Discrete-time Terminal Sliding Mode Control of Chaotic Lorenz System

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***Abstract-* The objective of this paper is to design a terminal sliding mode controller for Lorenz system in discrete-time. First, a discrete model is derived through Taylor series expansion. In the next step, a discrete terminal sliding mode controller (DTSMC) is developed to reach a fast and high precision control. The stability analysis of DTSMC is presented in the presence of external disturbance. Numerical simulations of Lorenz system are shown and compared to illustrate the effectiveness of the proposed control scheme. Finally, the sampling frequency effects on the closed loop system convergence are discussed.**

Keywords- Chaotic System; Terminal Sliding Mode; Discrete-Time Sliding Mode

I. INTRODUCTION

In recent years, chaos has been intensively studied [1-4], due to its powerful applications in chemical reactions, mechanical and biological systems, secure communications, etc. Chaotic behavior is complex, irregular, sensitive to initial conditions and usually undesirable in practical engineering. It often causes poor performance or even system instability. Thus, it is necessary to eliminate chaotic behaviour and stabilize the chaotic system at one of equilibrium points.

On the other hand, sliding mode control (SMC) is a particular type of variable structure control system that proposed by Emelyanov in the Soviet Union [5]. The SMC is one of most powerful tools to overcome the control problems of nonlinear dynamical systems [6]. This method is well known due to its high precision and robustness against model uncertainties, parameter variations and external disturbances [7]. In the field chaos control, SMC has been widely used to control of various chaotic systems like: Lorenz [8-10], Rossler [11] and Chua systems [12]. However, the main disadvantage of SMC scheme is that the system states reach the equilibrium point at low rate. In [13-18], to bring the system states to the equilibrium point in a finite time, terminal sliding mode control (DTSMC) has been proposed. But, this method has the singularity problem for some of nonlinear systems. In [15-16], some terminal sliding mode controllers have been proposed to overcome the singularity problem in the chaos and robot control.

Moreover, discrete-time sliding mode control (DSMC) has received a lot of attentions due to technological advances in digital electronics and computer control [19-26]. In general, DSMC have many advantages over SMC. Some of the advantages are: low power consumption, low cost, low weight, high accuracy and ease of making software and design changes [19]. In DSMC, the control signal is constant over the entire sampling period and is changed only at each sampling instant. As a result, some properties of continuous-time sliding mode controller are lost by discretization. To enhance the performance of DSMC, various methods have been proposed in [20-24]. Among these literatures, the proposed reaching law in [20] and stability conditions in [22] are common in comparison with others. Same as continuous-time one, DSMC can be designed in the finite-time form that is known as discrete-time terminal sliding mode control (DTSMC). In [27-28], DTSMC is applied to reach finite-time convergence.

In chaos control, DSMC is used to stabilizing unstable fixed points of discrete chaotic systems [29]. To our knowledge, there is no top literature in finite-time control of chaotic systems in discrete-time. Therefore, extra work is needed in this domain.

This paper presents the following main contributions: 1- DTSMC is implemented to control of chaotic systems for first time. 2- Stability of the closed loop system is proved for DTSMC in the presence of external disturbances. 3- Upper bound of sampling time is derived for the proposed control scheme. 4- Sampling time effects on the system outputs and sliding surfaces convergence are presented. 5- The proposed controller is nonsingular due to the non-derivative computations. 6- The main drawback of the terminal sliding mode control is illustrated.

The rest of this paper is organized as follows: Section 2 describes a canonical form for the Lorenz system. A discrete-time model is derived from the canonical form in Section 3. In Section 4, we design a discrete-time terminal sliding mode controller. The effectiveness of the proposed controller is illustrated by numerical simulations in Section 5. Finally, concluding remarks in Section 6 close the paper.

II. LORENZ SYSTEM AND ITS CANONICAL FORM

Consider the following chaotic Lorenz system with external disturbance [8-9],

$$\begin{cases} \dot{x}_1(t) = -\sigma x_1(t) + \sigma x_2(t) \\ \dot{x}_2(t) = r x_1(t) - x_2(t) - x_1(t)x_3(t) + d(t) + u(t) \\ \dot{x}_3(t) = x_1(t)x_2(t) - b x_3(t) \end{cases} \tag{1}$$

where $x_1(t)$, $x_2(t)$ and $x_3(t)$ are states of the system, $u(t)$ is the scalar control input, $|d(t)| < D$ is the bounded disturbance, D is positive constant.

