

Statistical inference on the basis of sequential order statistics under a linear trend for conditional proportional hazard rates

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Abstract

This paper deals with systems consisting of independent and heterogeneous exponential components. Since failures of components may change lifetimes of surviving components because of load sharing, a linear trend for conditionally proportional hazard rates is considered. Estimates of parameters, both point and interval estimates, are derived on the basis of observed component failures for $s(\geq 2)$ systems. Fisher information matrix of the available data is also obtained which can be used for studying asymptotic behaviour of estimates. The generalized likelihood ratio test is implemented for testing homogeneity of s systems. Illustrative examples are also given.

Key Words: Dynamic systems; Linear trend model; Exponential distribution; Hazard function

Mathematics Subject Classification: 62G30, 60K10, 62P30, 62N05.

1 Introduction

Let X_1, \dots, X_n be independent and identically distributed (i.i.d.) random variables with a common [distribution function \(DF\)](#), say F , and abbreviated by $X_1, \dots, X_n \stackrel{i.i.d.}{\sim} F$. Denote in magnitude order of X_1, \dots, X_n by $X_{1:n} \leq \dots \leq X_{n:n}$, which are known as *order statistics* (OSs). The theory of OSs has been widely studied in literature specially in system reliability analyses. For example, lifetimes of known r -out-of- n systems coincide to $X_{r:n}$ where X_1, \dots, X_n stand for component lifetimes; [For more information](#), see Barlow and Proschan [3] and David and Nagaraja [11] and references therein. In order to introduce more flexible models for analysing practical systems, various generalizations of OSs such as *fractional order statistics* and *generalized order statistics* have been proposed. The former is useful for providing more flexible tools and the later is a setting

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to unify similar results (David and Nagaraja [11], p. 21). In this paper, we deal with another unified concept, called *sequential order statistics* (SOS). There is also another motivation in reliability analyses for implementing SOS. Specifically, when component lifetimes are i.i.d., the OSs are suitable for describing the r -out-of- n system lifetime. Thus failing a component does not effect the DFs of lifetimes of surviving components. As motivated by Cramer and Kamps [7], in practice the failure of a component may result in a higher load on the surviving components and hence causes lifetime distributions change. This property may be due to load sharing and/or common working environments and hence dependent component lifetimes. More precisely, suppose that F_j and f_j , for $j = 1, \dots, n$, denote the common DF and **probability density function (PDF)** of the component lifetimes when $n-j+1$ components are jointly working. Then, the components begin to work independently at time $t = 0$ with the common DF F_1 . When at time x_1 , the first component failure occurs, the remaining $n - 1$ components are working with the (left truncated) common DF F_2 at x_1 . This process continues up to $n - r + 1$ components with the common DF F_r work until the r -th failure occurs at time x_r and hence the whole system fails. This system is called *sequential r -out-of- n system* (or dynamic system) and the system lifetime is then r -th observed component failure time, denoted by $X_{(r)}$. In the literature, $(X_{(1)}, \dots, X_{(r)})$ is called SOSs. Let $\mathbf{x} = (x_1, \dots, x_r)$ be the observed values from SOSs $(X_{(1)}, \dots, X_{(r)})$ with DFs (F_1, \dots, F_r) and PDFs (f_1, \dots, f_r) of component lifetimes. The joint PDF of $X_{(1)}, \dots, X_{(r)}$ ($r \leq n$) is

$$L(F_1, \dots, F_r; \mathbf{x}) = \left(\frac{n!}{(n-r)!} \right) \left\{ \prod_{j=1}^{r-1} f_j(x_j) \left(\frac{\bar{F}_j(x_j)}{\bar{F}_{j+1}(x_j)} \right)^{n-j} \right\} f_r(x_r) \bar{F}_r(x_r)^{n-r}, \quad (1)$$

