Discrete-time based sliding-mode control of robot manipulators

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Abstract

Purpose – The purpose of this paper is to develop sliding mode control with linear and nonlinear manifolds in discrete-time domain for robot manipulators.

Design/methodology/approach – First, a discrete linear sliding mode controller is designed to an n-link robot based on Gao's reaching law. In the second step, a discrete terminal sliding mode controller is developed to design a finite time and high precision controller. The stability analysis of both controllers is presented in the presence of model uncertainties and external disturbances. Finally, sampling time effects on the continuous-time system outputs and sliding surfaces are discussed.

Findings – Computer simulations on a three-link SCARA robot show that the proposed controllers are robust against model uncertainties and external disturbance. It was also shown that the sampling time has important effects on the closed loop system stability and convergence.

Practical implications – The proposed controllers are low cost and easily implemented in practice in comparison with continuous-time ones.

Originality/value – The novelty associated with this paper is the development of an approach to finite time and robust control of n-link robot manipulators in discrete-time domain. Also, obtaining an upper bound for the sampling time is another contribution of this work.

Keywords Sampling time, Discrete-time sliding mode, Terminal sliding mode, Linear and nonlinear manifolds, N-link robot manipulator, Robots, Control technology, Control systems

Paper type Research paper

1. Introduction

Sliding-mode control (SMC) is a particular type of variable structure control system that proposed by Emelyanov (1967) in the Soviet Union. The SMC is one of most powerful tools to overcome the control problems of nonlinear dynamical systems (Young *et al.*, 1999). This method is well known due to its high precision and robustness against model uncertainties, parameter variations and external disturbances (Yu and Kaynak, 2009). Thus, it has been widely used in variety fields such as robotic systems (Erbatur *et al.*, 1999; Islam and Liu, 2011), power converters (Tan *et al.*, 2008) and suspension systems (Yagiz *et al.*, 2008).

The SMC can be designed in two types: a linear sliding surface that is known as linear sliding-mode control (LSMC), and a nonlinear sliding manifold that is called as terminal sliding-mode control (TSMC). The TSMC have been proposed in Hui and Li (2009), Chang *et al.* (2008), Zhihong and O'Day (1999), Feng *et al.* (2002), Yu *et al.* (2005) and Jin *et al.* (2009), to bring the system states to the equilibrium point on the sliding surface in a finite time. This technique has relatively fast transient response in comparison with LSMC. However, this method has the singularity problem for some of nonlinear systems. Robot manipulator is one of these nonlinear systems. In Feng *et al.* (2002), Yu *et al.* (2005) and Jin *et al.* (2009), some terminal sliding-mode controllers have been proposed to overcome the singularity problem of robot manipulators.

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On the other hand, discrete-time sliding-mode control (DSMC) has received a lot of attentions due to technological advances in digital electronics and computer control (Wang, 2008; Gao *et al.*, 1995; Furuta, 1990; Spurgeon, 1992; Sarpturk *et al.*, 1987; Bartoszewicz, 1998; Monses, 2002; Bandyopadhyay, 2006). In general, DSMC have many advantages over SMC. Some of the advantages are: low power consumption, low cost, low weight, high accuracy and ease of making software and design changes (Wang, 2008). In DSMC, the control signal is constant over the entire sampling period and is changed only at each sampling instant. As a result, some properties of continuous-time sliding-mode controller are lost by discretization. To enhance the performance of DSMC, various methods have been proposed in Gao *et al.* (1995), Furuta (1990), Spurgeon (1992), Sarpturk *et al.* (1987) and Bartoszewicz (1998). Among these literatures, the proposed reaching law in Gao *et al.* (1995) and stability conditions in Sarpturk *et al.* (1987) are common in comparison with others. Same as continuous-time one, DSMC can be designed in discrete-time linear sliding-mode control (DLSMC) and discrete-time terminal sliding-mode control (DTSMC) forms. In Janardhanan and Bandyopadhyay (2006), Bandyopadhyay and Fulwani (2009) and Abidi *et al.* (2009), DTSMC is applied to reach finite time convergence.

In robotic control, DLSMC has been used to control a mobile robot in Corradini and Orlando (2002) and Corradini *et al.* (2002). In Sun *et al.* (2007, 2002), the DLSMC and Neuro-Fuzzy-DLSMC controllers are presented to control a rigid link robot manipulator. To our knowledge, there is no top literature in finite time control of robot manipulator in discrete-time. Therefore, extra work is needed in this domain.

This paper presents the following main contributions:

- DTSMC is implemented to control of robot manipulators for first time;
- stability of the closed loop system is proved for DLSMC and DTSMC in the presence of imperfect modeling and external disturbances;
- upper bound of sampling time is derived for the DLSMC and DTSMC;
- sampling time effects on the system outputs and sliding surfaces convergence are presented for DLSMC and DTSMC; and
- the proposed controller is non-singular due to the non-derivative computations.

The reminder of this paper is organized as follows. An n-link robot manipulator and equivalent discretized model are presented in Section 2. In Section 3, the DLSMC is designed to the discretized model. The DTSMC is developed in Section 4. Simulation results of a three-link SCARA robot and conclusions are given in Sections 5 and 6, respectively.

2. Dynamics of an n-link robot and model discretization

The dynamic model of an n-link robot manipulator can be expressed in the following Lagrange form as (Schilling, 2003; Wai *et al.*, 2004):

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) + F(\dot{q}) + u_d = u \tag{1}$$

where $D(q) \in R^{n\times n}$ is the inertia matrix which is symmetric and positive definite, $C(q,\dot{q})\dot{q} \in R^{n\times 1}$ is the vector of Coriolis and centripetal forces, $G(q)_{n\times 1}$ is the gravitational torque, $F(\dot{q})$ is the friction force vector, u_d is the disturbance torque vector ($|u_d| < U_D$, $U_D > 0$), $q = \begin{bmatrix} q_1 & \cdots & q_n \end{bmatrix}^T$ is the joint position vector and $u = \begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix}^T$ is the motor torque vector. The friction term in equation (1),

is a highly nonlinear force that is difficult to model accurately. In many cases it can have direct effects on robot arm dynamics. The frictional force model can be written as follows (Schilling, 2003):

$$F_i(\dot{q}) = F_i^v \dot{q}_k + \mathrm{sgn}(\dot{q}_i)\left[F_i^d + \left(F_i^s - F_i^d\right)\exp\frac{-|\dot{q}_i|}{\varepsilon}\right] \tag{2}$$

where for joint i, F_i^v is the coefficient of the viscous friction, F_i^d is the coefficient of dynamic friction and F_i^s is the coefficient of static friction and ε is a small positive constant. The dynamic and static frictions are discontinuous and nonlinear while the viscous friction is a linear function of $\dot{q}_i$.