The objective of this study is to drive the system states to a specified point in the state space. If $x_1(t) = x_{1d}$, then $\dot{x}_1(t) = -\sigma x_1(t) + \sigma x_2(t) = 0$ and then $x_2(t) = x_{1d}$. Moreover, by solving the third equation of (1) we have

$$x_3(t) = e^{-bt} x_3(0) + \frac{x_{1d}^2 (1 - e^{-bt})}{b} \tag{2}$$

Therefore, the states of Lorenz system will be steered to the equilibrium point $x_e = (x_{1d}, x_{2d}, x_{3d})^T = (x_{1d}, x_{1d}, \frac{x_{1d}^2}{b})^T$ by passing time.

Now let the error states be $e_1(t) = x_1(t) - x_{1d}$, $e_2(t) = x_2(t) - x_{2d}$ and $e_3(t) = x_3(t) - x_{3d}$, then we can get

$$\begin{cases} x_1(t) = e_1(t) + x_{1d} \\ x_2(t) = e_2(t) + x_{2d} \\ x_3(t) = e_3(t) + x_{3d} \end{cases} \rightarrow \begin{cases} \dot{x}_1(t) = \dot{e}_1(t) \\ \dot{x}_2(t) = \dot{e}_2(t) \\ \dot{x}_3(t) = \dot{e}_3(t) \end{cases} \tag{3}$$

and the following error dynamics can be derived from (1) and (3)

$$\begin{cases} \dot{e}_1(t) = -\sigma e_1(t) + \sigma e_2(t) \\ \dot{e}_2(t) = (r - x_{3d}) e_1(t) - e_1(t)e_3(t) - e_2(t) - x_{1d} e_3(t) + (r - 1 - x_{3d}) x_{1d} + d(t) + u(t) \\ \dot{e}_3(t) = -b e_3(t) + g(e_1(t), e_2(t)) \end{cases} \tag{4}$$

where $g(e_1(t), e_2(t)) = e_1(t)e_2(t) + x_{1d} e_1(t) + x_{1d} e_2(t)$.

To design the sliding mode control law, the reformulation of the error dynamics (4) into a controllable canonical form is needed. We applied the following state transformation to present (4) in the canonical form

$$E(t) = \begin{bmatrix} E_1(t) \\ E_2(t) \end{bmatrix} = P \begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix} = \begin{bmatrix} \sigma & 0 \\ \sigma & 1 \end{bmatrix} \begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix} \tag{5}$$

where $E(t)$ is the new state error vector and P is transformation matrix. Hence, the first two equations of (4) can be transferred to a standard form as

$$\begin{cases} \dot{E}_1(t) = E_2(t) \\ \dot{E}_2(t) = \sigma(r - 1 - x_{3d}) E_1(t) - \sigma E_1(t) e_3(t) - (1 + \sigma) E_2(t) - x_{1d} e_3(t) + (r - 1 - x_{3d}) x_{1d} + d(t) + u(t) \\ \dot{e}_3(t) = -b e_3(t) + \hat{g}(E_1(t), E_2(t)) \end{cases} \tag{6}$$

where $\hat{g}(E_1(t), E_2(t)) = \sigma^2 E_1^2(t) + \sigma E_1(t) E_2(t) + 2\sigma x_{1d} E_1(t) + x_{1d} E_2(t)$ and goes zero if $E_1, E_2 \to 0$. Then, $e_3 \to 0$ [8].

III. MODEL DISCRETIZATION

A possible block scheme of a discrete-time computer controlling a continuous-time system is sketched in Fig. 1. In this figure, the controller block is composed out of three parts [25]:

1- A digital central block

2- An analog to digital convertor

3- A continuous to discrete-time signal convertor.