where $x_1 < x_2 < \dots < x_n$ and $\bar{F}_j(x_j) = 1 - F_j(x_j)$, $j = 1, \dots, r$. Statistical properties of SOSs have been studied by Kamps [15, 16], Cramer and Kamps [7, 8], Balakrishnan *et al.* [1], Beutner and Kamps [6], Esmailian and Doostparast [12], Bedbur [5], Hashempour and Doostparast [13] and references therein. In Equation (1), F_j ($j = 1, \dots, r$) depends usually on some unknown parameters. Let the number of independent parameters of F_j be k_j . Thus, Equation (1) depends on $\sum_{j=1}^r k_j$ parameters. We consider a simplified statistical model to reduce the dimension of the parameter space in Equation (1) which is called proportional hazard rate model.

A large family of models introduced by Cox (1972) focuses directly on the hazard rate function. The simple member of the family is the proportional hazard rate model. Different kinds of proportional hazard models may be obtained by making different assumptions about the baseline survival function, or equivalently, the baseline hazard rate function. Let $F_0(\cdot)$ be a absolutely continuous DF with a corresponding PDF $f_0(\cdot)$. The hazard rate function is defined by $h_0(t) = f_0(t)/\bar{F}_0(t)$ for $t > 0$, where $\bar{F}_0(t) = 1 - F_0(t)$ is the survival function of the DF $F_0(\cdot)$. If X is a member of proportional hazard family with the baseline DF $F_0(\cdot)$, then the survival function of X becomes $\bar{F}(t; \theta) = \bar{F}_0^\theta(t)$, $t \in S$, where θ is the proportional parameter and $F_0(\cdot)$ is the baseline DF and S is the support of the baseline DF. In this case, the hazard rate function of X is given by $h(t; \theta) = f(t; \theta)/\bar{F}(t; \theta) = \theta h_0(t)$ for $t > 0$.

In this paper, we consider the problem of estimating the parameters on the basis of s (≥ 2) independent SOSs samples under a proposed linear trend conditional propor-

tional hazard rates (LTCPHR) model, defined by $\bar{F}_j(t) = \bar{F}_0^{aj}(t)$ for $j = 1, \dots, r$, where $aj = a \times j$, $a > 0$ and $F_0(t)$ is the underlying DF. Remember that the hazard rate function of the DF F defined by $h(t) = f(t)/\bar{F}(t)$ for $t > 0$, where $f(t) = \partial F(t)/\partial t$ is the probability density function (PDF) of the DF $F(t)$. Therefore, $h_j(t) = ajh_0(t)$, for $j = 1, \dots, r$, is the proportional hazard rate function of the DF F_j , where $h_0(t)$ is hazard rate function of the baseline DF F_0 for all t . The LTCPHR model is a new defined statistical concept for modelling engineering systems in which components share the system load. In fact, impact of failing a component on the surviving components are modeled via hazard rate components. Notice that F_i is the common component distribution function when $n - i + 1$ components are jointly working. The connection between F_i is done by assuming a proportional hazard rate among them. Hence, it is called conditionally proportional hazard rate models.

In this paper, we consider the problem of the estimation parameters of the LTCPHR model with independent multiple SOS samples coming from heterogeneous exponential populations. Thus, this paper is organized as follows: In Section 2, the maximum likelihood estimates (MLEs) of parameters are derived and the generalized likelihood ratio test (GLRT) is used for testing homogeneity of the parent exponential populations. In Section 3, we analyse a simulated data set. Finally, some concluding remarks are given in Section 4.

2 Statistical inference for the LTCPHR model parameters

In this section, we obtain MLEs of LTCPHR model parameters. Also, GLRT is derived for testing homogeneity of populations. To do these, two scenarios, namely, (i) the parameter a is known; and (ii) the parameter a is unknown, are considered.

