2-Capability of 2-Generator 2-Groups of Class Two

F. Gharibi Monfared, S. Kayvanfar & F. Johari

Bulletin of the Malaysian Mathematical Sciences Society

ISSN 0126-6705

Bull. Malays. Math. Sci. Soc.
DOI 10.1007/s40840-020-00915-w

ONLINE FIRST

Springer

Bull. Malays. Math. Sci. Soc.
https://doi.org/10.1007/s40840-020-00915-w

BULLETIN *of the* MALAYSIAN MATHEMATICAL SCIENCES SOCIETY
http://www.springer.com/mathematics/journal/40840

2-Capability of 2-Generator 2-Groups of Class Two

F. Gharibi Monfared[1] · S. Kayvanfar[1] · F. Johari[1]

Received: 2 July 2019 / Revised: 1 February 2020

Abstract

The aim of this paper is to classify all 2-capable 2-generator 2-groups of class two. Obtaining the structure of the 2-nilpotent multipliers of these 2-groups is the other aim.

Keywords Capable group · 2-Capable group · 2-Nilpotent multiplier · 2-Group

Mathematics Subject Classification 20C25 · 20D15

1 Introduction

A group G is called capable if there exists some group H such that $G \cong H/Z(H)$. Capability of groups was first appeared in [3], where Baer succeeds to characterize all capable abelian groups among the direct sums of cyclic groups. The concept of capability for p-groups is used in the classification of p-groups into isoclinism classes by Hall [7].

The notion of varietal capability with respect to any variety of groups was introduced by Moghaddam and Kayvanfar [13]. Moreover, Burns and Ellis [6] generalized the concept of capability of groups to the varietal capability with respect to the variety of nilpotent groups of class at most c for $c \geq 1$. Recall that a group G is called c-capable if $G \cong H/Z_c(H)$ for some group H, where $Z_c(H)$ is the c-th term of the upper central series of H for $c \geq 1$. As a result, every c-capable group is also 1-capable.

Communicated by Peyman Niroomand.

✉ S. Kayvanfar
skayvanf@um.ac.ir

F. Gharibi Monfared
fr.gharibimonfared@mail.um.ac.ir

F. Johari
farangis.johari@mail.um.ac.ir

[1] Department of Pure Mathematics, Ferdowsi University of Mashhad, Mashhad, Iran

Published online: 25 February 2020

When $c = 1$, 1-capable groups are indeed capable groups. Later in [6], Burns and Ellis generalized the same results of Baer for the capability of finitely generated abelian groups to c-capability. It is shown that there exists a finite 2-group that is capable but not 2-capable [6, Theorem 1.4]. Therefore, the concepts of capability and c-capability for groups are different from each other.
Let G be the quotient of a free group F by a normal subgroup R. Then, the 2-nilpotent multiplier of G is defined as the abelian group

$$\mathcal{M}^{(2)}(G) \cong R \cap \gamma_3(F)/[R, F, F],$$

where $\gamma_3(F) = [F, F, F]$. This is a lesser extent of the Baer invariant of a group G with respect to the variety of nilpotent groups of class at most 2, which has been introduced in [4]. The 1-nilpotent multiplier of G is more known as the Schur multiplier of G, $\mathcal{M}(G)$, and it is much more studied, for instance in [9,15]. Information about the 2-nilpotent multiplier of groups may be used as an instrument in the connection to the 2-capability of groups. Niroomand and Parvizi proved that all extra-special p-groups are capable and 2-capable simultaneously by obtaining explicit structure of the 2-nilpotent multipliers of all extra-special p-groups in [16]. Recently, Niroomand et al. [17] showed that "capability" and "c-capability" are equivalent for these groups.

A new classification for the 2-generator p-groups of nilpotency class two is presented in [1] that corrects and simplifies previous classifications for these groups [2,8,18]. For the case $p = 2$, computations of the non-abelian tensor square are stated in [8]. Determination of the capable groups among the 2-generator 2-groups of class two is stated in [10]. This computation is based on a classification that is incomplete. Thus, Magidin and Morse determined all capable 2-generator 2-groups of class two by a new classification in [11]. They have also described the Schur multiplier of these groups in [11, Theorem 50].

The main result of the present paper is to identify which of the 2-generator 2-groups of class two are 2-capable. At first, we compute the 2-nilpotent multipliers of all capable 2-generator 2-groups of class two. Then, we show that the concepts "capable" and "2-capable" are equivalent for these 2-groups.

2 Preliminaries

This section is devoted to state concepts and results which will be used in the next section. We use techniques involving the concept of the basic commutators. Here is the definition.

Definition 2.1 Let X be an arbitrary subset of a free group and select an arbitrary total order for X. The basic commutators on X, their weight wt, and the ordering among them are defined as follows.

(i) The elements of X are basic commutators of weight one, ordered according to the total order previously chosen.
(ii) Having defined the basic commutators of weight less than n, a basic commutator of weight n is $d = [s, k]$, where:

(a) s and t are basic commutators and $wt(s) + wt(k) = n$, and
(b) $s > k$, and if $s = [s_1, s_2]$, then $k \geq s_2$.

(iii) The basic commutators of weight n follow those of weight less than n. The basic commutators of weight n are ordered among themselves in any total order, but the most common used total order is lexicographic order, that is if $[b_1, a_1]$ and $[b_2, a_2]$ are basic commutators of weight n, then $[b_1, a_1] < [b_2, a_2]$ if and only if $b_1 < b_2$ or $b_1 = b_2$ and $a_1 < a_2$.