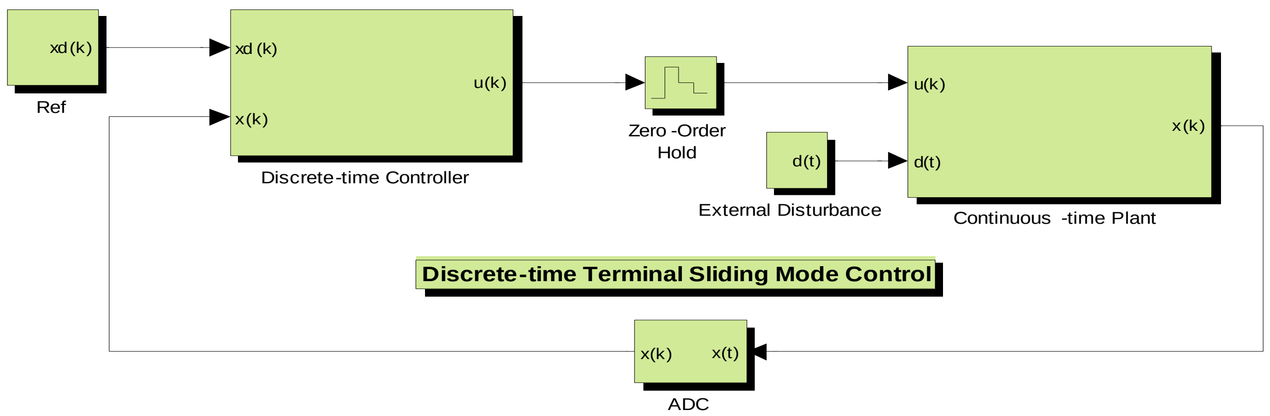

Fig. 1 Closed loop system composed of a discrete time controller and a continuous time plant

From above figure, to design a discrete control law, the continuous-time model (6) should be transferred to a discrete-time one. By using the following Taylor series expansion [30],

$$e(k+1) = e(k) + \dot{e}(k)\big|_{t=kT} T + O(T) \tag{7}$$

where T is the sampling period and $O(T)$ is the high order term of the above expansion, i.e.

$$O(T) = \frac{1}{2!} e^{(2)}(k) T^2 + \cdots + \frac{1}{n!} e^{(n)}(k) T^n + \cdots \tag{8}$$

the higher order terms of $O(T)$ will be small for the high sampling rates. By substituting (6) into the first and second equations of (7), the discrete-time model will be as

$$\begin{cases} E_1(k+1) = E_1(k) + TE_2(k) \\ E_2(k+1) = E_2(k) + T\sigma(r-1-x_{3d})E_1(k) - T\sigma E_1(k)e_3(k) - T(1+\sigma)E_2(k) - Tx_{1d}e_3(k) \\ \qquad\qquad + T(r-1-x_{3d})x_{1d} + T\Delta f(X) + Td(k) + Tu(k) \end{cases} \tag{9}$$

Remark 1: The discretization model of Lorenz system in (9) is an approximation model for the original uncertain model in (6). In order to have an approximately similar behaviour between the discrete-time (9) and original continuous-time (6) systems, the sampling rate should be sufficiently fast in comparison with the system dynamics and uncertainties.

IV. TERMINAL SLIDING MODE CONTROL

To obtain a small transient time convergence for (6), the following continuous-time terminal sliding surface is defined

$$s(t) = E_2(t) + \lambda E_1(t)^p \tag{10}$$

where λ is positive constant and $0 < p < 1$. p should be rational with odd numerator and denominator.

On the terminal sliding manifold ($s = 0$), the system dynamics are determined by the following nonlinear differential equation

$$E_2(t) + \lambda E_1(t)^p = 0 \tag{11}$$

by solving (11), the reaching time (t_{reach}) is given by

$$[\, t_{reach} = -\int_{E_1(0)}^{0} \frac{dE_1}{\lambda E_1{}^p} = \frac{|E_1(0)|^{1-p}}{\lambda(1-p)} \tag{12}$$

this means that the system states converge to zero in a finite time. Same as (10), we can define the discrete-time terminal sliding surface as

$$s(k) = E_2(k) + \lambda E_1{}^p(k) \tag{13}$$

According to the reaching law that proposed in [20, 31],

$$s(k+1) = (I_{n\times n} - Th)s(k) - T\varepsilon \,\mathrm{sgn}(s(k)) \qquad \varepsilon > 0,\ \ h > 0,\ \ 1 - Th > 0 \tag{14}$$

we can define below Theorem to stabilize the Lorenz system. Here T, ε and h are the sampling time, approximation rate and reaching rate, respectively.

Theorem. Considering the Lorenz system discretized Equation (6), and with the combination of (13) and (14), the following control law

$$u(k)=\sigma(x_{3d}+1-r)E_1(k)+(\sigma+1)E_2(k)+x_{1d}e_3(k)+\sigma E_1(k)e_3(k)+(x_{3d}+1-r)x_{1d} \\ \frac{\lambda}{T}\left(E_1{}^p(k)-(E_1(k)+TE_2(k))^p\right)-hs(k)-\varepsilon\,\mathrm{sgn}(s(k)) \tag{15}$$

with Inequalities (23) and (25) guarantees boundedness of closed loop system.