2.1 Maximum likelihood estimation

Suppose that we observed s (≥ 2) independent heterogeneous SOS samples. The available data may be represented as

$$\mathbf{x} = [[x_{ij}]]_{i=1, \dots, s, j=1, \dots, r}, \quad (2)$$

where the i -th row of the matrix \mathbf{x} in (2) denotes the SOS sample coming from the i -th system. The likelihood function (LF) of the available data given by (2) is then derived from (4) as

$$L(\mathcal{F}; \mathbf{x}) = \left(\frac{n!}{(n-r)!} \right)^s \prod_{i=1}^s \left(\prod_{j=1}^{r-1} \left[f_j^{[i]}(x_{ij}) \left(\frac{\bar{F}_j^{[i]}(x_{ij})}{\bar{F}_{j+1}^{[i]}(x_{ij})} \right)^{n-j} \right] f_r^{[i]}(x_{ir}) \bar{F}_r^{[i]}(x_{ir})^{n-r} \right), \quad (3)$$

where $\mathcal{F} = \{F_j^{[i]}, i = 1, \dots, s, j = 1, \dots, r\}$ and for $i = 1, \dots, s, j = 1, \dots, r$, $\bar{F}_j^{[i]}(x) = 1 - F_j^{[i]}(x)$, $F_j^{[i]}$ and $f_j^{[i]}$ denote the survival function, DF and PDF of j th component lifetime of the i -th dynamic system, respectively. For more details, see Cramer and Kamps [8, 9] and Hashempour and Doostparast [13]. Suppose that the baseline DF of the

i -th dynamic system ($i = 1, \dots, s$) follows the exponential distribution with the mean σ_i , i.e.

$$F_0(t) = 1 - \exp\left\{-\frac{t}{\sigma_i}\right\}, \quad t > 0, \quad \sigma_i > 0. \quad (4)$$

It should be mentioned that the baseline DFs for the considered LTCPHR model for component lifetimes are heterogeneous with different scale parameters. Therefore under the earlier mentioned LTCPHR model in Section 1, we have $\bar{F}_j^{[i]}(x_{ij}) = \exp\{-ajx_{ij}/\sigma_i\}$ and $f_j^{[i]}(x_{ij}) = aj\sigma_i^{-1} \exp\{-ajx_{ij}/\sigma_i\}$ for $i = 1, \dots, s$ and $j = 1, \dots, r$. Then, Equation(3) yields the LF of the available data as

$$L(\sigma_1, \dots, \sigma_s, a; \mathbf{x}) = \left(\frac{r!n!}{(n-r)!}\right)^s (a)^{rs} \left(\prod_{i=1}^s \sigma_i\right)^{-r} \exp\left\{-\sum_{i=1}^s \sum_{j=1}^r \left(\frac{x_{ij}m_j}{\sigma_i}\right)\right\}, \quad (5)$$

where $a > 0$, and $m_j = (n-j+1)ja - (n-j)(j+1)a$ with convention $(n-r)(r+1)a \equiv 0$. For sake of brevity, we assumed that the proportional parameter a are the same among the s sequential r -out-of- n systems. Following Cramer and Kamps [8] and Hashempour and Doostparast [13], two cases are considered in sequel: (i) a is known, and (ii) a is unknown.

Case I: The parameter a is known

Suppose that the parameter a in Equation (5) is known. If $\sigma_1 = \dots = \sigma_s$, the ML estimate of the common mean of the s baseline exponential populations, say σ_0 , is derived by maximizing (5) with respect to σ_0 as

$$\begin{aligned} \hat{\sigma}_0 &= \frac{a}{rs} \sum_{i=1}^s \sum_{j=1}^r (n-j+1)jD_{ij} \\ &= \frac{1}{rs} \sum_{i=1}^s \sum_{j=1}^r x_{ij}m_j, \end{aligned} \quad (6)$$

where $D_{ij} = x_{ij} - x_{i,j-1}$, for $j = 1, \dots, r$, with convention $x_{i0} := 0$ for $i = 1, \dots, s$. Notice that $\sum_{j=1}^r x_{ij}m_j = \sum_{j=1}^r j(n-j+1)aD_{ij}$, for $i = 1, \dots, s$.