By defining the joint position and joint velocity vectors of n-link robot:

$$x_{2N-1}(t) = [x_1(t), x_3(t), \ldots, x_{2n-1}(t)]^T = [q_1, q_2, \ldots, q_n]^T \tag{3}$$

$$x_{2N}(t) = [x_2(t), x_4(t), \ldots, x_{2n}(t)]^T = [\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n]^T \tag{4}$$

the state-space representation of equation (1) can be expressed in the following form:

$$\begin{cases} \dot{x}_{2N-1}(t) = x_{2N}(t) \\ \dot{x}_{2N}(t) = D^{-1}(x_{2N-1})[u - C(x_{2N-1}, x_{2N})x_{2N} - G(x_{2N-1}) - F(x_{2N}) - u_d] \end{cases} \tag{5}$$

The continuous system model (5) should be transferred to a discrete model, in order to design a discrete control law. Hence, using the Euler approximation under the assumption of a sufficiently small sampling period (Lincolny and Veresyz, 2010):

$$\begin{cases} x_{2N-1}(k+1) = x_{2N-1}(k) + Tx_{2N}(k) \\ x_{2N}(k+1) = x_{2N}(k) + TD^{-1}(x_{2N-1})[u - C(x_{2N-1}, x_{2N})x_{2N} - G(x_{2N-1}) \\ -F(x_{2N}) - u_d] \end{cases} \tag{6}$$

where T is the sampling rate, $x_{2N-1}(k)$ is the joint position vector and $x_{2N}(k)$ is the joint velocity vector in the discrete-time domain.

Remark 1. The discretization model of the n-link robot in equation (6) is an approximation model for the original uncertain model in equation (5). In order to have an approximately similar behavior between the discrete-time (6) and original continuous-time (5) systems, the sampling rate should be sufficiently fast in comparison with system dynamics.

3. Discrete LSMC

A possible block scheme of a discrete-time computer controlling a continuous-time system is sketched in Figure 1. In this figure, the controller block is composed out of three parts (Monses, 2002):

(1) a digital central block;

(2) an analog to digital convertor; and

(3) a continuous to discrete-time signal convertor.

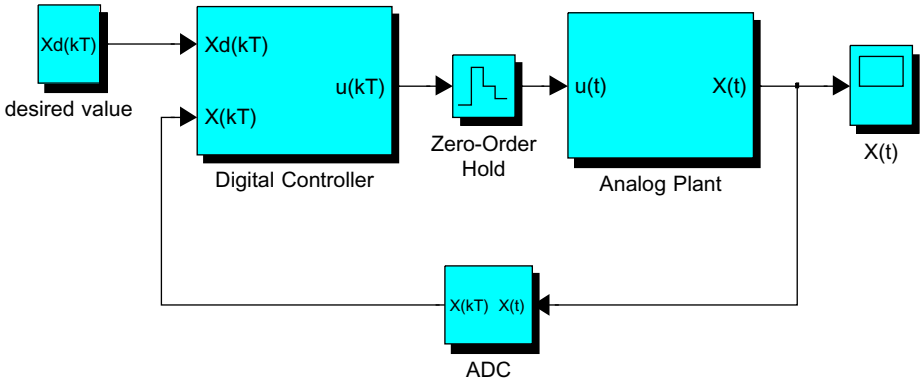


Figure 1. Closed loop system composed of a discrete-time controller and a continuous-time plant

In order to design the digital central block, consider the linear discrete sliding surface in the following form:

$$s(k) = e_{2N}(k) + \lambda e_{2N-1}(k) \tag{7}$$

where $\lambda = diag\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ is a positive constant diagonal matrix. $e_{2N-1}(k)$ and $e_{2N}(k)$ are the vectors of position and velocity error, respectively. The sliding surface vector $s(k)$ is in the form of:

$$s(k) = [s_1(k), s_2(k), \ldots, s_n(k)]^T \tag{8}$$

the details in equations (7) and (8) are:

$$e_{2N}(k) = X_{2N}(k) - X_{(2N)d}(k) \tag{9}$$

$$e_{2N-1}(k) = X_{2N-1}(k) - X_{(2N-1)d}(k) \tag{10}$$

$$X_{2N}(k) = [x_2(k), x_4(k), \ldots, x_{2n}(k)]^T \tag{11}$$

$$X_{(2N)d}(k) = [x_{2d}(k), x_{4d}(k), \ldots, x_{2nd}(k)]^T \tag{12}$$

$$X_{2N-1}(k) = [x_1(k), x_3(k), \ldots, x_{2n-1}(k)]^T \tag{13}$$

$$X_{(2N-1)d}(k) = [x_{1d}(k), x_{3d}(k), \ldots, x_{2n-1d}(k)]^T \tag{14}$$

where $X_{(2N-1)d}(k)$ and $X_{(2N)d}(k)$ are the vectors of desired position and velocity, respectively. According to Gao *et al.* (1995) and Wu and Gao (2008), the DSMC reaching law can be rewritten to the n-link robot (6) as follows:

$$s(k+1) = (I_{n\times n} - Th)s(k) - T\varepsilon\,\mathrm{sgn}(s(k)) \tag{15}$$

$\mathrm{sgn}(s(k)) = [\mathrm{sgn}(s_1(k)), \mathrm{sgn}(s_2(k)), \ldots, \mathrm{sgn}(s_n(k))]^T, \varepsilon > 0, h > 0$ and $I_{n\times n} - Th > 0$. Here T, $\varepsilon = diag\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n\}$ and $h = diag\{h_1, h_2, \ldots, h_n\}$ are the sampling time, approximation rate and reaching rate, respectively. In equation (15), the sliding manifolds are bounded as:

$$|s_i(k)| \le \Delta_i, \quad i = 1, \ldots, n \tag{16}$$

where Δ_i is called quasi-sliding mode band width and is:

$$\Delta_i = \frac{\varepsilon_i T}{1 - h_i T}, \quad i = 1, \ldots, n \tag{17}$$

For equation (15), the system states will converge to the desired values only if $s_i(k) \to 0$. On the other hand, sliding manifolds will approach to zero only if $\varepsilon_i T \to 0$. As we know, $\varepsilon_i T$ is nonzero, then the tracking errors will not converge to the origin. However, it will approach near the origin if T is given a small value. Moreover, there is no guarantee for inequality (16) in the presence of model uncertainty and external disturbance. In the Theorem 1 we will derive the closed loop stability bounds.

