mathbf{z_n}|\,|\mathbf{f_{n2}}\left(\bar{x}_\mathbf{n},\theta\right)|-$$

$$\mathbf{z_n sat}\left(\mathbf{z_n}/\mu\right)\gamma_{\mathbf{n}}\left(\bar{x}_{\mathbf{n}}\right)\hat{\Theta}+\mathbf{z_n f_{n3}^{T}}\left(\bar{x}_{\mathbf{n}},\mathbf{u}\right)\tilde{\theta}-\tilde{\Theta}\dot{\hat{\Theta}}-\tilde{\theta}^{\mathbf{T}}\mathbf{\Gamma^{-1}}\dot{\hat{\theta}}*^{3}\frac{\partial\mathbf{\Xi}\left(\hat{\theta}\right)}{\partial\hat{\theta}}$$

$$\leq\left(\sum_{\mathbf{j=1}}^{\mathbf{n}}-\mathbf{k_j z_j^2}\right)+\left(\sum_{\mathbf{j=1}}^{\mathbf{n}}\left|\mathbf{z_j}\right|\gamma_{\mathbf{j}}\left(\bar{x}_{\mathbf{j}}\right)-\dot{\hat{\Theta}}\right)\tilde{\Theta}-$$

$$\tilde{\theta}^{\mathbf{T}}[\mathbf{\Gamma^{-1}}\dot{\hat{\theta}}*^{3}\frac{\partial\mathbf{\Xi}\left(\hat{\theta}\right)}{\partial\hat{\theta}}-\sum_{\mathbf{j=1}}^{\mathbf{n-1}}\mathbf{f_{j3}}(\bar{x}_{\mathbf{j}},\mathbf{x_{j+1}})\mathbf{z_j}-\mathbf{f_{n3}}\left(\bar{x}_{\mathbf{n}},\mathbf{u}\right)\mathbf{z_n}]. \tag{29}$$

Finally, we can eliminate the $\tilde{\Theta}$ and $\tilde{\theta}^{\mathbf{T}}$ terms from Eq. (29) by designing the adaptation law as

$$\dot{\hat{\Theta}}=\sum_{\mathbf{j=1}}^{\mathbf{n}}\left|\mathbf{z_j}\right|\gamma_{\mathbf{j}}\left(\bar{x}_{\mathbf{j}}\right), \tag{30}$$

$$\dot{\hat{\theta}}=\Gamma[\sum_{j=1}^{n-1}f_{j3}(\bar{x}_j,x_{j+1})z_j+f_{n3}\left(\bar{x}_n,u\right)z_n]\Big/{}^{4}\frac{\partial\Xi(\hat{\theta})}{\partial\hat{\theta}}. \tag{31}$$

Therefore, the derivative of $\mathbf{V_n}$ is

$$\dot{V}_{\mathbf{n}}\leq\sum_{\mathbf{j=1}}^{\mathbf{n}}-\mathbf{k_j z_j^2}. \tag{32}$$

By using LaSalle's theorem, this Lyapunov function guarantees the asymptotic stability of the origin of the reduced system of Eqs. (13), (20), and (27).

Remark 4 *The unknown parameters for each subsystem of the system of Eq. (4) are considered as a constant vector; that is, $\theta\in\mathbf{R^s}$, while it was considered as a constant scalar for each subsystem in [18].*

Remark 5 *In [18], first a state predictor is developed for deriving adaptive laws of unknown parameters and then the combination of time-scale separation and backstepping is applied to this state estimator for obtaining the virtual/actual controls. In this paper, designing the state predictor is excluded and a combination of time-scale separation and backstepping is applied on the original system for deriving the virtual/actual controls.*

For the stability analysis of this proposed control system, we present the following theorem using Theorem 1.