If baseline exponential populations are heterogeneous, the (unique) ML estimate of σ_i ($i = 1, \dots, s$) is derived from Equation (5) as

$$\hat{\sigma}_i = \frac{a}{r} \sum_{j=1}^r (n-j+1)jD_{ij} = \frac{1}{r} \sum_{j=1}^r x_{ij}m_j. \quad (7)$$

Under the LTCPHR with the one-parameter exponential baseline DF, we have

$$Q_i := \sum_{j=1}^r (n-j+1)jaD_{ij} \sim \Gamma(r, \sigma_i), \quad i = 1, \dots, s, \quad (8)$$

where $\Gamma(m, n)$ calls for the gamma distribution with the shape and the scale parameters m and n , respectively. From Equation (8) and for $i = 1, \dots, s$, $\hat{\sigma}_i \sim \Gamma(r, \sigma_i/r)$, and then $E(\hat{\sigma}_i) = \sigma_i$ and $Var(\hat{\sigma}_i) = \sigma_i^2/r$. Notice that the ML estimate $\hat{\sigma}_0$ in Equation (6) is the arithmetic mean of the ML estimates $\hat{\sigma}_i$ for the mean populations given by Equation (7), i.e. $\hat{\sigma}_0 = \sum_{i=1}^s \hat{\sigma}_i/s$. So, $E(\hat{\sigma}_0) = \bar{\sigma}$ and $Var(\hat{\sigma}_0) = \bar{\sigma}^2/sr$ where $\bar{\sigma} = \sum_{i=1}^s \sigma_i/s$ and $\bar{\sigma}^2 = \sum_{i=1}^s \sigma_i^2/s$.

Case II: The parameter a is unknown

Suppose that the parameter a in Equation (5) is unknown. In this case, calculations are complicated. The logarithm of LF in Equation (5) can be written as

$$l(\sigma_1, \dots, \sigma_s, a; \mathbf{x}) = rs \ln a - r \sum_{i=1}^s \ln \sigma_i - a \sum_{i=1}^s \sum_{j=1}^r \frac{j(n-j+1)D_{ij}}{\sigma_i}. \quad (9)$$

The ML estimates of the parameters which are shown by $\hat{\sigma}_i$ and \hat{a} for $i = 1, \dots, s$, (if exist) are obtained (numerically) by solving the following likelihood equations:

$$\frac{\partial l}{\partial a} = \frac{rs}{a} - \sum_{i=1}^s \sum_{j=1}^r \frac{j(n-j+1)D_{ij}}{\sigma_i} = 0, \quad (10)$$

and

$$\frac{\partial l}{\partial \sigma_k} = \frac{a}{\sigma_k^2} \sum_{j=1}^r j(n-j+1)D_{kj} - \frac{r}{\sigma_k} = 0, \quad k = 1, \dots, s. \quad (11)$$

From Equations (10) and (11), we have

$$\hat{a}(\sigma_1, \dots, \sigma_{k-1}, \sigma_{k+1}, \dots, \sigma_s) = \frac{r(s-1)}{\sum_{i=1, i \neq k}^s \sum_{j=1}^r \frac{j(n-j+1)D_{ij}}{\sigma_i}}, \quad (12)$$

and

$$\hat{\sigma}_k(\sigma_1, \dots, \sigma_{k-1}, \sigma_{k+1}, \dots, \sigma_s) = \frac{(s-1) \sum_{j=1}^r j(n-j+1)D_{kj}}{\sum_{i=1, i \neq k}^s \sum_{j=1}^r \frac{j(n-j+1)D_{ij}}{\sigma_i}}, \quad k = 1, \dots, s. \quad (13)$$