The following theorem gives a formula for the 2-nilpotent multiplier of a direct product of two finite groups.

Theorem 2.2 [14] *Let G and H be two finite groups. Then,*

$$\mathcal{M}^{(2)}(G \times H) \cong \mathcal{M}^{(2)}(G) \oplus \mathcal{M}^{(2)}(H) \oplus \big((G/G' \otimes G/G') \otimes H/H'\big) \oplus \big((H/H' \otimes H/H') \otimes G/G'\big).$$

Let $\mathbb{Z}_t^{(r)}$ denote the direct sum of r copies of $\mathbb{Z}_t$ in which $\mathbb{Z}_t$ is the cyclic group of order t. The following theorem determines the structure of the 2-nilpotent multipliers for finite abelian p-groups, which can be found in [12, Theorem 2.4] and [16, Theorem 2.3].

Theorem 2.3 *Let $G \cong \mathbb{Z}_{p^{m_1}} \oplus \cdots \oplus \mathbb{Z}_{p^{m_k}}$, where $m_1 \geq m_2 \geq \cdots \geq m_k$. Then,*

$$\mathcal{M}^{(2)}(G) \cong \bigoplus_{i=2}^{k} \mathbb{Z}_{p^{m_i}}^{(i^2-i)}.$$

The notion of the epicenter $Z^*(G)$ is defined for a group G by Beyl et al. [5]. They have proved that every group G possesses a uniquely determined central subgroup $Z^*(G)$ which is the smallest central subgroup of G whose factor group is capable. It gives a criterion for detecting capable groups. In fact G is capable if and only if $Z^*(G) = 1$. To prove the 2-capability of groups, we need the following results, which can be found in [6,13].

Let F/R be a free presentation for a group G, and $\pi : F/[R, F, F] \to G$ be the canonical surjection. The subgroup $Z_2^*(G)$ of G is defined as follows:

$$Z_2^*(G) = \pi(Z_2(F/[R, F, F])).$$

Lemma 2.4 [6, Proposition 1.2] *A group G is 2-capable if and only if $Z_2^*(G) = 1$.*

Theorem 2.5 ([13, Theorem 4.4] and [6, Lemma 2.1 (vii)]) *Let N be a normal subgroup of a group G contained in $Z_2(G)$. Then, $N \subseteq Z_2^*(G)$ if and only if the natural map $\mathcal{M}^{(2)}(G) \longrightarrow \mathcal{M}^{(2)}(G/N)$ is a monomorphism.*

Our computations in this paper are based on the classification of the 2-capable 2-generator 2-groups of class two. The following theorem determines all capable 2-generated 2-groups of class two and is an immediate consequence of [11, Theorems 1 and 67].

Theorem 2.6 *Let G be a capable 2-generated 2-group of class two. Then, G is isomorphic to one of the following groups.*

(i) $G_1 = \langle a, b \mid [a,b]^{2^\beta} = a^{2^\beta} = b^{2^\beta} = [a,b,b] = [a,b,a] = 1\rangle$,
(ii) $G_2 = \langle a, b \mid [a,b]^{2^\beta} = a^{2^{\beta+1}} = b^{2^\beta} = [a,b,b] = [a,b,a] = 1\rangle$,
(iii) $G_3 = \langle a, b \mid [a,b]^{2^\gamma} = a^{2^\alpha} = b^{2^\alpha} = [a,b,b] = [a,b,a] = 1, \alpha > \gamma\rangle$,
(iv) $G_4 = \langle a, b \mid [a,b]^{2^\gamma} = a^{2^\alpha} = [a,b,b] = [a,b,a] = 1, b^{2^\beta} = [a,b]^{2^\sigma}\rangle$, *where* $\alpha - \beta > \delta_{\beta\gamma}$, $\alpha - \beta = \gamma - \sigma$, $\alpha > \beta$ *and* $\gamma > \sigma$, δ_{ij} *is the Kronecker delta.*

3 2-Nilpotent Multipliers of All Capable 2-Generator 2-Groups of Class Two

We know that every 2-capable group is capable. Theorem 2.6 shows that there are only four capable 2-generator 2-groups of class two. Now, we just need to discuss the 2-capability of these 2-groups. Here, we intend to obtain the structure of the 2-nilpotent multipliers of all capable 2-generator 2-groups of class two, and then in the next section, we will show that capability and 2-capability for these groups are equivalent. In what follows, by considering every capable 2-generator 2-group G of class two, we compute the 2-nilpotent multiplier of G depending on the nilpotency class and the free presentation of G.

Let G be the quotient of a free group F by a normal subgroup R. Then, we have $\gamma_3(F) \subseteq R$. Hence,

$$\mathcal{M}^{(2)}(G) \cong \frac{R \cap \gamma_3(F)}{[R, F, F]} \cong \frac{\gamma_3(F)/\gamma_5(F)}{[R, F, F]/\gamma_5(F)}.$$

Since G is 2-generator, we conclude that F is the free group on the set $\{a, b\}$. We know that $\gamma_3(F)/\gamma_5(F)$ is a free abelian group with the basis of all basic commutators of weights 3 and 4 on $\{a, b\}$ that is the set

$$\{[a,b,b], [a,b,a], [a,b,a,a], [a,b,a,a], [a,b,b,b]\}, \text{ in which } a > b.$$

The following lemma is obtained by Theorem 2.6.