Remark 2. The precise values of robot parameters are unknown due to measuring errors. Then, the parameters are supposed uncertain with bounded uncertainties:

$$D(X_{2N-1}) = \hat{D}(X_{2N-1}) + \Delta D(X_{2N-1}) \tag{18}$$

$$C(X_{2N-1}, X_{2N}) = \hat{C}(X_{2N-1}, X_{2N}) + \Delta C(X_{2N-1}, X_{2N}) \tag{19}$$

$$G(X_{2N-1}) = \hat{G}(X_{2N-1}) + \Delta G(X_{2N-1}) \tag{20}$$

$$F(X_{2N}) = \hat{F}(X_{2N}) + \Delta F(X_{2N}) \tag{21}$$

with $\hat{D}$, $\hat{C}$, $\hat{G}$, $\hat{F}$ estimated values, and:

$$|\Delta D_{ij}| \le \delta_{d_{ij}}, \ \ |\Delta C_{ij}| \le \delta_{c_{ij}} X_{2Ni}(k), \ \ |\Delta G_i| \le \delta_{g_i}, \ \ |\Delta F_i| \le \delta_{fd} + \delta_{fv} X_{2Ni}(k) \tag{22}$$

$\delta_{d_{ij}}$, $\delta_{c_{ij}}$, δ_{g_i}, δ_{fd} and δ_{fv} are known constants.

Theorem 1. Consider the n-link robot described by equation (6) and the discrete sliding manifold (7), by using the exponential reaching law (15), control law:

$$\begin{aligned} u(k) =& \hat{C}(X_{2N-1}, X_{2N}) X_{2N}(k) + \hat{G}(X_{2N-1}) + \hat{F}(X_{2N}) + \hat{D}(X_{2N-1}) \\ & \times \left(\frac{1}{T}(X_{(2N)d}(k) - X_{(2N)d}(k-1)) - \lambda e_{2N}(k) - hs(k) - \varepsilon(k)\mathrm{sgn}(s(k)) \right) \end{aligned} \tag{23}$$

with inequalities (31) and (34) guarantees boundedness of closed loop system. Here $u = [u_1, u_2, \ldots, u_n]^T$ is the control torque vector, $\varepsilon = diag\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n\}$ and $h = diag\{h_1, h_2, \ldots, h_n\}$ are positive diagonal matrixes.

Proof. By substituting the control law (23) in equation (6), the error dynamics will be in the following form:

$$e_{2N-1}(k+1) = e_{2N-1}(k) + T e_{2N}(k) \tag{24}$$

$$\begin{aligned} e_{2N}(k+1) =& (I_{n\times n} - T\lambda) e_{2N}(k) + (I_{n\times n} - D^{-1}\hat{D})(-\Delta X_{2Nd}(k) + T\lambda e_{2N}(k)) \\ & + TD^{-1}(\Delta A - u_d) - TD^{-1}\hat{D}(hs(k) + \varepsilon(k)\mathrm{sgn}(s(k))) \end{aligned} \tag{25}$$

where $\Delta A = \Delta C X_{2N}(k) + \Delta G + \Delta F$ and $|u_d| < U_D$, $U_D > 0$ are bounded, $I_{n\times n}$ is a identical matrix and $\Delta X_{2Nd}(k) = X_{2Nd}(k) - X_{2Nd}(k-1) \cong X_{2Nd}(k+1) - X_{2Nd}(k)$.

By putting equations (24) and (25) in equation (7), we get:

$$s(k+1) = s(k) + TD^{-1}(\Delta A - u_d) + (I_{n\times n} - D^{-1}\hat{D})(-\Delta X_{2Nd}(k) + T\lambda e_{2N}(k)) - TD^{-1}\hat{D}(hs(k) + \varepsilon(k)\mathrm{sgn}(s(k))) \tag{26}$$

from Sarpturk *et al.* (1987), stability conditions for the DSMC can be illustrated in this way:

$$P_1 = s^T(k)(s(k+1) - s(k)) < 0 \tag{27}$$

$$P_2 = s^T(k)(s(k+1) + s(k)) > 0 \tag{28}$$

that are known as Sarpturk's reaching laws.

First condition: the sliding gain $\varepsilon(k)$ can be determined from the first condition. By substituting equation (26) in the first condition, we have:

$$P_1 = s^T(k)[TD^{-1}(\Delta A - u_d) + (I_{n\times n} - D^{-1}\hat{D})(-\Delta X_{2Nd}(k) + T\lambda e_{2N}(k)) - TD^{-1}\hat{D}(hs(k) + \varepsilon(k)\mathrm{sgn}(s(k)))] \tag{29}$$

for $s(k) > 0$ ($\mathrm{sgn}(s(k)) = 1,\ s(k+1) - s(k) < 0$) the sliding gain lower bound should be as:

$$\varepsilon(k) > \left(\hat{D}^{-1}(\Delta A - u_d) + (\hat{D}^{-1}D - I_{n\times n})\left(-\frac{\Delta X_{2Nd}(k)}{T} + \lambda e_{2N}(k)\right)\right) - hs(k) \tag{30}$$

where $|\hat{D}^{-1}(\Delta A - u_d)|_i < \delta_{Ai},\ |\hat{D}^{-1}D - I_{n\times n}|_{ij} < \delta_{Dij},\ i = 1,2,\ldots,n.$ $\delta_{Ai},\ \delta_{Dij}$ is bounded and known. Then, $\varepsilon(k)$ should be as:

$$\varepsilon(k) > \delta_A + \delta_D\left|-\frac{\Delta X_{2Nd}(k)}{T} + \lambda e_{2N}(k)\right| + h|s(k)| \tag{31}$$

where $\delta_A = [\delta_{A1}, \delta_{A2}, \ldots, \delta_{An}]^T$ and:

$$\delta_D = \begin{bmatrix} \delta_{D_{11}} & \delta_{D_{12}} & \cdots & \delta_{D_{1n}} \\ \delta_{D_{21}} & \delta_{D_{22}} & \cdots & \delta_{D_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{D_{n1}} & \delta_{D_{n2}} & \cdots & \delta_{D_{nn}} \end{bmatrix}.$$

On the other hand, if $s(k) < 0$. By putting equation (31) in $s(k+1) - s(k)$:

$$\begin{aligned} s(k+1) - s(k) = {} & TD^{-1}\hat{D}(\delta_A + \hat{D}^{-1}(\Delta A - u_d)) + TD^{-1}\hat{D} \\ & \times \left(\delta_D \left| -\frac{\Delta X_{2Nd}(k)}{T} + \lambda e_{2N}(k) \right| - (\hat{D}^{-1}D - I_{n\times n}) \right. \\ & (-\Delta X_{2Nd}(k) + T\lambda e_{2N}(k))) + TD^{-1}\hat{D}h(|s(k)|) - s(k)) > 0 \end{aligned} \tag{32}$$

$s(k+1) - s(k)$ will be positive. Therefore, inequality (31) satisfies equation (27).