Equations (12) and (13) cannot be solved analytically. The the matrix of second derivatives of the likelihood with respect to the parameters is called Hessian matrix (**HM**), that is $\mathbf{HM} = [(\partial^2 \log(L)/\partial \theta_i \partial \theta_j)_{1 \leq i, j \leq s+1}]$, where $\theta_i = \sigma_i, (1 \leq i, j \leq s)$ and $\theta_i = a, (i, j = s+1)$. For more information, see Khuri [17]. After some algebraic calculations, we have

$$\mathbf{HM} = \begin{pmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{pmatrix}, \quad (14)$$

where

$$\begin{aligned}\mathbf{B}_{11} &= \left\{ {}_d \frac{r}{\sigma_i^2} - \frac{2a \sum_{k=1}^r k(n-k+1)D_{ij}}{\sigma_i^3} \right\}_{i=1, \dots, s}, \\ \mathbf{B}_{22} &= \left\{ \frac{-sr}{a^2} \right\}_{j=1, \dots, r}, \\ \mathbf{B}_{12} &= \mathbf{B}_{21}^T = \left[\left[\frac{\sum_{j=1}^r (n-j+1)D_{ij}}{\sigma_i^2} \right] \right]_{i=1, \dots, s, j=1, \dots, r},\end{aligned}$$

and $\{ {}_d a_i \}$ denotes a diagonal matrix with element a_i on the main diagonal. The Hessian matrix (14) is not necessarily negative definite on the parameter space. Therefore, one needs to use numerically methods for maximizing the LF (9) with respect to a and $\sigma_1, \dots, \sigma_s$ by using Equations (12) and (13).

2.2 Generalized likelihood ratio test

In this section, we consider the problem of homogeneity testing on the basis of independent SOS samples from different exponential populations, i.e.,

$$H_0 : \sigma_1 = \dots = \sigma_s \quad v.s \quad H_1 : \sigma_i \neq \sigma_j \quad \exists i \neq j. \quad (15)$$

Case I: The parameter a is known

The GLRT statistic for testing the problem of hypotheses (15) is

$$\begin{aligned}\Lambda_1 &= \frac{\sup_{\Omega_0} L(\sigma_1, \dots, \sigma_s; \mathbf{x})}{\sup_{\Omega} L(\sigma_1, \dots, \sigma_s; \mathbf{x})} \\ &= \prod_{i=1}^s \left(\frac{\hat{\sigma}_i}{\hat{\sigma}_0} \right)^r \exp \left\{ \sum_{i=1}^s \sum_{j=1}^r \left(\frac{1}{\hat{\sigma}_i} - \frac{1}{\hat{\sigma}_0} \right) m_j x_{ij} \right\},\end{aligned} \quad (16)$$

where $\Omega = \{(\sigma_1, \dots, \sigma_s) : \sigma_i > 0, i = 1, \dots, s\} =: \mathbb{R}^{+s}$ is the whole parameter space and $\Omega_0 = \{(\sigma_1, \dots, \sigma_s) : \sigma_1 = \dots = \sigma_s, \sigma_i > 0, i = 1, \dots, s\}$ denotes the parameter space under the null hypothesis H_0 . After some algebraic manipulations, the logarithm of the GLRT statistic Λ_1 given by Equation (16) reduces to

$$\log \Lambda_1 = r \sum_{i=1}^s \log \left(\frac{sQ_i}{\sum_{j=1}^s Q_j} \right), \quad (17)$$

where Q_i is defined by Equation (8) and “log” stands for the natural logarithm. Then, the null hypothesis H_0 is rejected if

$$A(\mathbf{Q}, \mathbf{a}) > c, \quad (18)$$

where $\mathbf{Q} = (Q_1, \dots, Q_s)$ and

$$A(\mathbf{Q}, \mathbf{a}) = - \sum_{i=1}^s \log \left(\frac{Q_i}{\sum_{j=1}^s Q_j} \right).$$

The constant c in Equation (18) is obtained subject to the level of the test, say γ . To derive the constant c in Equation (18), we need the distribution of the vector \mathbf{Q} under the null hypothesis H_0 in (15). To do this, we need the following lemma.