Theorem 2 *Consider the singular perturbation problem of the pure-feedback system of Eq. (4) and the controllers of Eqs. (11), (18), and (25). Assume that the following conditions are satisfied for all $(\bar{z}_{\mathbf{i+1}},\alpha_{\mathbf{i}}-\mathbf{h_i}(\bar{z}_{\mathbf{i+1}},\hat{\Theta},\hat{\theta}))\in\mathbf{D}_{\bar{z}_{\mathbf{i+1}}}\times\mathbf{D}_{\mathbf{y_i}}$ for some domains $\mathbf{D}_{\bar{z}_{\mathbf{i+1}}}\subset\mathbf{R^{i+1}}$ and $\mathbf{D}_{\mathbf{y_i}}\subset\mathbf{R}$, which contain their respective origins, where $\mathbf{i}=\mathbf{1},\ldots,\mathbf{n}$, $\bar{z}_{\mathbf{n+1}}=\bar{z}_{\mathbf{n}}$, $\mathbf{D}_{\bar{z}_{\mathbf{n+1}}}=\mathbf{D}_{\bar{z}_{\mathbf{n}}}$, and $\alpha_{\mathbf{n}}=\mathbf{u}$.*

[3] This product is defined as the element-by-element multiplication of array $\dot{\hat{\theta}}$ by $\frac{\partial\Xi(\hat{\theta})}{\partial\hat{\theta}}$.

[4] This calculation is defined as the element-by-element division of array $\Gamma[\sum_{j=1}^{n-1}f_{j3}\left(\bar{x}_j,x_{j+1}\right)z_j+f_{n3}\left(\bar{x}_n,u\right))z_n]$ by $\frac{\partial\Xi(\hat{\theta})}{\partial\hat{\theta}}$.

B1) $\mathbf{f_{i1}}(\mathbf{0},\mathbf{0})=\mathbf{0}, \mathbf{f_{i2}}(\mathbf{0},\theta)=\mathbf{0}, \mathbf{f_{i3}}(\mathbf{0},\mathbf{0})=\mathbf{0}, \mathbf{Q_i}\left(\mathbf{0},\ \mathbf{0},\hat{\Theta},\hat{\theta}\right)=\mathbf{0}$

B2) On any compact subset of $\mathbf{D}_{\bar{z}_{i+1}}\times\mathbf{D}_{\mathbf{y_i}}$*, the equation* $\mathbf{0}=\mathbf{Q_i}(\bar{z}_{\mathbf{i+1}},\alpha_{\mathbf{i}},\hat{\Theta},\hat{\theta})$ *has an isolated root* $\alpha_{\mathbf{i}}=\mathbf{h_i}(\bar{z}_{\mathbf{i+1}},\hat{\Theta},\hat{\theta})$ *such that* $\mathbf{h_i}(\mathbf{0},\hat{\Theta},\hat{\theta})=\mathbf{0}$

B3) The functions $\mathbf{Q_i h_i}$ *and their first partial derivatives with respect to their arguments are bounded.*

B4) $(\bar{z}_{\mathbf{i+1}},\mathbf{y_i})\mapsto(\partial\mathbf{Q_i}/\partial\alpha_{\mathbf{i}})(\bar{z}_{\mathbf{i+1}}\mathbf{y_i}+\mathbf{h_i}(\bar{z}_{\mathbf{i+1}},\hat{\Theta},\hat{\theta}),\hat{\Theta},\hat{\theta})$ *is bounded below by some positive constant for all* $\bar{z}_{\mathbf{i+1}}\in\mathbf{D}_{\bar{z}_{i+1}}$*. Therefore, the origins of Eqs. (14), (21), and (28) are exponentially stable.*

Then there exists a positive constant ε^* *such that for all* $\varepsilon<\varepsilon^*$*, the origin of Eq. (4) is asymptotically stable.*