Second condition: in addition to the sliding gain, big values of the sampling time can cause system instability in discrete-time. Then, the second condition (28) checks the stability bounds of $T(0 < T < T_{\max})$. While, deriving exact value of $T_{\max}$ for a nonlinear system is not easy. By substituting equation (26) in the second condition, we can find an approximation inequality:

$$\begin{aligned} & [D^{-1}\hat{D}(h|s(k)| + \varepsilon(k)) - D^{-1}(\Delta A - u_d) - (I_{n\times n} - D^{-1}\hat{D})\lambda e_{2N}(k)]T \\ & < 2|s(k)| - (I_{n\times n} - D^{-1}\hat{D})\Delta X_{2Nd}(k) \end{aligned} \tag{33}$$

assume that $D = \hat{D}$. Hence, we have:

$$T < \min\left\{\frac{2\Delta_{1\max}}{\varepsilon_{1\max} + h_1\Delta_{1\max} + \delta_{A1}}, \frac{2\Delta_{2\max}}{\varepsilon_{2\max} + h_2\Delta_{2\max} + \delta_{A2}}, \ldots, \frac{2\Delta_{n\max}}{\varepsilon_{n\max} + h_n\Delta_{n\max} + \delta_{An}}\right\} \tag{34}$$

where $\varepsilon_{1\max}, \varepsilon_{2\max}, \ldots, \varepsilon_{n\max}$ and $\Delta_{1\max}, \Delta_{2\max}, \ldots, \Delta_{n\max}$ are maximum bounds of $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ and $|s_1|, |s_2|, \ldots, |s_n|$ for different values of k in the steady-state. Therefore, control law (23) with inequalities (31) and (34) guarantees the closed loop system convergence.

4. Discrete TSMC

To obtain a small transient time convergence, the following continuous-time terminal sliding surface is defined:

$$s(t) = \dot{e}(t) + \lambda e(t)^p \tag{35}$$

where $e = q - q_d$ is the tracking error, $\lambda = diag\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ is positive matrix, $0 < p < 1$ and it should be rational with odd numerator and denominator. On the terminal sliding manifold ($s = 0$), it follows that:

$$\dot{e}_i(t) + \lambda_i e_i(t)^p = 0 \quad i = 1, 2, \ldots, n \tag{36}$$

by solving equation (36), the reaching time ($t_{reach-i}$) for link-i can be driven as:

$$t_{reach-i} = -\int_{e_i(0)}^{0} \frac{de_i}{\lambda e_i^p} = \frac{|e_i(0)|^{1-p}}{\lambda_i(1-p)} \tag{37}$$

this means that the system states converge to zero in a finite time. Now, to obtain the finite time convergence of the tracking error in discrete domain, the discrete terminal sliding surface can be defined as:

$$s(k) = e_{2N}(k) + \lambda e_{2N-1}^{p}(k) \tag{38}$$

where $p = p_1/p_2,\ \ p_1, p_2 > 0,\ \ p_2 > p_1,\ \ p_1, p_2$ are odd integers.

Theorem 2. Considering the robotic system discretized equation (6), and with the combination of equations (15) and (38), the control law is expressed by:

$$\begin{aligned} u(k) = {} & \hat{C}(X_{2N-1}, X_{2N})X_{2N}(k) + \hat{G}(X_{2N-1}) + \hat{F}(X_{2N}) + \hat{D}(X_{2N-1}) \\ & \left(\frac{1}{T}(X_{(2N)d}(k) - X_{(2N)d}(k-1)) - \frac{\lambda}{T}\left((e_{2N-1}(k) + Te_{2N}(k))^{p} - e_{2N-1}^{p}(k)\right)\right. \\ & \left. -hs(k) - \varepsilon(k)\mathrm{sgn}(s(k))\right) \end{aligned} \tag{39}$$

Proof. By substituting equation (39) into equation (6), we have:

$$e_{2N-1}(k+1) = e_{2N-1}(k) + Te_{2N}(k) \tag{40}$$

$$\begin{aligned} e_{2N}(k+1) = {} & e_{2N}(k) + \lambda e_{2N-1}^{p}(k) - \lambda(e_{2N-1}(k) + Te_{2N}(k))^{p} \\ & + (I_{n\times n} - D^{-1}\hat{D})(-\Delta X_{2Nd}(k) + \lambda(e_{2N-1}(k) + Te_{2N}(k))^{p}) \\ & + TD^{-1}(\Delta A - u_d) - TD^{-1}\hat{D}(hs(k) + \varepsilon(k)\mathrm{sgn}(s(k))) \end{aligned} \tag{41}$$

putting equation (40) and (41) into equation (38):

$$\begin{aligned} s(k+1) = {} & s(k) + TD^{-1}(\Delta A - u_d) + (I_{n\times n} - D^{-1}\hat{D}) \times (-\Delta X_{2Nd}(k) \\ & + \lambda(e_{2N-1}(k) + Te_{2N}(k))^{p}) - TD^{-1}\hat{D}hs(k) - TD^{-1}\hat{D}\varepsilon(k)\mathrm{sgn}(s(k)) \end{aligned} \tag{42}$$

same as the Theorem 1, substituting equation (42) into the Sarpturk's first stability condition, we have:

$$\begin{aligned} P_1 = {} & s^{T}(k)[TD^{-1}(\Delta A - u_d) + (I_{n\times n} - D^{-1}\hat{D}) \times (-\Delta X_{2Nd}(k) + \lambda(e_{2N-1}(k) \\ & + Te_{2N}(k))^{p}) - TD^{-1}\hat{D}(hs(k) + \varepsilon(k)\mathrm{sgn}(s(k)))] \end{aligned} \tag{43}$$

for $s(k) > 0$ ($\mathrm{sgn}(s(k)) = 1,\ s(k+1) - s(k) < 0$). In order to have a negative values for P_1, the sliding gain lower bound can be driven as:

$$\begin{aligned} \varepsilon(k) > {} & \left(\hat{D}^{-1}(\Delta A - u_d) + (\hat{D}^{-1}D - I_{n\times n})\left(-\frac{\Delta X_{2Nd}(k)}{T} + \frac{\lambda}{T}(e_{2N-1}(k) + Te_{2N}(k))^{p}\right)\right) \\ & - hs(k) \end{aligned} \tag{44}$$

δ_{Ai} and δ_{Dij} are mentioned in Theorem 1. Then, $\varepsilon(k)$ can be expressed as:

$$\varepsilon(k) > \delta_A + \delta_D\left|-\frac{\Delta X_{2Nd}(k)}{T} + \frac{\lambda}{T}(e_{2N-1}(k) + Te_{2N}(k))^{p}\right| + h|s(k)| \tag{45}$$

On the other hand, if $s(k) < 0$ then $\mathrm{sgn}(s(k)) = -1$. By putting equation (45) in $s(k+1) - s(k)$:

$$\begin{aligned} s(k+1) - s(k) = {} & TD^{-1}\hat{D}(\delta_A + \hat{D}^{-1}(\Delta A - u_d)) + TD^{-1}\hat{D}\times \\ & \left(\delta_D \left| -\frac{\Delta X_{2Nid}(k)}{T} + \frac{\lambda}{T}(e_{2N-1}(k) + Te_{2N}(k))^p \right| - (\hat{D}^{-1}D - I_{3\times 3}) \right. \\ & \left. \left(-\Delta X_{2Nd}(k) + \frac{\lambda}{T}(e_{2N-1}(k) + Te_{2N}(k))^p\right)\right) \\ & + TD^{-1}\hat{D}h(|s(k)| - s(k)) > 0 \end{aligned} \tag{46}$$

$s(k+1) - s(k)$ is positive. Then, inequality (45) satisfies first stability condition. Therefore, control law (39) with inequalities (34) and (45) guarantees the closed loop system convergence.