Proof. By substituting (15) into (9), the error dynamics will be as

$$E_1(k+1)=E_1(k)+TE_2(k) \tag{16}$$

$$E_2(k+1)=E_2(k)+\lambda E_1{}^p(k)-\lambda(E_1(k)+TE_2(k))^p-Ths(k)+Td(k)-T\varepsilon\,\mathrm{sgn}(s(k)) \tag{17}$$

putting (16) and (17) into (13), we can get

$$s(k+1)=(1-Th)s(k)+Td(k)-T\varepsilon\,\mathrm{sgn}(s(k)) \tag{18}$$

from [22], stability conditions for the discrete-time sliding mode control can be illustrated in this way

$$P_1=s^T(k)(s(k+1)-s(k))<0 \tag{19}$$

$$P_2=s^T(k)(s(k+1)+s(k))>0 \tag{20}$$

that are known as Sarpturk's reaching laws. Now, by putting (18) in (19) and (20),

$$P_1=s(k)[-Ths(k)+Td(k)-T\varepsilon\,\mathrm{sgn}(s(k))] \tag{21}$$

$$P_2=s(k)[(2-Th)s(k)+Td(k)-T\varepsilon\,\mathrm{sgn}(s(k))] \tag{22}$$

to have a negative values for P_1, the sliding gain lower bound should be as

$$\varepsilon>D>D-h|s(k)| \tag{23}$$

also, for the second condition

$$\begin{cases} s(k)>0\xrightarrow{sign(s(k))=1}(2-Th)s(k)+Td(k)-T\varepsilon>0\xrightarrow{|d(k)|<D,\varepsilon>D}s(k)>\dfrac{T\varepsilon+Td(k)}{2-Tq} \\ s(k)<0\xrightarrow{sign(s(k))=-1}(2-Th)s(k)+Td(k)+T\varepsilon<0\xrightarrow{|d(k)|<D,\varepsilon>D}s(k)<-\dfrac{T\varepsilon+Td(k)}{2-Tq} \end{cases} \tag{24}$$

and from the above inequalities

$$|s(k)|>\frac{T(\varepsilon+D)}{2-Tq} \quad or \quad T<\frac{2|s(k)|}{\varepsilon+D+h|s(k)|} \tag{25}$$

then for (15), the system states will converge to the desired values only if $s(k)\to 0$. On the other hand, sliding manifold will approach to zero only if $T(\varepsilon+D)\to 0$. As we know, the parameters ε, T, D are nonzero. Then, the tracking errors will not converge to the origin. However, it will approach near the origin if T is given a small value. Therefore, Control Law (15) with Inequality (23) guarantees the closed loop convergence for a small sampling time.

Remark 2: To design the sliding gain ε and sampling time T values for DTSM controller, we should act as below algorithm:

1- Select oscillation range (needed precision) of sliding surface $|s(k)|$.

2- Evaluate ε from inequality (23) for DTSMC.

3- Calculate T from inequality (25).

Remark 3: From Equation (15), it can be seen that the controller demonstrates chattering phenomenon. Chattering can cause vibration and system failure due to excitation the high frequency un-modelled dynamics [6]. Hence, to reduce the chattering from (15), the control law ($u_{Switching} = -\varepsilon sign(s(k))$) is modified in the form of

$$u_{Switching-term} = -\varepsilon \frac{s(k)}{|s(k)| + \delta} \qquad \delta > 0 \tag{26}$$

where δ is sufficiently small.

V. SIMULATION RESULTS

For numerical simulations, consider the Lorenz system described by (1), whose parameters are chosen as $\sigma = 10$, $b = 8/3$ and $r = 28$. The initial and desired values of $X = [x_1, x_2, x_3]^T$ are given by $X = [12,2,9]^T$ and $X = [8.5,8.5,27.1]^T$, respectively.