Lemma 3.1 *Let G be a capable 2-generator 2-group of class two. Then, the group G is isomorphic to one of the following groups:*

(i) $G_1 = \langle a, b \mid [a,b]^{2^\beta} = a^{2^\beta} = b^{2^\beta} = [a,b,b] = [a,b,a] = 1\rangle \cong F/R_1$,
(ii) $G_2 = \langle a, b \mid [a,b]^{2^\beta} = a^{2^{\beta+1}} = b^{2^\beta} = [a,b,b] = [a,b,a] = 1\rangle \cong F/R_2$,
(iii) $G_3 = \langle a, b \mid [a,b]^{2^\gamma} = a^{2^\alpha} = b^{2^\alpha} = [a,b,b] = [a,b,a] = 1, \alpha > \gamma\rangle \cong F/R_3$,
(iv) $G_4 = \langle a, b | [a,b]^{2^\gamma} = a^{2^\alpha} = [a,b,b] = [a,b,a] = 1, b^{2^\beta} = [a,b]^{2^\sigma}\rangle \cong F/R_4$,

in which F is the free group on the set $\{a, b\}$ and R_i denotes normal closure generated by the relations of G_i, for all $1 \leq i \leq 4$.

In what follows, to compute $\mathcal{M}^{(2)}(G_i)$, we will find a suitable basis for the free abelian group $[R_i, F, F]/\gamma_5(F)$ for all $1 \leq i \leq 4$.

Proposition 3.2 *Let G be a capable 2-generator 2-group of class two.*

(i) *If $G_1 = \langle a, b \mid [a,b]^{2^\beta} = a^{2^\beta} = b^{2^\beta} = [a,b,b] = [a,b,a] = 1\rangle$, then $[R_1, F, F] \equiv \langle [a,b,b]^{2^\beta}[a,b,a]^{-2^\beta}, [a,b,a,a]^{2^{\beta-1}}[a,b,a]^{2^\beta}, [a,b,a]^{2^\beta}[a,b,b,b]^{2^{\beta-1}}, [a,b,a]^{2^{\beta+1}}, [a,b,a]^{2^\beta}[a,b,b,a]^{2^{\beta-1}}\rangle$ mod $\gamma_5(F)$*

(ii) *If $G_2 = \langle a, b \mid [a,b]^{2^\beta} = a^{2^{\beta+1}} = b^{2^\beta} = [a,b,b] = [a,b,a] = 1\rangle$, then $[R_2, F, F] \equiv \langle [a,b,b]^{2^{\beta+1}}, [a,b,a]^{2^{\beta+1}}, [a,b,a,a]^{2^\beta}, [a,b,b]^{2^\beta}[a,b,b,b]^{2^{\beta-1}}, [a,b,a]^{2^\beta}[a,b,b,a]^{2^{\beta-1}}\rangle$, mod $\gamma_5(F)$.*

(iii) *If $G_3 = \langle a, b \mid [a,b]^{2^\gamma} = a^{2^\alpha} = b^{2^\alpha} = [a,b,b] = [a,b,a] = 1, \alpha > \gamma\rangle$, then $[R_3, F, F] \equiv \langle [a,b,b]^{2^\alpha}, [a,b,a]^{2^\alpha}, [a,b,a,a]^{2^\gamma}, [a,b,b,b]^{2^\gamma}, [a,b,b,a]^{2^\gamma}\rangle$, mod $\gamma_5(F)$, where $\alpha > \gamma$.*

(iv) *If $G_4 = \langle a, b \mid [a,b]^{2^\gamma} = a^{2^\alpha} = [a,b,b] = [a,b,a] = 1, b^{2^\beta} = [a,b]^{2^\sigma}\rangle$, then $[R_4, F, F] \equiv \langle [a,b,a]^{2^\alpha}, [a,b,b,a]^{2^\sigma}, [a,b,b,b]^{2^\sigma}, [a,b,a]^{2^\beta}[a,b,b,a]^{2^\sigma}, [a,b,b]^{2^\beta}\rangle$, mod $\gamma_5(F)$, where $\alpha - \beta > \delta_{\beta\gamma}$, $\alpha - \beta = \gamma - \sigma$, $\alpha > \beta$ and $\gamma > \sigma$, δ_{ij} is Kronecker delta.*

Proof (i) Clearly,

$$[R_1, F, F]/\gamma_5(F) = \langle [a^{2^\beta}, f_1, f_2], [b^{2^\beta}, f_3, f_4], [[a,b]^{2^\beta}, f_5, f_6] \mid f_i \in F, 1 \leq i \leq 6\rangle \gamma_5(F)/\gamma_5(F).$$

By using [16, Lemma 3.3], we have the following relations

$$\begin{aligned}
[a^{2^\beta}, f_1, f_2] &\equiv [a, f_1, f_2]^{2^\beta}[a, f_1, a, f_2]^{\binom{2^\beta}{2}} \\
&\equiv ([a, f_1, f_2]^2[a, f_1, a, f_2]^{(2^\beta - 1)})^{2^{\beta-1}} \pmod{\gamma_5(F)}, \\
[b^{2^\beta}, f_3, f_4] &\equiv [b, f_3, f_4]^{2^\beta}[b, f_3, b, f_4]^{\binom{2^\beta}{2}} \\
&\equiv ([b, f_3, f_4]^2[b, f_3, b, f_4]^{(2^\beta - 1)})^{2^{\beta-1}} \pmod{\gamma_5(F)}, \\
[[a,b]^{2^\beta}, f_5, f_6] &\equiv [a, b, f_5, f_6]^{2^\beta} \pmod{\gamma_5(F)}.
\end{aligned}$$