Remark 3. In practical applications, SMC and DSMC suffer from an important disadvantage that is known as chattering phenomenon. Chattering can cause vibration and system failure due to excitation the high frequency unmodelled dynamics (Young *et al.*, 1999; Wang, 2008). From equations (23) and (39), it can be seen that the controllers are demonstrate chattering. In order to remove the chattering from equation (23) and reduce from equation (39), the switching term of DLSMC an DTSMC control laws ($u_{Switching} = -\varepsilon[sign(s_1(k)), sign(s_2(k)), \ldots, sign(s_n(k))]^T$) are modified in the form of:

$$u_{Switching} = -\varepsilon \begin{bmatrix} \frac{s_1(k)}{|s_1(k)|+\delta_1} \\ \vdots \\ \frac{s_n(k)}{|s_n(k)|+\delta_n} \end{bmatrix} \quad \delta_1, \delta_2, \ldots, \delta_n > 0 \tag{47}$$

$\delta_1, \delta_2, \ldots, \delta_n$ have sufficiently small values.

Remark 4. To design the sliding gain $\varepsilon(k)$ and sampling time T values for DLSM and DTSM controllers, we should act as below algorithm:

- select oscillation range (needed precision) of sliding surface $|s(k)|$;
- choose $\varepsilon(k)$ from inequalities (31) and (45) for DLSMC and DTSMC, respectively; and
- calculate T from inequality (34).

5. Discrete TSMC

A three-link SCARA robot is simulated to show the effectiveness of the DSMC and DTSMC methods in discrete-time domain. The dynamic equations are given as follows (Schilling, 2003):

$$\begin{bmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{bmatrix} + \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} + \begin{bmatrix} G_1 \\ G_2 \\ G_3 \end{bmatrix} + F(\dot{q}) + u_d(t) = \begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix} \tag{48}$$

where:

$$D_{11} = l_1^2\left(\frac{m_1}{3} + m_2 + m_3\right) + l_1 l_2(m_2 + 2m_3)\cos(q_2) + l_2^2\left(\frac{m_2}{3} + m_3\right)$$

$$D_{12} = -l_1 l_2\left(\frac{m_2}{2} + m_3\right)\cos(q_2) - l_2^2\left(\frac{m_2}{3} + m_3\right) = D_{21}$$

$$D_{22} = l_2^2\left(\frac{m_2}{3} + m_3\right) \quad D_{33} = m_{33} \quad D_{13} = D_{31} = D_{23} = D_{32} = 0$$

$$C_{11} = -\dot{q}_2(m_2 + 2m_3)l_1 l_2\sin(q_2) \quad C_{12} = -\dot{q}_2\left(\frac{m_2}{2} + m_3\right)l_1 l_2\sin(q_2)$$

$$C_{21} = -\dot{q}_1\left(\frac{m_2}{2} + m_3\right)l_1 l_2\sin(q_2) \quad C_{22} = C_{13} = C_{31} = C_{23} = C_{32} = C_{33} = 0$$

$$G_1 = G_2 = 0 \quad G_3 = -m_3 g$$

in which q_1, q_2 and q_3 are the angle of joints 1, 2 and 3; m_1, m_2 and m_3 are the mass of links 1, 2 and 3; l_1, l_2, l_3 and are the length of links 1, 2 and 3; and g is the gravity acceleration. The system parameters of the robot are selected as (Wai *et al.*, 2004):

$$m_1 = 1.0\,\text{kg} \quad m_2 = 0.8\,\text{kg} \quad m_3 = 0.5\,\text{kg} \quad g = 9.8 \quad l_1 = 1.0\,\text{m} \quad l_2 = 0.8\,\text{m}$$
$$l_3 = 0.6\,\text{m}$$

The friction term and the external disturbance are the most important parameters that affect the control performance of the robotic system. Therefore, the friction forces are considered as follows:

$$F(\dot{q}) = \begin{bmatrix} f_1^v\dot{q}_1 + f_1^d\,\text{sgn}(\dot{q}_1) \\ f_2^v\dot{q}_2 + f_2^d\,\text{sgn}(\dot{q}_2) \\ f_3^v\dot{q}_3 + f_3^d\,\text{sgn}(\dot{q}_3) \end{bmatrix}^T \tag{49}$$

where $f_1^v = f_2^v = f_3^v = 12 \quad f_1^d = f_2^d = f_3^d = 0.2$. Moreover, the disturbance that affected into the robotic system are given as:

$$u_d(t) = [0.2\sin(3t) \quad 0.2\sin(3t) \quad 0.2\sin(3t)]^T \tag{50}$$

DLSMC design

In order to design DLSMC, controller parameters are selected as:

$$\hat{M} = 0.95\,\text{M} \quad \hat{C} = 0.8\,\text{C} \quad \hat{G} = 0.9\,\text{G} \quad \hat{F} = 0.95\,\text{F} \quad T = 0.002\,\text{s}$$
$$h = diag\{2, 2, 2\} \quad \lambda = diag\{3, 3, 3\}$$

moreover, sliding gain vector is:

$$\varepsilon(k) = \delta_A + \delta_D \left| -\frac{\Delta X_{2Nd}(k)}{T} + \lambda e_{2N}(k) \right| + h|s(k)| \tag{51}$$

where:

$$\delta_A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \quad \delta_D = \begin{bmatrix} 1 & 0.1 & 0 \\ 0.1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The tracking position, control torque and sliding gain variations of joints 1, 2 and 3 are shown in Figures 2 and 3. From Figure 2, we can see that the DLSMC has a high precision tracking and robust behavior.

Figure 3 shows the sliding gain vector $\varepsilon(k)$ variation, and is a proof for equation (51). In the other word, $\varepsilon(k)$ has high value in starting time due to maximum magnitude of tracking errors, and it decreases by passing time.

DTSMC design

In this method, sliding gain is selected as:

$$\varepsilon(k) = \delta_A + \delta_D \left| -\frac{\Delta X_{2Nd}(k)}{T} + \frac{\lambda}{T}(e_{2N-1}(k) + Te_{2N}(k))^p \right| + h|s(k)| \tag{52}$$

where $p = 3/5$ and other parameters are similar to DLSMC design. The tracking position, control torque and sliding gain variation of joints 1, 2 and 3 are shown in Figures 4 and 5.

From Figures 2, 4 and 3, 5, it can be seen that DTSMC presents a better transient response and smaller tracking error than the DLSMC, but there is some chattering in the control signal that saturation function could not remove it completely.