The disturbance that affected into the chaotic system is $d(t) = 0.2\sin(3\pi t)$. Moreover, controller parameters are selected as

$$T = 0.002\sec \quad h = 10 \quad \varepsilon = 2 \quad \lambda = 6 \quad \delta = 0.1$$

Fig. 2 shows the phase portrait of Lorenz system without any controllers. Fig. 3, 4 and 5 present system states convergence for two different values of p ($p = 1$: linear manifold and $p = 7/11$: non-linear manifold). Also, Figure 6 shows control input. From Fig. 3, 4, 5 and 6, it can be seen that:

1- When *p* is close to zero, the transient response is fast but the chattering phenomenon is not removed completely from control signal even in the presence of saturation function.

2- When *p* is approaching one, the transient response is slow, but the chattering is eliminated.

Then, *p* should be chosen in a way that we reach a tradeoff between the transient response and control signal oscillations.

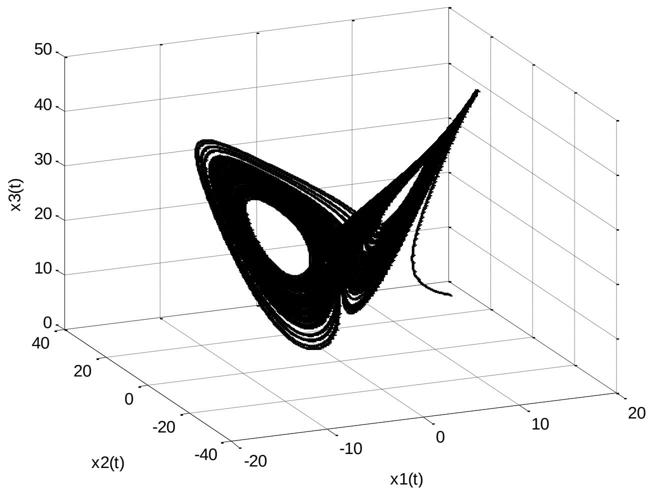