Lemma 2.1. (Balakrishnan and Nevzorov [2])

Let Z_1, \dots, Z_k be independent random variables and $Z_i \sim \Gamma(a_i, 1)$, for $i = 1, \dots, k$. Then $(U_1, \dots, U_k) \sim D(a_1, \dots, a_k)$ where $U_i = Z_i / \sum_{j=1}^k Z_j$, for $i = 1, \dots, k$. Here, $D(a_1, \dots, a_k)$ stands for the Dirichlet distribution with PDF

$$f(u_1, \dots, u_k) = \frac{\Gamma(\sum_{i=1}^k a_i)}{\prod_{i=1}^k \Gamma(a_i)} \left(\prod_{i=1}^k u_i^{a_i-1} \right), \quad 0 \leq u_i \leq 1, \quad \sum_{i=1}^k u_i = 1.$$

Let $V_i = Q_i / \sum_{j=1}^s Q_j$, for $i = 1, \dots, s-1$ and $V_s = -\sum_{i=1}^s \log(Q_i / \sum_{j=1}^s Q_j)$. The Jacobian transformation is then

$$J = \exp\{-v_s\} / \prod_{i=1}^{s-1} v_i. \quad (19)$$

The joint PDF of (V_1, \dots, V_s) under the homogeneity hypothesis H_0 in (15) is derived from Equations (8) and (19) and Lemma 2.1 as

$$\begin{aligned} f_{V_1, \dots, V_s}(v_1, \dots, v_s) &= \frac{\Gamma(sr)}{\Gamma(r)^s} \left(\prod_{i=1}^{s-1} v_i^{r-1} \right) \frac{\exp\{-(r-1)v_s\} \exp\{-v_s\}}{\prod_{i=1}^{s-1} v_i^{r-1} \prod_{i=1}^{s-1} v_i} \\ &= \frac{\Gamma(sr)}{\Gamma(r)^s} \frac{\exp\{-rv_s\}}{\prod_{i=1}^{s-1} v_i}, \end{aligned} \quad (20)$$

for $v_i \geq 0$, $1 \leq i \leq s$ and $\sum_{i=1}^{s-1} v_i + \exp\{-v_s\} / \prod_{i=1}^{s-1} v_i = 1$. Therefore, the marginal PDF of V_s is readily obtained from Equation (20) as

$$f_{V_s}(v_s) = \int \int \cdots \int \frac{\Gamma(sr)}{\Gamma(r)^s} \frac{\exp\{-rv_s\}}{\prod_{i=1}^{s-1} v_i} dv_1 \cdots dv_{s-1}, \quad (21)$$

where $B = \{(v_1, \dots, v_{s-1}) | v_i \geq 0, \sum_{i=1}^{s-1} v_i + \exp\{-v_s\} / \prod_{i=1}^{s-1} v_i = 1\}$. In practice, one may use numerical methods such as Monte Carlo simulation to derive the threshold c in the rejection region (18). For more details, see Hashempour [14].

Remark 2.2. It is easy to verify that the distribution family (5) is invariant with respect to the group of the scale transformations

$$\mathcal{G} = \{g_d : g_d(\mathbf{x}) = d\mathbf{x} = \{dx_{ij}^*\}_{1 \leq i \leq s, 1 \leq j \leq r}, d > 0\}. \quad (22)$$

Also, the problem of hypotheses testing (15) remains invariant under \mathcal{G} in (22) since $\bar{G}(\Omega) = \Omega$ and $\bar{G}(\Omega_0) = \Omega_0$ where $\Omega = \{(\sigma_1, \dots, \sigma_s) : \sigma_i > 0, i = 1, \dots, s\} = \mathbb{R}^{+s}$, $\Omega_0 =$

$\{(\sigma_1, \dots, \sigma_s) : \sigma_1 = \dots = \sigma_s, \sigma_i > 0, i = 1, \dots, s\}$ and $\bar{G} = \bar{g}_d(\sigma_1, \dots, \sigma_s) = (d\sigma_1, \dots, d\sigma_s)$ is the induced group of transformations by the group of transformations \mathcal{G} in Equation (22) on the parameter space Ω . Finding the uniformly most powerful invariant test (if exists) for the problem (15) remains as an open problem.