For any element $f \in F$, $f = a^n b^m [a,b]^t s$, such that $s \in \gamma_3(F)$ and $n, m, t \in \mathbb{Z}$. By using the Hall–Witt identity and easy commutator calculations, we have

$$\begin{aligned}
&[b,a,b] \equiv [a,b,b]^{-1}, [b,a,b,b] \equiv [a,b,b,b]^{-1} \pmod{\gamma_5(F)}, \\
&[b,a,a] \equiv [a,b,a]^{-1}, [b,a,b,a] \equiv [a,b,b,a]^{-1}, [a,b,a,b] \equiv [a,b,b,a] \\
&\qquad \pmod{\gamma_5(F)}.
\end{aligned}$$

Now, we conclude that $[R_1, F, F]/\gamma_5(F)$ is generated by the set

$$\{[a,b,b]^{2^\beta}[a,b,b,a]^{(2^\beta-1)2^{\beta-1}}, [a,b,a]^{2^\beta}[a,b,a,a]^{(2^\beta-1)2^{\beta-1}}, [a,b,a,a]^{2^\beta},\\ [a,b,b,a]^{2^\beta}, [a,b,b,b]^{2^\beta},\\ [a,b,b]^{-2^\beta}[a,b,b,b]^{-2^{\beta-1}(2^\beta-1)}, [a,b,a]^{-2^\beta}[a,b,b,a]^{-2^{\beta-1}(2^\beta-1)}\}$$

modulo $\gamma_5(F)$.

We also know that

$$[a,b,b,a]^{(2^\beta-1)2^{\beta-1}}[a,b,b,a]^{2^\beta} = [a,b,b,a]^{2^{\beta-1}(2^\beta+1)},$$

and so

$$[a,b,b]^{2^\beta}[a,b,b,a]^{(2^\beta-1)2^{\beta-1}}[a,b,b,a]^{2^\beta}\\ = [a,b,b]^{2^\beta}[a,b,b,a]^{2^{\beta-1}}([a,b,b,a]^{2^\beta})^{2^{\beta-1}}.$$

By a similar process, we have

$$[a,b,a]^{2^\beta}[a,b,a,a]^{(2^\beta-1)2^{\beta-1}}[a,b,a,a]^{2^\beta}\\ = [a,b,a]^{2^\beta}[a,b,a,a]^{2^{\beta-1}}([a,b,a,a]^{2^\beta})^{2^{\beta-1}},\\ [a,b,b]^{2^\beta}[a,b,b,b]^{2^{\beta-1}(2^\beta-1)}[a,b,b,b]^{2^\beta}\\ = [a,b,b]^{2^\beta}[a,b,b,b]^{2^{\beta-1}}([a,b,b,b]^{2^\beta})^{2^{\beta-1}},\\ [a,b,a]^{2^\beta}[a,b,b,a]^{2^{\beta-1}(2^\beta-1)}[a,b,b,a]^{2^\beta}\\ = [a,b,a]^{2^\beta}[a,b,b,a]^{2^{\beta-1}}([a,b,b,a]^{2^\beta})^{2^{\beta-1}}.$$

Therefore, $[R_1, F, F]/\gamma_5(F)$ is generated by the set

$$\{[a,b,b]^{2^\beta}[a,b,b,a]^{2^{\beta-1}}, [a,b,a]^{2^\beta}[a,b,a,a]^{2^{\beta-1}}, [a,b,a,a]^{2^\beta}, [a,b,b,b]^{2^\beta},\\ [a,b,b,a]^{2^\beta}, [a,b,b]^{-2^\beta}[a,b,b,b]^{-2^{\beta-1}}, [a,b,a]^{-2^\beta}[a,b,b,a]^{-2^{\beta-1}},\\ [a,b,a,a]^{2^\beta}\}.$$

By a similar method, we can obtain that

$$\{([a,b,b][a,b,a]^{-1})^{2^\beta}, ([a,b,a]^2[a,b,a,a])^{2^{\beta-1}}, ([a,b,a]^2[a,b,b,b])^{2^{\beta-1}},\\ [a,b,a]^{2^{\beta+1}}, ([a,b,a]^2[a,b,b,a])^{2^{\beta-1}}\}.$$

is a basis of $[R_1, F, F]/\gamma_5(F)$.

Now the proof is complete.