Fig. 2 The phase plot of Lorenz system

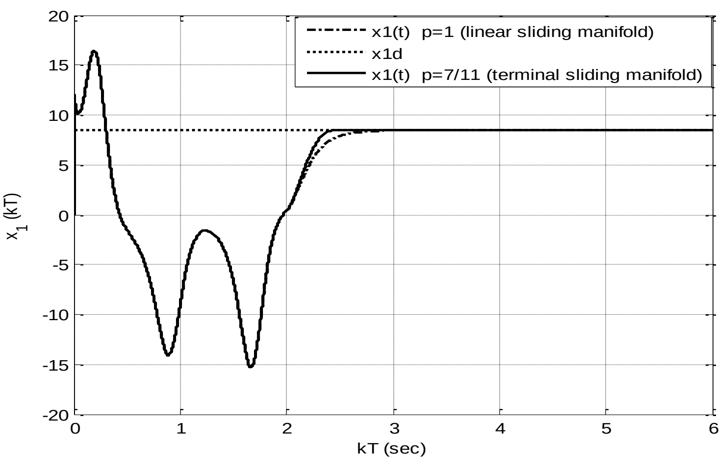

Fig. 3 Time response of $x_1(kT)$ for $p = 1, 7/11$

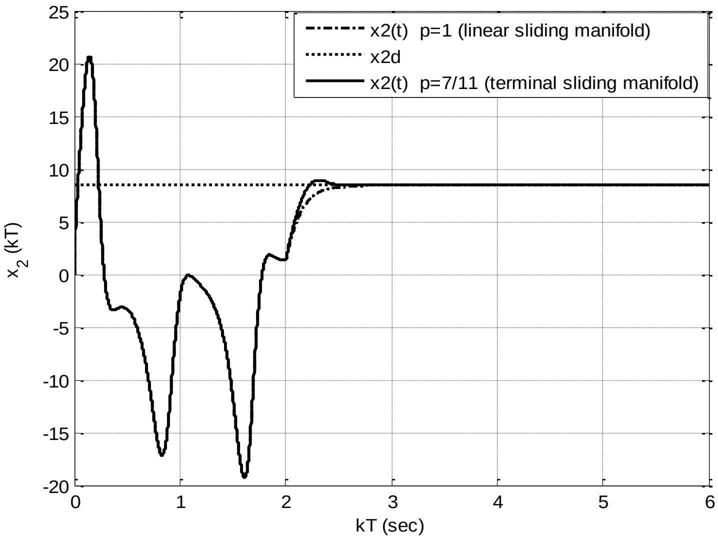

Fig. 4 Time response of $x_2(kT)$ for $p=1, 7/11$

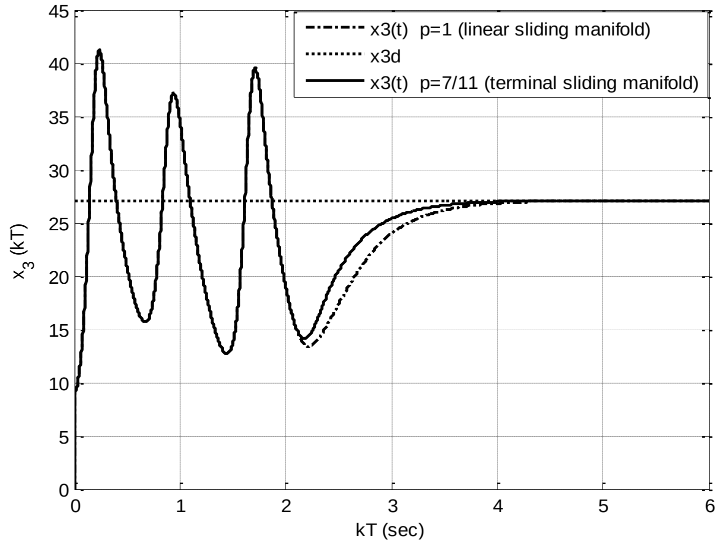

Fig. 5 Time response of $x_3(kT)$ for $p=1, 7/11$

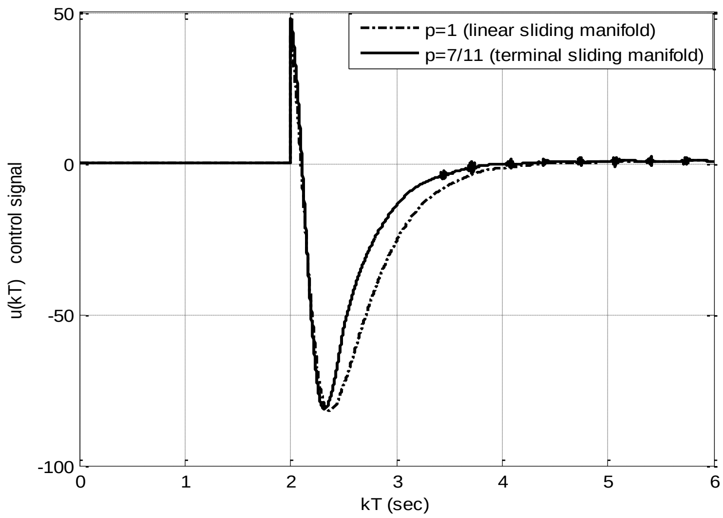

Fig. 6 Time response of $u(kT)$ for $p=1, 7/11$

Finally, to study the sampling period effects on the DTSMC controller, the computer simulations are repeated for different sampling rates. The tracking position and discrete sliding surface convergence for $T = 0.05\text{sec}$ and $T = 0.005\text{sec}$ are shown in Fig. 7 with additional zoom on the right side. From this Figure, it can be seen that the sliding mode control in discrete domain is depended on the sampling time values. Then, to have a high precision controller the sampling time should be as small as possible.