From Equation (8), one can see that $2r(\hat{\sigma}_i/\sigma_i) \sim \chi_{2r}^2$, where χ_ν^2 stands for the chi-square distribution with ν degrees of freedom. So, an equi-tailed confidence interval at level $100\gamma\%$ for σ_i ($i = 1, \dots, s$) is

$$\left(\frac{2r\hat{\sigma}_i}{\chi_{2r, (1+\gamma)/2}^2}, \frac{2r\hat{\sigma}_i}{\chi_{2r, (1-\gamma)/2}^2} \right), \quad (23)$$

where $\chi_{\nu, p}^2$ calls for the p -th quantile of the χ_ν^2 -distribution.

Note that the observed Fisher Information (FI), denoted by $I_{jk}(\hat{\sigma}_1, \dots, \hat{\sigma}_s)$, on the basis of the available SOSs data (2) is equal to minus of the Hessian matrix (HM) at the point of MLEs, i.e.

$$I_{jk}(\hat{\sigma}_1, \dots, \hat{\sigma}_s) = [(-\partial^2 \log(L)/\partial\sigma_i\partial\sigma_j)_{1 \leq i, j \leq s}]|_{\sigma_1=\hat{\sigma}_1, \dots, \sigma_s=\hat{\sigma}_s}.$$

It is well known that the unique MLEs follow asymptotically the multivariate normal distribution with mean vector $(\sigma_1, \dots, \sigma_s)$ and the variance-covariance matrix $[I_{jk}(\hat{\sigma}_1, \dots, \hat{\sigma}_s)]^{-1}$ (see, e.g., Lehmann and Romano, [18]). Therefore, an approximate equi-tailed $100\gamma\%$ confidence interval for σ_i is

$$\left(\hat{\sigma}_i - z_{(1-\gamma)/2} \sqrt{\frac{\hat{\sigma}_i^2}{r}}, \hat{\sigma}_i + z_{(1-\gamma)/2} \sqrt{\frac{\hat{\sigma}_i^2}{r}} \right), \quad (24)$$

where z_γ stands for the γ -quantile of the standard normal distribution.

Case II: The parameter a is unknown

It is easy to verify that the unique ML estimates of the parameters under the null hypothesis H_0 are

$$\hat{\sigma}_0 = \frac{\sum_{i=1}^s \sum_{j=1}^r x_{ij} \hat{m}_{0,j}}{rs} = \frac{\hat{a}_0}{rs} \sum_{i=1}^s \sum_{j=1}^r (n-j+1)jD_{ij}, \quad (25)$$

and

$$\hat{a}_0 = \frac{rs}{\sum_{i=1}^s \sum_{j=1}^r (n-j+1)jD_{ij}} \hat{\sigma}_0, \quad (26)$$

where $\hat{m}_{0,j} = (n-j+1)j\hat{a}_0 - (n-j)(j+1)\hat{a}_0$, with convention $\hat{a}_0(r+1) \equiv 0$. Therefore, the GLRT statistic for the hypotheses testing problem (15) is

$$\Lambda_2 = \prod_{j=1}^r \left(\frac{\hat{a}_0}{\hat{a}} \right)^{rs} \prod_{i=1}^s \left(\frac{\hat{\sigma}_i}{\hat{\sigma}_0} \right)^r \exp \left\{ \sum_{i=1}^s \sum_{j=1}^r \left(\frac{\hat{m}_j}{\hat{\sigma}_i} - \frac{\hat{m}_{0,j}}{\hat{\sigma}_0} \right) x_{ij} \right\}, \quad (27)$$