(ii) We have

$$[R_2, F, F] = \langle [a^{2^{\beta+1}}, f_1, f_2], [b^{2^\beta}, f_3, f_4], [[a, b]^{2^\beta}, f_5, f_6] \mid f_i \in F, 1 \le i \le 6\rangle$$

modulo $\gamma_5(F)$. Using a similar method to the proof of part (i), $[R_2, F, F]/\gamma_5(F)$ is generated by the set

$$\begin{aligned}&\{\big([a, b, b]^2[a, b, a, b]^{(2^{\beta+1}-1)}\big)^{2^\beta}, \big([a, b, a]^2[a, b, a, a]^{(2^{\beta+1}-1)}\big)^{2^\beta},\\ &\quad [a, b, a, a]^{2^\beta}, [a, b, b, a]^{2^\beta},\\ &[a, b, b, b]^{2^\beta}, \big([b, a, b]^2[b, a, b, b]^{(2^\beta-1)}\big)^{2^{\beta-1}}, \big([b, a, a]^2[b, a, b, a]^{(2^\beta-1)}\big)^{2^{\beta-1}}\}\end{aligned}$$

modulo $\gamma_5(F)$. One can easily check that

$$\begin{aligned}[R_2, F, F] \equiv &\langle [a, b, b]^{2^{\beta+1}}, [a, b, a]^{2^{\beta+1}}, [a, b, a, a]^{2^\beta}, ([a, b, b]^2[a, b, b, b])^{2^{\beta-1}},\\ &([a, b, a]^2[a, b, b, a])^{2^{\beta-1}}\rangle\end{aligned}$$

modulo $\gamma_5(F)$. Now the proof is complete.
For (iii) and (iv), we use a similar method to parts (i) and (ii). Then, for $\alpha > \gamma$,

$$[R_3, F, F] \equiv \langle [a, b, b]^{2^\alpha}, [a, b, a]^{2^\alpha}, [a, b, a, a]^{2^\gamma}, [a, b, b, b]^{2^\gamma}, [a, b, b, a]^{2^\gamma}\rangle,$$

mod $\gamma_5(F)$ and for $\alpha - \beta > \delta_{\beta\gamma}, \alpha - \beta = \gamma - \sigma, \alpha > \beta$ and $\gamma > \sigma$,

$$\begin{aligned}[R_4, F, F] \equiv &\langle [a, b, b]^{2^\beta}, [a, b, a]^{2^\alpha}, [a, b, b, a]^{2^\sigma}, [a, b, b, b]^{2^\sigma},\\ &[a, b, a]^{2^\beta}[a, b, b, a]^{2^\sigma}\rangle\end{aligned}$$

mod $\gamma_5(F)$, where δ_{ij} is the Kronecker delta. The proof is completed. □

So far, all necessary information is gathered and now we are ready to compute the 2-nilpotent multipliers of the groups G_1, G_2, G_3 and G_4, which are introduced in Lemma 3.1.

Theorem 3.3 *Let G be isomorphic to the group G_i, $1 \le i \le 4$, which is mentioned in Lemma 3.1. Then,*

(i) $\mathcal{M}^{(2)}(G_1) \cong \mathbb{Z}^{(3)}_{2^{\beta-1}} \oplus \mathbb{Z}_{2^\beta} \oplus \mathbb{Z}_{2^{\beta+1}}$.
(ii) $\mathcal{M}^2(G_2) \cong \mathbb{Z}^{(2)}_{2^{\beta+1}} \oplus \mathbb{Z}^{(2)}_{2^{\beta-1}} \oplus \mathbb{Z}_{2^\beta}$.
(iii) $\mathcal{M}^2(G_3) \cong \mathbb{Z}^{(2)}_{2^\alpha} \oplus \mathbb{Z}^{(3)}_{2^\gamma}$.
(iv) $\mathcal{M}^{(2)}(G_4) \cong \mathbb{Z}_{2^\alpha} \oplus \mathbb{Z}_{2^\beta} \oplus \mathbb{Z}^{(3)}_{2^\sigma}$.

Proof (i) We know that the set

$$\begin{aligned}&\{[a, b, a]^2[a, b, a, a], [a, b, a]^2[a, b, b, b], [a, b, a], [a, b, a]^2[a, b, b, a],\\ &\quad [a, b, b][a, b, a]^{-1}\}\end{aligned}$$

is a basis for the free abelian group $\gamma_3(F)/\gamma_5(F)$ and also

$$[R_1, F, F] \equiv \langle ([a,b,b][a,b,a]^{-1})^{2^\beta}, [a,b,a]^{2^{\beta+1}}, ([a,b,a]^2[a,b,a,a])^{2^{\beta-1}},\\ ([a,b,a]^2[a,b,b,a])^{2^{\beta-1}},\\ ([a,b,a]^2[a,b,b,b])^{2^{\beta-1}}\rangle \bmod \gamma_5(F)$$

according to Proposition 3.2(i). Therefore,

$$\mathcal{M}^{(2)}(G_1) \cong \frac{\langle [a,b,a]^2[a,b,a,a]\rangle}{\langle \big([a,b,a]^2[a,b,a,a]\big)^{2^{\beta-1}}\rangle} \oplus \frac{\langle [a,b,a]^2[a,b,b,b]\rangle}{\langle \big([a,b,a]^2[a,b,a,a]\big)^{2^{\beta-1}}\rangle}\\ \oplus \frac{\langle [a,b,a]^2[a,b,b,a]\rangle}{\langle \big([a,b,a]^2[a,b,b,a]\big)^{2^{\beta-1}}\rangle} \oplus \frac{\langle [a,b,b][a,b,a]^{-1}\rangle}{\langle ([a,b,b][a,b,a]^{-1})^{2^\beta}\rangle} \oplus \frac{\langle [a,b,a]\rangle}{\langle [a,b,a]^{2^{\beta+1}}\rangle}\\ \cong \mathbb{Z}^{(3)}_{2^{\beta-1}} \oplus \mathbb{Z}_{2^\beta} \oplus \mathbb{Z}_{2^{\beta+1}}.$$

To prove (ii), (iii) and (iv), we use the same process as the proof of part (i). It can be concluded that the results hold for the remaining groups G_2, G_3 and G_4. □

4 2-Capability of 2-Generator 2-Groups of Class Two

In this section, we detect the 2-generator 2-groups of class two which are 2-capable. The following results are used in the proof of Theorem 4.3.