Proof For using Theorem 1, it should be verified that the conditions in Theorem 2 satisfy assumptions (A1)–(A5). First, assumptions (B1)–(B3) directly imply that assumptions (A1) and (A3) hold, respectively. Second, we show from assumption 1 that assumption (A4) holds. The exponential stability of the boundary layer system of Eqs. (14), (21), and (28) can be easily obtained locally by linearization with respect to $\mathbf{y_i}$. Using assumption 1 and (B4) yields

$$\mathbf{sign}\left(\frac{\partial\mathbf{Q_i}}{\partial\alpha_{\mathbf{i}}}\right)=\mathbf{sign}\left(\frac{\partial\mathbf{f_{i1}}}{\partial\alpha_{\mathbf{i}}}+\frac{\partial(\mathbf{f_{i3}^T})}{\partial\alpha_{\mathbf{i}}}\mathbf{\Xi}(\hat{\theta})\right)>\mathbf{0}. \tag{33}$$

This confirms that the boundary layer system has a locally exponentially stable origin.

Finally, since in the previous section we showed the asymptotic stability of the origin of the reduced system of Eqs. (13), (20), and (27), by considering Remark 1, assumption $(\mathbf{A5})'$ is satisfied.

According to Theorem 1 and Remark 1, there exists a constant $\varepsilon_{\mathbf{i}}^*>\mathbf{0}$ such that for $\mathbf{0}<\varepsilon<\varepsilon^*$, the origins of the systems of Eqs. (9), (11), (16), (18), (23), and (25) are asymptotically stable. It follows that $\mathbf{z_i}\to\mathbf{0}$ and $\alpha_{\mathbf{i}}\to\mathbf{0}$ as $\mathbf{t}\to\infty$. Since $\mathbf{x_1}=\mathbf{z_1}$ and $\mathbf{x_i}=\mathbf{z_i}+\alpha_{\mathbf{i-1}}$, it can be concluded that the origin of the pure-feedback system of Eq. (4) is asymptotically stable. □

Remark 6 *The idea of using positive function* $\mathbf{\Xi}(\hat{\theta})$ *instead of* $\hat{\theta}$ *is just for satisfying Eq. (33).*

4. Simulation results

Consider the following pure-feedback system:

$$\dot{x}_1 = x_2+\frac{x_2^3}{5}+\ln\left(1+(\theta_1x_1)^2\right)+\theta_1e^{x_2}+\theta_2x_1^2,$$

$$\dot{x}_2 = 0.01u+\frac{u^3}{7}+x_2^2u+\frac{\theta_2x_1^2}{(1+\theta_1x_2)^2+x_2^2}+\theta_1x_1x_2+\theta_2u^3. \tag{34}$$

θ_1 and θ_2 are unknown linearly/nonlinearly connected parameters. The control object is to design an adaptive control law $u(t)$ such that the origin of the system is asymptotically stable. To obtain the bounding function of the nonlinearly connected parameter term, a direct calculation gives $\left|\frac{\theta_2x_1^2}{(1+\theta_1x_2)^2+x_2^2}\right|\le|\theta_2|(1+\theta_1^2)x_1^2$. On the other hand, by the mean value theorem, $\left|\ln\left(1+(\theta_1x_1)^2\right)\right|\le|x_1||\theta_1|$. We define $\gamma_1(x_1)=|x_1|$, $\gamma_2(x_1,x_2)=x_1^2$, and $\Theta=|\theta_1|+|\theta_2|(1+\theta_1^2)$.

In order to show the effectiveness of the proposed approach, the simulation results are evaluated for different parameter values in two cases.

Case 1: Here, the unknown parameters are taken as $\theta_1 = 1.6$ and $\theta_2 = 2.5$. The initial conditions are set to $x_1(0) = 0.5$, $x_2(0) = -1$, $u(0) = 0$, $\hat{\Theta}(0) = 1, \hat{\theta}_1(0) = 2$, $\hat{\theta}_2(0) = 3$. The design parameters for the proposed control system are adopted as follows: $k_1 = k_1 = 2$, $\mu = 0.00001$, $\varepsilon_1 = \varepsilon_2 = 0.01$. Furthermore, $\Gamma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\Xi\left(\hat{\theta}_1\right) = 1 + \tanh\left(\hat{\theta}_1\right)$, $\Xi\left(\hat{\theta}_2\right) = 1 + \tanh\left(\hat{\theta}_2\right)$ are chosen. The adaptation laws are designed as:

$$\dot{\hat{\Theta}} = |z_1|\ \gamma_1(x_1) + |z_2|\ \gamma_2(x_1, x_2) \tag{35}$$

$$\dot{\hat{\theta}}_1 = (z_1 e^{x_2} + z_2 x_1 x_2)/(1 - tanh^2(\hat{\theta}_1)), \tag{36}$$

$$\dot{\hat{\theta}}_2 = (z_1 x_1^2 + z_2 u^3)/(1 - tanh^2(\hat{\theta}_2)), \tag{37}$$

where Q_1 and Q_2 are defined in Eqs. (12) and (19), respectively.

Case 2: In this case, the unknown parameters, initial conditions, and positive functions parameters are chosen as: $\theta_1 = 1$, $\theta_2 = 0.5$, $x_1(0) = 2$, $x_2(0) = -2$, $u(0) = 0$, $\hat{\Theta}(0) = 0.1\hat{\theta}_1(0) = 0.1$, $\hat{\theta}_2(0) = 0$, $\Xi\left(\hat{\theta}_1\right) = \exp\left(\hat{\theta}_1\right)$, $\Xi\left(\hat{\theta}_2\right) = \exp\left(\hat{\theta}_2\right)$ and the other design parameters are similar to case 1.

The simulation results in Figures 1 and 2 show transient response of states, actual control effort $u(t)$, and adaptive parameters for case 1. Furthermore, transient response of states, control effort, and adaptive parameters for case 2 are illustrated in Figures 3 and 4. These figures indicate that, despite the linear and nonlinear parameterization and the nonaffine property of this system, the stability of the closed-loop system as well as asymptotic state regulation, with satisfactory dynamic performance of the system states and control input, can be achieved by the proposed approach.

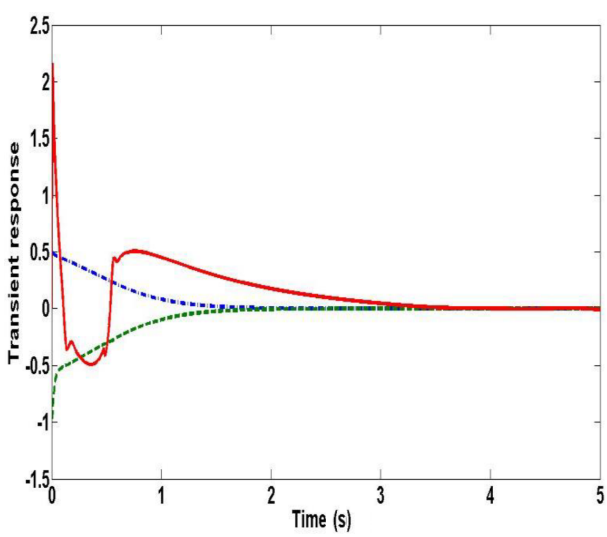

Figure 1. Transient response of the states x_1 (dashed-dotted line) and x_2 (dashed line), and control input u (solid line).

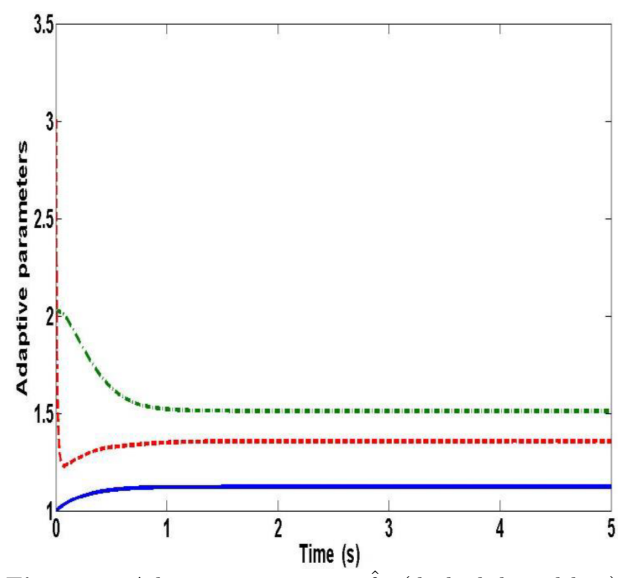

Figure 2. Adaptive parameters $\hat{\theta}_1$ (dashed-dotted line), $\hat{\theta}_2$ (dashed line), and $\hat{\Theta}$ (solid line).