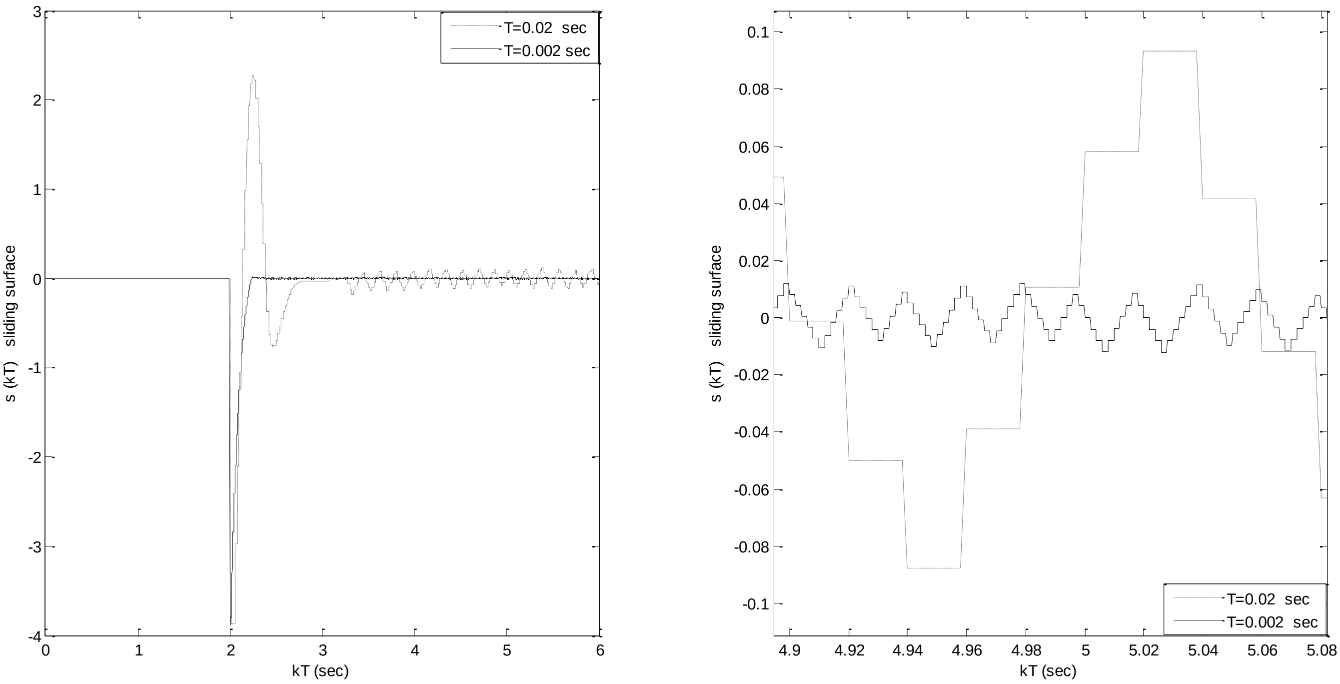

Fig. 7 Sliding surface $s(kT)$ variation for $T = 0.02$ and $T = 0.002$

VI. CONCLUSIONS

In this paper, the terminal sliding mode control of Lorenz system has been developed in discrete-time. Through the computer simulations, it has been shown that by using DTSMC, system states have been steered fast to the equilibrium point. Also, stability of the closed loop system has been theoretically proved. Meanwhile, the proposed controller behaviour is highly depended on the sampling rate. Therefore, to have a high performance and stable closed loop system, the sampling frequency should be small enough. Moreover, the most important advantages of DTSMC in comparison with TSMC are:

1-There is no singularity problem for the DTSMC controller due to differential (non-derivative) computations.

2-The DTSMC controller is low cost and easily implemented in practice in comparison with continuous-time one.

Such a control algorithm could be easily extended to the Rossler and Chua chaotic systems.

REFERENCES

[1] T. Kapitaniak, "Chaotic oscillations in mechanical systems," New York, Manchester University Press, 1991.

[2] A. Nayfeh, "Applied nonlinear dynamics," New York, Wiley, 1995.

[3] G. Chen, X. Dong, "From chaos to order: methodologies, perspectives and applications," Singapore, *World Scientific*, 1998.

[4] J. Lu, X. Wu, J. L, "Synchronization of a unified chaotic system and the application in secure communication," *Physics Letters A*, vol. 305, pp. 365–370, 2002.

[5] S. V. Emelyanov, *Variable structure control systems*, Nauka, Moscow, 1967.

[6] K. D. Young, V. I. Utkin, U. Uzguner, "A Control Engineer's Guide to Sliding Mode Control," *IEEE Trans. On Control System Technology*, Vol. 7, No. 3, 1999.

[7] X. Yu, O. Kaynak, "Sliding-Mode Control with Soft Computing: A Survey," *IEEE Trans. On Industrial Electronics*, Vol. 56, No. 9, September, 2009.

[8] H. Guo, S. Lin, J. Liu, "A radial basis function sliding mode controller for chaotic Lorenz system," *Physics Letters A*, 351, pp. 257–261, 2006.

[9] W. Ligang, W. Changhong, "Adaptive fuzzy sliding mode control of Lorenz chaotic system," *Front Electr Electron Eng*, Vol. 2, No. 3, pp. 277–281, 2007.

[10] H. T. Yau, J. J. Yan, "Design of sliding mode controller for Lorenz chaotic system with nonlinear input," *Chaos, Solitons and Fractals*, Vol. 19, pp. 891–898, 2004.