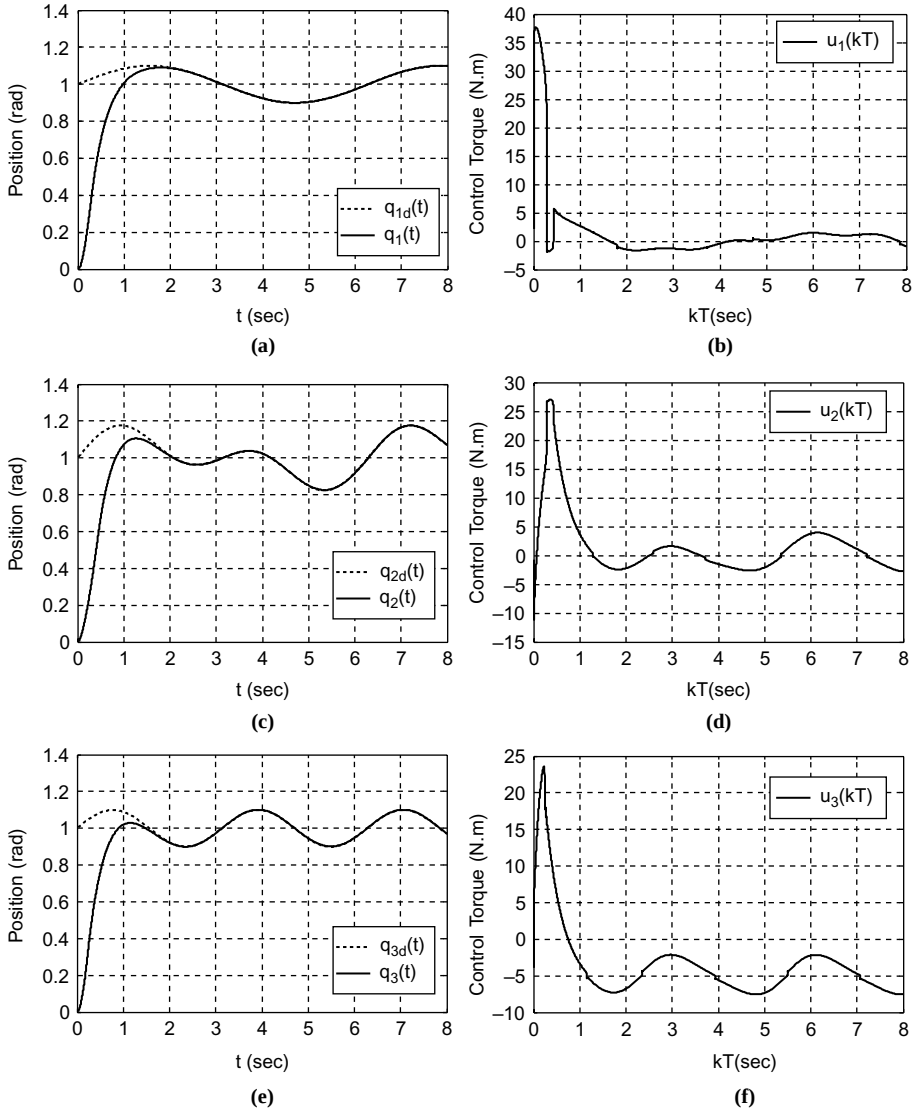
Tracking error of joint 2 is shown in Figure 6 to illustrate another advantage of DTSMC against DLSMC. This figure shows that the tracking error magnitude of DTSMC is lower than DLSMC.

Sampling time effects on the closed loop system convergence: finally, in order to study the sampling times effects on the DLSMC and DTSMC controllers, the computer simulations are repeated for different sampling times. The tracking position and discrete-time sliding surface convergence for $T = 0.04$ and $T = 0.004$ are shown in Figures 7 and 8 for joint 2.

From Figures 7 and 8, it can be seen that the SMC in discrete domain is depended on the sampling time values. Then, in addition to the joint friction and the external disturbance, the sampling time can have important effects on the closed loop convergence. Therefore, the sampling time should be as small as possible to have a high precision controller.

6. Conclusion

In this paper, two robust nonlinear controllers for robotic manipulators have been developed based on discrete-time case. A discrete model of an n-link robot has been used



Notes: (a) q_1 angle of joint 1; (b) $u_1(k)$ control torque of joint 1; (c) q_2 angle of joint 2; (d) $u_2(k)$ control torque of joint 2; (e) q_3 angle of joint 3; (f) $u_3(k)$ control torque of joint 3

Figure 2.
Tracking position and control torque of joints 1, 2 and 3 (DLSMC)

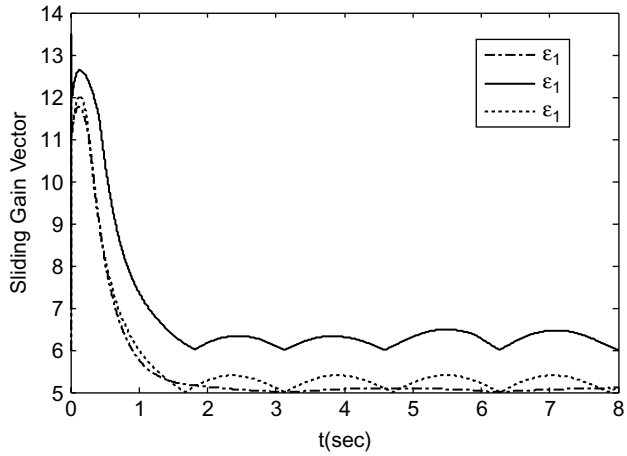


Figure 3.
Sliding gain vector variation ($\varepsilon(k)$) of joints 1, 2 and 3 (DLSMC)

to design these controllers. Through the comparative computer simulations on the three-link SCARA robot, the DLSMC and DTSMC advantages and similarities can be demonstrated as:

- The presented control schemes are robust against the model uncertainties and external disturbance.
- Boundedness of the controlled variables has been theoretically proved by using the Sarpturk inequalities.
- The control torque and sliding manifold (linear and terminal) values are depended on the sampling period. In order to have a high performance and stable closed loop system, the sampling frequency should be small.