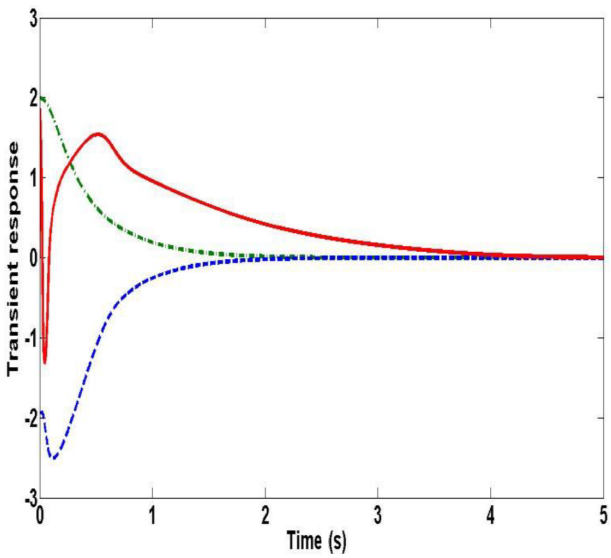

Figure 3. Transient response of the states x_1 (dashed-dotted line) and x_2 (dashed line), and control input u (solid line).

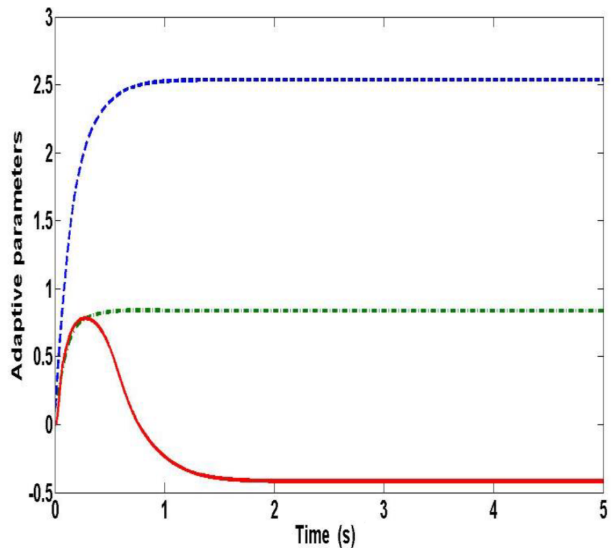

Figure 4. Adaptive parameters $\hat{\theta}_1$ (dashed-dotted line), $\hat{\theta}_2$ (solid line), and $\hat{\Theta}$ (dashed line).

5. Conclusion

In this paper, an adaptive control method has been developed for completely nonaffine pure-feedback systems in the presence of parametric uncertainties. These uncertainties include nonlinear parameterization, which has been known as a challenging problem in the field of nonlinear adaptive control, and linear parameterization, which makes the class of systems much more general. By combination of backstepping and the singular perturbation concept, and coupling it effectively with the parameter separation technique and taking into account the idea of the positive function of linear connected parameters, virtual/actual control inputs as well as the adaptation law of unknown parameters have been derived. The proposed control approach can overcome the linear/nonlinear parameterization and nonaffine property. The stability proof is carried out by presenting the theorem in singular perturbation theory and the asymptotic stability of the origin of these systems is achieved. The simulation results show that the proposed approach works reasonably well.

References

[1] Kokotović P, Arcak M. Constructive nonlinear control: a historical perspective. Automatica 2001; 37: 637-662.

[2] Hunt LR, Meyer G. Stable inversion for nonlinear systems. Automatica 1997; 33: 1549-1554.