Proposition 4.1 *Let G be isomorphic to one of the following groups*

(i) $G_1 = \langle a, b \mid [a,b]^{2^\beta} = a^{2^\beta} = b^{2^\beta} = [a,b,b] = [a,b,a] = 1\rangle$,
(ii) $G_2 = \langle a, b \mid [a,b]^{2^\beta} = a^{2^{\beta+1}} = b^{2^\beta} = [a,b,b] = [a,b,a] = 1\rangle$,
(iii) $G_3 = \langle a, b \mid [a,b]^{2^\gamma} = a^{2^\alpha} = b^{2^\alpha} = [a,b,b] = [a,b,a] = 1, \alpha > \gamma\rangle$.

Then, $Z_2^*(G)$ is trivial.

Proof Let $G \cong G_1$. Then, $G_1/G_1' \cong \mathbb{Z}^{(2)}_{2^\beta}$ and so [6, Theorem 1.3] implies that G_1/G_1' is 2-capable. Now, using [13, Theorem 2.3], we conclude that $Z_2^*(G_1) \subseteq \langle [a,b]\rangle$. Assume $x = [a,b]^r$ such that $1 \le r \le 2^\beta - 1$. If $gcd(2,r) = 1$, then $\langle x\rangle = \langle [a,b]\rangle$. Let $gcd(2,r) \neq 1$. Then $r = 2^t r_1$ for some $t > 1$ such that $gcd(r_1, 2) = 1$ and so $\langle x\rangle = \langle [a,b]^{2^t}\rangle$ for some $t > 1$.

Consider $\overline{y}$ to denote the image of $y \in G_1$ in $G_1/\langle x\rangle$. Then,

$$G_1/\langle x\rangle = \langle \overline{a}, \overline{b} | \overline{a}^{2^\beta} = \overline{b}^{2^\beta} = [\overline{a}, \overline{b}]^{2^{\beta-t}} = [\overline{a}, \overline{b}, \overline{a}] = [\overline{a}, \overline{b}, \overline{b}] = 1, t > 1\rangle$$

or $G_1/\langle x\rangle = G_1/G_1'$. According to Theorems 2.3, 3.3(i) and (iii), we conclude that $\mathcal{M}^{(2)}(G_1/G_1') \cong \mathbb{Z}^{(2)}_{2^\beta}$, $\mathcal{M}^{(2)}(G_1/\langle x\rangle) \cong \mathbb{Z}^{(2)}_{2^\beta} \oplus \mathbb{Z}^{(3)}_{2^{\beta-t}}$ and $\mathcal{M}^{(2)}(G_1) \cong \mathbb{Z}^{(3)}_{2^{\beta-1}} \oplus \mathbb{Z}_{2^\beta} \oplus \mathbb{Z}_{2^{\beta+1}}$. Since $|\mathcal{M}^{(2)}(G_1)| > |\mathcal{M}^{(2)}(G_1/\langle x\rangle)|$, we get $\mathcal{M}^{(2)}(G_1) \longrightarrow \mathcal{M}^{(2)}(G_1/\langle x\rangle)$

is not a monomorphism and so $x \notin Z_2^*(G_1)$, by Theorem 2.5. Hence, $Z_2^*(G_1) = 1$. Now, let $G \cong G_2$. Then using [11, Section 5.1 p.28], $Z(G_2) = \langle a^{2^\beta}, [a, b]\rangle$. Therefore, $\langle a^{2^\beta}\rangle \trianglelefteq G$ and $\langle a^{2^\beta}\rangle \cong \mathbb{Z}_2$. We denote the image of $y \in G_2$ in $G_2/\langle a^{2^\beta}\rangle$ by $\overline{y}$. Since

$$G_2/\langle a^{2^\beta}\rangle = \langle \overline{a}, \overline{b} \mid \overline{a}^{2^\beta} = \overline{b}^{2^\beta} = [\overline{a}, \overline{b}]^{2^\beta} = [\overline{a}, \overline{b}, \overline{a}] = [\overline{a}, \overline{b}, \overline{b}] = 1\rangle \cong G_1,$$

$Z_2^*(G_2/\langle a^{2^\beta}\rangle) = 1$, and so Lemma 2.4 implies that $G_2/\langle a^{2^\beta}\rangle$ is 2-capable. Now by [13, Theorem 2.3], $Z_2^*(G_2) \subseteq \langle a^{2^\beta}\rangle$. Using the results of Theorem 3.3, it is straightforward that $|\mathcal{M}^{(2)}(G_2)| > |\mathcal{M}^{(2)}(G_2/\langle a^{2^\beta}\rangle)|$. As a consequence, the map $\mathcal{M}^{(2)}(G_2) \longrightarrow \mathcal{M}^{(2)}(G_2/\langle a^{2^\beta}\rangle)$ is not a monomorphism and so $Z_2^*(G_2) \neq \langle a^{2^\beta}\rangle$, by Theorem 2.5. Hence, $Z_2^*(G_2)$ is trivial. By a similar process, we can obtain $Z_2^*(G_3) = 1$. The result follows. □

Proposition 4.2 *Let*

$$G_4 = \langle a, b \mid [a, b]^{2^\gamma} = a^{2^\alpha} = [a, b, b] = [a, b, a] = 1, b^{2^\beta} = [a, b]^{2^\sigma}\rangle,$$

where $\alpha - \beta > \delta_{\beta\gamma}$, $\alpha - \beta = \gamma - \sigma$, $\alpha > \beta$ and $\gamma > \sigma$, δ_{ij} is the Kronecker delta. Then, $Z_2^*(G_4)$ is trivial.