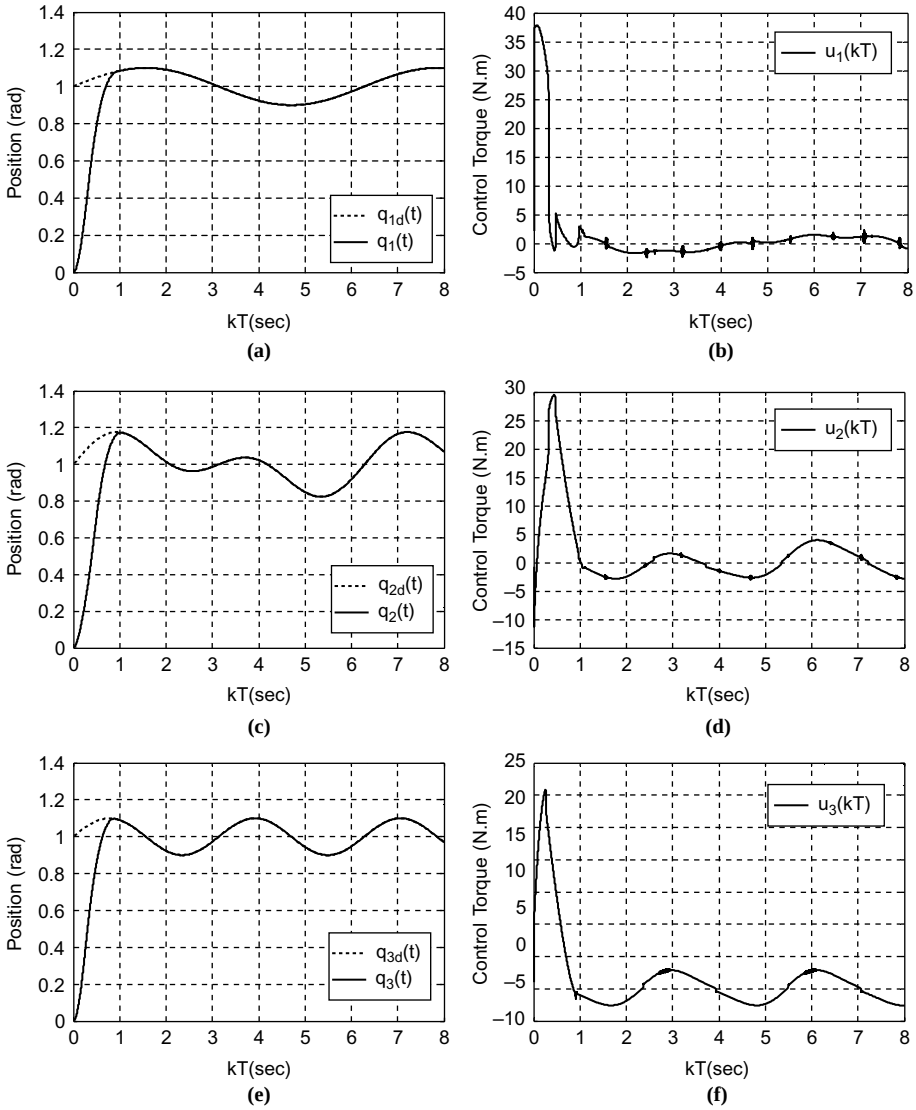
The main differences of DLSMC and DTSMC controllers are deduced as:

- The DTSMC presents a better transient response and smaller tracking error than the DLSMC, while the control signal values are high in the first steps.
- The chattering phenomenon does not removed completely from the DTSMC control signal.

Moreover, the most important advantages of DLSMC and DTSMC in comparison with SMC and TSMC are:

- The DLSMC and DTSMC controllers are low cost, low weight, high accuracy and ease of making software and design changes in comparison with continuous-time ones.
- There is no singularity problem for DTSMC due to differential (non-derivative) computations.

The proposed controllers are applicable to large class of two-rational order systems.



Notes: (a) q_1 angle of joint 1; (b) $u_1(k)$ control torque of joint 1; (c) q_2 angle of joint 2; (d) $u_2(k)$ control torque of joint 2; (e) q_3 angle of joint 3; (f) $u_3(k)$ control torque of joint 3

Figure 4.
Tracking position and control torque of joints 1, 2 and 3 (DTSMC)

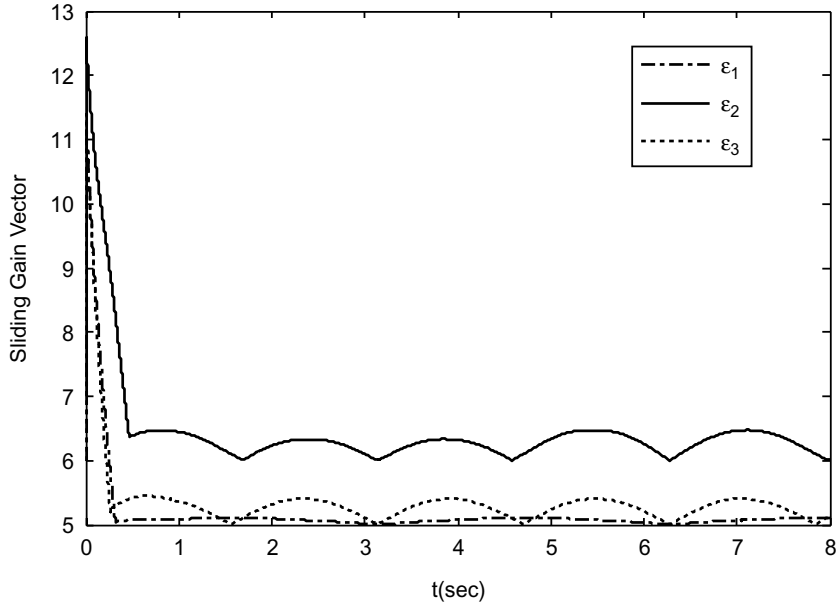


Figure 5.
Sliding gain vector variation ($\varepsilon(k)$) of joints 1, 2 and 3 (DTSMC)