[11] J. F. Chang, M. L. Hung, Y. S. Yang, T. L. Liao, J. J. Yan, "Controlling chaos of the family of Rossler systems using sliding mode

control," *Chaos, Solitons and Fractals*, Vol. 37, pp. 609–622, 2008.

[12] M. J. JANG, C. L. CHEN, C. K. CHEN, "SLIDING MODE CONTROL OF CHAOS IN THE CUBIC CHUA'S CIRCUIT SYSTEM," *International Journal of Bifurcation and Chaos*, Vol. 12, No. 6, pp. 1437-1449, 2002.

[13] L. Hui and J. Li, "Terminal Sliding Mode Control for Spacecraft Formation Flying," *IEEE Trans. On Aerospace And Electronic Systems*, Vol. 45, No. 3, July, 2009.

[14] R. M. Zhihong and M. O'Day, "A Robust Adaptive Terminal Sliding Mode Control for Rigid Robotic Manipulators," *Journal of Intelligent and Robotic Systems*, 24: 23-41, 1999.

[15] H. Wang, Z. Z. Han, Q. Y. Xie, W. Zhang, "Finite-time chaos control via nonsingular terminal sliding mode control," *Communications in Nonlinear Science and Numerical Simulation*, Vol. 14, pp. 2728–2733, 2009.

[16] Y. Feng, X. Yu and Z. Man, "Non-singular terminal sliding mode control of rigid manipulators," Automatica, 38(12), 2159-2167, July 2002.

[17] L. Yang, J. Yang, "Robust finite-time convergence of chaotic systems via adaptive terminal sliding mode scheme," *Communications in Nonlinear Science and Numerical Simulation*, Vol. 16, pp. 2405–2413, 2011.

[18] S. Dadras, H. R. Momeni, "Fractional terminal sliding mode control design for a class of dynamical systems with uncertainty," *Communications in Nonlinear Science and Numerical Simulation*, Vol. 17, pp. 367–377, 2012.

[19] B. Wang, On Discretization of Sliding Mode Control Systems, Ph.D thesis, RMIT University, Melbourne, Australia, 2008.

[20] W. Gao, "Discrete time variable structure control systems," *IEEE Trans. On Industrial Electronics*, 45: 633-637, 1995.

[21] K. Furuta, "Sliding mode control of a discrete system," *Systems and Control letters*, 14: 144-152, 1990.

[22] S. K. Spurgeon, "Hyperplane design techniques for discte-time variable structure control system," Int. J. of Control 55(2), pp. 445-456, 1992.

[23] S. T. Sarpturk, Y. Istefanopolos and O. Kaynak, "On the stability of Discrete-Time Sliding Mode Control," *IEEE Trans. On Automatic Control*, Vol. 32, No. 10, 930-932, October 1987.

[24] A. Bartoszewicz, "Discrete time quasi-sliding-mode-control strategies," *IEEE Trans. On Industrial Electronics*, 45: 633-637, 1998.

[25] G. Monses, discrete time sliding mode control, Ph.D thesis, 2002.

[26] B. Bandyopadhyay, Discrete-time Sliding Mode Control: A Multirate Output Feedback Approach, Springer-Verlag Berlin Heidelberg New York, 2006.

[27] S. Janardhanan and B. Bandyopadhyay, "on discretization of continuous time terminal sliding mode," *IEEE Trans. On Automatic Control*, Vol. 51, No. 9, pp. 1532-1536, September, 2006.

[28] K. Abidi, J. X. Xu, and J. H. She, "A Discrete-Time Terminal Sliding-Mode Control Approach Applied to a Motion Control Problem," *IEEE Trans. On Industrial Electronics*, Vol. 56, No. 9, September, 2009.

[29] H. Salarieh, S. M. M. Kashani, G. Vossoughi, A. Alasty, "Stabilizing unstable fixed points of discrete chaotic systems via quasi-sliding mode method," *Communications in Nonlinear Science and Numerical Simulation*, Vol. 14, pp. 839-849, 2009.

[30] K. C. Veluvolu, Y. C. Soh, W. Cao, "Robust discrete-time nonlinear sliding mode state estimation of uncertain nonlinear systems," *INTERNATIONAL JOURNAL OF ROBUST AND NONLINEAR CONTROL*, Vol. 17, pp. 803-828, 2007.

[31] L. Wu, H. Gao, "Sliding mode control of two dimensional systems in rossler model," *IET Control Theory and Appl*, Vol. 2, No. 4, pp. 352-364, 2008.

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