Proof Using [11, Section 5.1 p.28], we have $Z(G_4) = \langle a^{2^\gamma}, [a, b], b^{2^\gamma}\rangle$. Therefore, $\langle a^{2^\beta}, b^{2^\beta}\rangle \trianglelefteq G_4$. We denote the image of $y \in G_4$ in $G_4/\langle a^{2^\beta}, b^{2^\beta}\rangle$ by $\tilde{y}$. As a result,

$$G_4/\langle a^{2^\beta}, b^{2^\beta}\rangle = \langle \tilde{a}, \tilde{b} \mid \tilde{a}^{2^\beta} = \tilde{b}^{2^\beta} = [\tilde{a}, \tilde{b}]^{2^\sigma} = [\tilde{a}, \tilde{b}, \tilde{a}] = [\tilde{a}, \tilde{b}, \tilde{b}] = 1, \beta > \sigma\rangle \cong G_3.$$

By Proposition 4.1, we know that $G_4/\langle a^{2^\beta}, b^{2^\beta}\rangle$ is 2-capable and consequently by [13, Theorem 2.3], $Z_2^*(G_4) \subseteq \langle a^{2^\beta}, b^{2^\beta}\rangle$. We claim that $Z_2^*(G_4) = 1$. By contrary, let $1 \neq d \in Z_2^*(G_4)$ be an arbitrary element. Then, $d = a^{i2^\beta} b^{j2^\beta}$ such that i and j are integers. Then,

$$\begin{aligned} H = G_4/\langle d\rangle \cong \langle a_1, b_1 \mid a_1^{2^\alpha} = b_1^{2^\alpha} = [a_1, b_1]^{2^\gamma} = 1, b_1^{2^\beta} = [a_1, b_1]^{2^\sigma}, \\ a_1^{i2^\beta} = b_1^{-j2^\beta}, [a_1, b_1, b_1] = [a_1, b_1, a_1] = 1\rangle \end{aligned}$$

Suppose that $i = i'2^{k_2}$, $j = j'2^{k_1}$ and $gcd(2, i') = gcd(2, j') = 1$, where k_1 and k_2 are integers. As a consequence,

$$\begin{aligned} H = \langle a_1, b_1 \mid a_1^{2^\alpha} = b_1^{2^\alpha} = [a_1, b_1]^{2^\gamma} = 1, b_1^{2^\beta} = [a_1, b_1]^{2^\sigma}, a_1^{i'2^{\beta+k_2}} = b_1^{-j'2^{\beta+k_1}}, \\ [a_1, b_1, b_1] = [a_1, b_1, a_1] = 1\rangle \end{aligned}$$

Without loss of generality, take $i' = j' = 1$, by [11, Proposition 3.1]. Therefore,

$$\begin{aligned} H = \langle a_1, b_1 \mid a_1^{2^\alpha} = b_1^{2^\alpha} = [a_1, b_1]^{2^\gamma} = 1, b_1^{2^\beta} = [a_1, b_1]^{2^\sigma}, a_1^{2^{\beta+k_2}} = b_1^{-2^{\beta+k_1}}, \\ [a_1, b_1, b_1] = [a_1, b_1, a_1] = 1\rangle. \end{aligned}$$

Since $|a_1^{2^{\beta+k_2}}| = |b_1^{2^{\beta+k_1}}|$, we obtain $k_1 = k_2$. Putting $k_1 = k_2 = k$, we have

$$H = \langle a_1, b_1 \mid a_1^{2^\alpha} = b_1^{2^\alpha} = [a_1, b_1]^{2^\gamma} = 1, b_1^{2^\beta} = [a_1, b_1]^{2^\sigma}, a_1^{2^{\beta+k}} = b_1^{-2^{\beta+k}},$$
$$[a_1, b_1, b_1] = [a_1, b_1, a_1] = 1\rangle.$$

On the other hand, $\langle a_1^{2^{\beta+k}}\rangle \trianglelefteq H$, since $a_1^{2^{\beta+k}} \in Z(H)$. We denote the image of $y \in H$ in $H/\langle a_1^{2^{\beta+k}}\rangle$ by $\overline{y}$. As a result,

$$H_1 = H/\langle a_1^{2^{\beta+k}}\rangle = \langle \overline{a_1}, \overline{b_1} \mid \overline{a_1}^{2^{\beta+k}} = [\overline{a_1}, \overline{b_1}]^{2^{\sigma+k}} = 1, \overline{b_1}^{-2^\beta} = \overline{[a_1, b_1]}^{2^\sigma},$$
$$[\overline{a_1}, \overline{b_1}, \overline{b_1}] = [\overline{a_1}, \overline{b_1}, \overline{a_1}] = 1\rangle.$$