[3] Ge SS, Zhang J. Neural-network control of non-affine nonlinear system with zero dynamics by state and output feedback. IEEE T Neural Networ 2003; 14: 900-918.

[4] Yang BJ, Calise AJ. Adaptive control of a class of non-affine systems using neural networks. IEEE T Neural Networ 2007; 18: 1149-1159.

[5] Labiod S, Guerra TM. Adaptive fuzzy control of a class of SISO non-affine nonlinear systems. Fuzzy Set Syst 2007; 158: 1126-1137.

[6] Park JH, Park GT, Kim SH, Moon CJ. Direct adaptive self-structuring fuzzy controller for non-affine nonlinear system. Fuzzy Set Syst 2005; 153: 429-445.

[7] Labiod S, Guerra TM. Indirect adaptive fuzzy control for a class of non-affine nonlinear systems with unknown control directions. Int J Control Autom 2010; 8: 903-907.

[8] Park JY, Park GT. Robust adaptive fuzzy controller for non-affine nonlinear systems with dynamic rule activation. Int J Robust Nonlin 2003; 13: 117-139.

[9] Hovakimyan N, Lavretsky E, Cao CH. Adaptive dynamic inversion via time-scale separation. IEEE T Neural Networ 2008; 19: 1702-1711.

[10] Lavretsky E, Hovakimyan N. Adaptive dynamic inversion for non-affine-in-control uncertain systems via time-scale separation. Part II. J Dyn Control Syst 2008; 14: 33-41.

[11] Krstic M, Kokotovic PV, Kanellakopoulos I. Nonlinear and Adaptive Control Design. New York, NY, USA: Wiley, 1995.

[12] Ferrara A, Giacomini L. Control of a class of mechanical systems with uncertainties via a constructive adaptive/second order VSC approach. J Dyn Syst-T ASME 2000; 122: 33-39.

[13] Kokotovic PV. The joy of feedback: nonlinear and adaptive. IEEE Contr Syst Mag 1992; 12: 7-17.

[14] Ge SS, Wang C. Adaptive NN control of uncertain nonlinear pure-feedback systems. Automatica 2002; 38: 671-682.

[15] Wang D, Huang G. Adaptive neural network control for a class of uncertain nonlinear systems in pure-feedback from. Automatica 2002; 38: 1365-1372.

[16] Wang C, Hill DJ, Ge SS, Chen GR. An ISS-modular approach for adaptive neural control of pure-feedback systems. Automatica 2000; 42: 723-731.

[17] Gao D, Sun Z, Xu B. Fuzzy adaptive control for pure-feedback system via time scale separation. Int J Control Autom 2013; 11: 147-158.

[18] Yoo SJ. Adaptive control of non-linearly parameterized pure-feedback systems. IET Control Theory A 2012; 6: 467-473.

[19] Yu J. Adaptive fuzzy stabilization for a class of pure-feedbacks with unknown dead-zones. Int J Fuzzy Syst 2013; 15: 289-296.

[20] Li X, Wang D, Li T, Peng ZH, Sun G, Wang N. Adaptive NN control of uncertain non-affine pure-feedback systems with unknown time-delay. In: American Control Conference; 29 June–1 July 2011. New York, NY, USA: IEEE, 2011.

[21] Ren B, Ge SS, Su CY, Lee TH. Adaptive neural control for a class of uncertain nonlinear systems in pure-feedback form with hysteresis input. IEEE T Syst Man Cy B 2009; 39: 431-443.

[22] Kim B, Yoo S. Adaptive control of nonlinear pure-feedback systems with output constraints: integral barrier Lyapunov functional approach. Int J Control Autom 2015; 13: 249-256.

[23] Khalil HK. Nonlinear systems. Upper Saddle River, NJ, USA: Prentice Hall, 1996.

[24] Lin W, Qian CJ. Adaptive control of non-linearly parameterized systems: the smooth feedback case. IEEE T Automat Contr 2002; 47: 1249-1266.