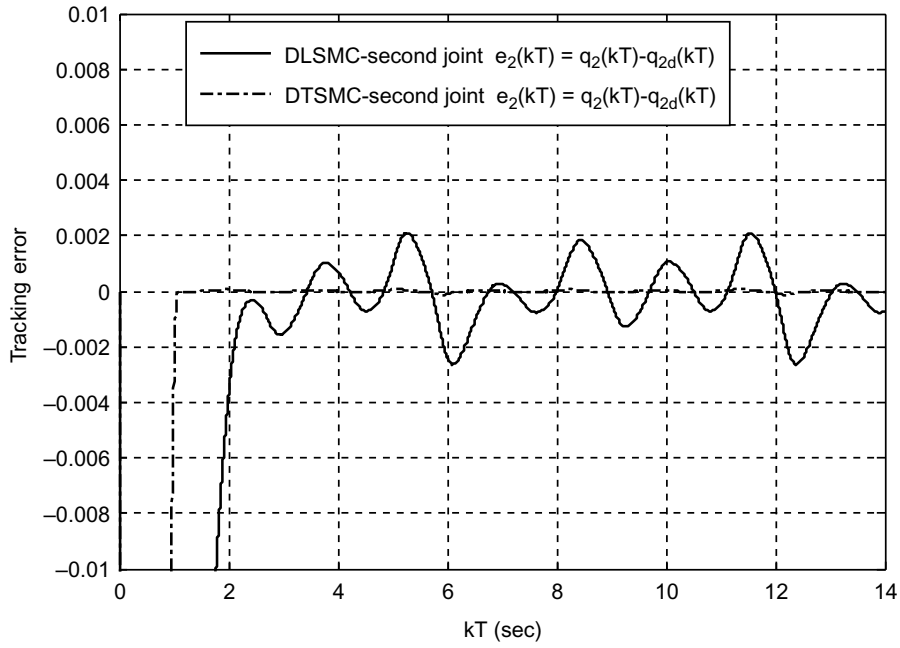
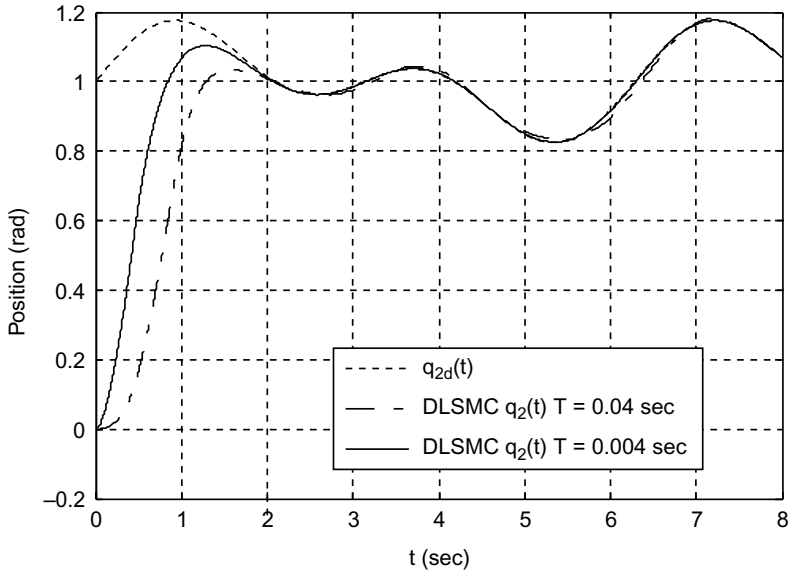
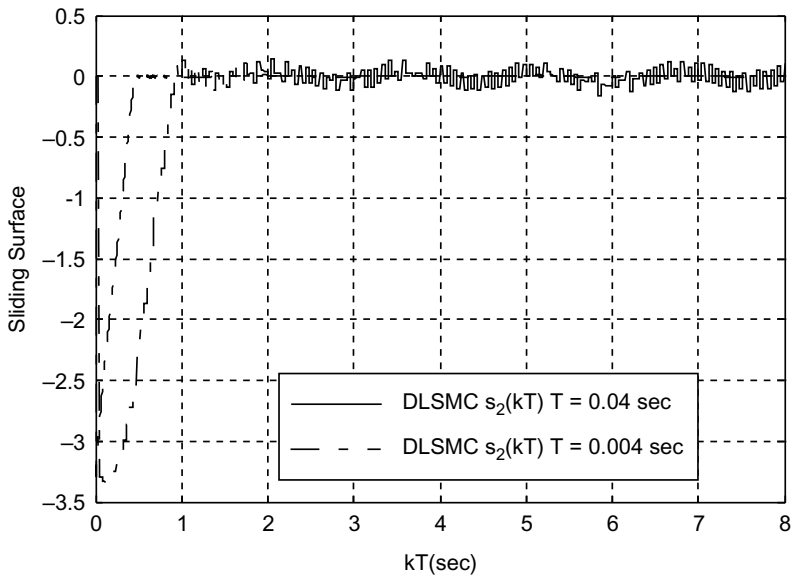


Figure 6.
Position tracking error of joint 2 (DLSMC and DTSMC)



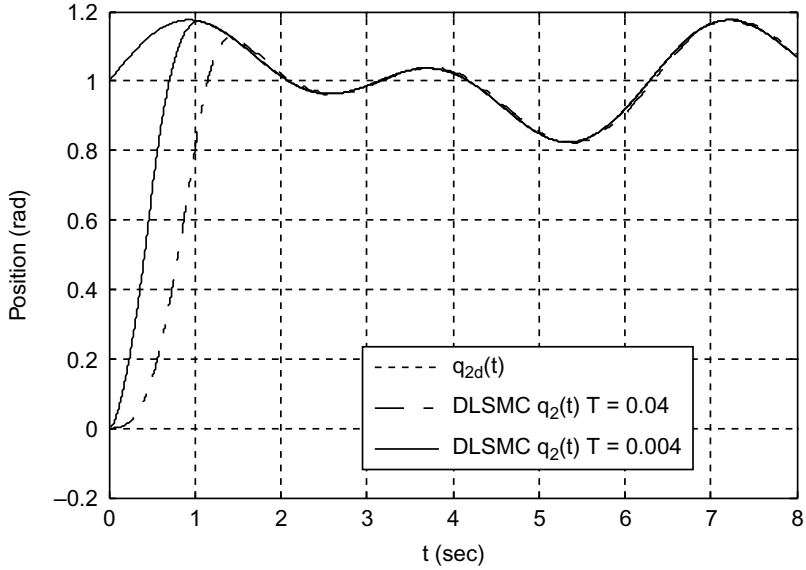
(a)



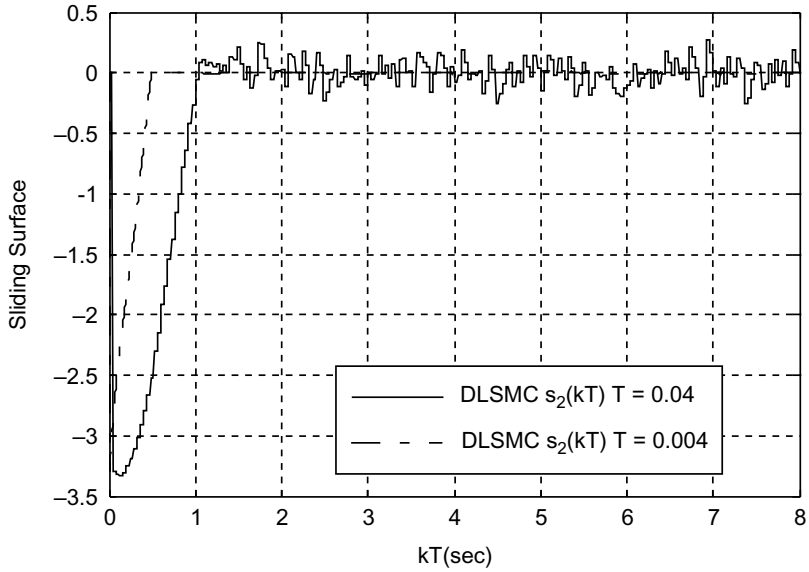
(b)

Notes: (a) Tracking position of joint 2; (b) sliding surface of joint 2

Figure 7.
Tracking position and sliding surface for two different sampling times (DLSMC)



(a)



(b)

Notes: (a) Tracking position of joint 2; (b) sliding surface of joint 2

Figure 8.
Tracking position and sliding surface for two different sampling times (DTSMC)

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