If $\beta + k = \alpha$, then $H = H_1 = G_4$, so $d = 1$. Thus it quickly becomes apparent that $\beta + k < \alpha$. It is a contradiction. Now by using [16, Theorem 2.1 (i) (b)], the sequence

$$\mathcal{M}^{(2)}(H) \xrightarrow{\eta} \mathcal{M}^{(2)}(H_1) \longrightarrow \langle a_1^{2^{\beta+k}}\rangle \cap \gamma_3(H) \longrightarrow 1$$

is exact. Since H is of class two, we have $\langle a_1^{2^{\beta+k}}\rangle \cap \gamma_3(H) = 1$ and so the homomorphism η is surjective. By using [5, Proposition 1.1] and [6, Lemma 2.1 (viii)], we get $a_1^{2^{\beta+k}} \in Z^*(H) \subseteq Z_2^*(H)$. We simply note that the homomorphism η is the monomorphism, by applying Theorem 2.5. As a result,

$$\mathcal{M}^{(2)}(G_4/\langle d\rangle) \cong \mathcal{M}^{(2)}(H) \cong \mathcal{M}^{(2)}(H_1) \cong \mathbb{Z}_{2^{\beta+k}} \oplus \mathbb{Z}_{2^\beta} \oplus \mathbb{Z}_{2^\sigma}^{(3)}$$

and

$$\mathcal{M}^{(2)}(G_4) \cong \mathbb{Z}_{2^\alpha} \oplus \mathbb{Z}_{2^\beta} \oplus \mathbb{Z}_{2^\sigma}^{(3)},$$

by Theorem 3.3(iv). Now it can be simply deduced that $|\mathcal{M}^{(2)}(G_4)| > |\mathcal{M}^{(2)}(G_4/\langle d\rangle)|$. Theorem 2.5 implies that $d \notin Z_2^*(G_4)$ which is a contradiction since $1 \neq d \in Z_2^*(G_4)$. Consequently, $Z_2^*(G_4) = 1$, as required. □

In what follows, we determine all 2-capable 2-generated 2-groups of class two.

Theorem 4.3 *Let G be a 2-generated 2-group of class two. Then, G is 2-capable if and only if $G \cong G_1$, $G \cong G_2$, $G \cong G_3$ or $G \cong G_4$.*

Proof Let G be 2-capable. Then, G is capable. Theorem 2.6 implies that $G \cong G_1$, $G \cong G_2$, $G \cong G_3$ or $G \cong G_4$. The converse holds by Lemma 2.4 and Propositions 4.1 and 4.2 . □

Corollary 4.4 *Let G be a 2-generator 2-group of class two. Then, G is capable if and only if G is 2-capable.*

References

1. Ahmad, A., Magidin, A., Morse, R.F.: Two generator p-groups of nilpotency class 2 and their conjugacy classes. Publ. Math. Debr. **81**(1–2), 145–166 (2012)
2. Bacon, M., Kappe, L.-C.: The non-abelian tensor square of a 2-generator p-group of class 2. Arch. Math. **61**, 508–516 (1993)
3. Baer, R.: Groups with preassigned central and central quotient group. Trans. Am. Math. Soc. **44**, 387–412 (1938)
4. Baer, R.: Representations of groups as quotient groups, I, II, and III. Trans. Am. Math. Soc. **58**, 295–419 (1945)
5. Beyl, F.R., Felgner, U., Schmid, P.: On groups occurring as center factor groups. J. Algebra **61**, 161–177 (1979)
6. Burns, J., Ellis, G.: On the nilpotent multipliers of a group. Math. Z. **226**, 405–428 (1997)
7. Hall, P.: The classification of prime-power groups. J. Reine Angew. Math. **182**, 130–141 (1940)
8. Kappe, L.C., Visscher, M.P., Sarmin, N.H.: Two-generator 2-groups of class 2 and their non-abelian tensor squares. Glasg. Math. J. **41**, 417–430 (1999)
9. Karpilovsky, G.: The Schur Multiplier, London Mathematical Society Monographs, New Series, vol. 2. Clarendon Press, Oxford (1987)
10. Magidin, A.: Capable 2-generator 2-groups of class two. Commun. Algebra **34**, 2183–2193 (2006)
11. Magidin, A., Morse, R.F.: Certain homological functors of 2-generator p-groups of class 2. Contemp. Math. **511**, 127–166 (2010)
12. Mashayekhy, B., Moghaddam, M.R.R.: Higher Schur multiplicator of a finite abelian group. Algebra Colloq. **4**(3), 317–322 (1997)
13. Moghaddam, M.R.R., Kayvanfar, S.: A new notion derived from varieties of groups. Algebra Colloq. **4**(1), 1–11 (1997)
14. Moghaddam, M.R.R.: The Baer-invariant of a direct product. Arch. Math. **33**, 504–511 (1980)
15. Niroomand, P.: On the order of Schur multiplier of non-abelian p-groups. J. Algebra **322**, 4479–4482 (2009)
16. Niroomand, P., Parvizi, M.: On the 2-nilpotent multiplier of finite p-groups. Glasg. Math. J. **57**(1), 201–210 (2015)
17. Niroomand, P., Johari, F., Parvizi, M.: On the c-nilpotent multiplier of finite p-groups. Bull. Malays. Math. Sci. Soc. (2019). https://doi.org/10.1007/s40840-019-00723-x
18. Trebenko, D.Ya.: Nilpotent groups of class two with two generators. Current analysis and its applications, "Naukova Dumka". Kiev **201–208**, 228 (1989). (Russian)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.