where $\hat{m}_j = (n - j + 1)\hat{\alpha}_j - (n - j)\hat{\alpha}_{j+1}$. The logarithm of the GLRT statistic Λ_2 in Equation (27) reads

$$\log \Lambda_2 = rs \log \left(\frac{\hat{a}_0}{\hat{a}} \right) + r \sum_{i=1}^s \log \left(\frac{\hat{\sigma}_i}{\hat{\sigma}_0} \right) + \sum_{i=1}^s \sum_{j=1}^r \left(\frac{\hat{m}_j}{\hat{\sigma}_i} - \frac{\hat{m}_{0,j}}{\hat{\sigma}_0} \right) x_{ij}. \quad (28)$$

The null hypothesis H_0 rejects if

$$-2 \log \Lambda_2 > c, \quad (29)$$

where $-2 \log \Lambda_2$ follows asymptotically chi-square distribution. Exact distribution of the statistic $\log \Lambda_2$ in Equation (28) under the null hypothesis H_0 is complicated and we could not obtain an explicit expression. This remains as an open problem. In practice, one may use numerical methods such as the Monte Carlo simulation to derive the threshold c in the rejection region (29). Bedbur [5] obtained the uniformly most powerful unbiased tests under the conditionally proportional hazard rates model based on multiple homogeneous SOS samples from a common exponential distribution.

3 An illustrative example

In order to assess the performance of the derived estimates in the preceding sections a simulation study was conducted. In the case of unknown parameter a , $N = 10^4$ multiple SOS samples were generated for some selected values parameters parameter. For this purpose, we assume that $a = 0.9, 1, 1.1$, $n = 10, 20$, $s = 4$, $r = 4, 6, 15$ and $(\sigma_1, \dots, \sigma_s) = (1, \dots, 1)$. Here, Bias and MSE stand for estimated bias and mean squared error of estimators. The Bias and MSE of the ML estimates of parameters $(\sigma_1, \dots, \sigma_s)$ and a are reported in rows of Table 1 from up to down, respectively.

For $n = 10$, $s = 1$, $a = 1.1$ and $r = 4$, the equi-tailed confidence intervals at level 95% for unknown parameters σ_i ($i = 1, \dots, n$) in Equation (23) are (0.4570, 3.6768), (0.4658, 3.7471), (0.4477, 3.6018) and (0.4614, 3.7120), respectively. The length of aforementioned confidence intervals for unknown parameters σ_i for $i = 1, \dots, s$ are 3.2208, 3.2813, 3.1540 and 3.2506, respectively. Also, from Equation (24), the approximate 95% confidence intervals for the unknown parameters σ_i for $i = 1, \dots, s$ are given by (0.0200, 1.9835), (0.0200, 1.9835), (0.0200, 1.9835) and (0.0200, 1.9835), respectively. The length of approximate confidence intervals for unknown parameters σ_i for $i = 1, \dots, s$ are 1.9635, 2.0010, 1.9234 and 1.9823, respectively.

Empirical evidences from Table 1 are summarized as follow;

- The MSEs are decreasing in n and r .
- The Biases are negligible and negative.
- The MSE of the obtained numerical estimator of parameter a is increasing in a .

Table 1: The MSE and $Bias$ for unknown a .

n	r	$a = 0.9$		$a = 1$		$a = 1.1$	
		$Bias$	MSE	$Bias$	MSE	$Bias$	MSE
10	4	-0.0121	0.0662	-0.0129	0.0541	-0.0114	0.0504
		-0.0259	0.0579	-0.0165	0.0533	-0.0151	0.0537
		-0.0277	0.0538	-0.0209	0.0466	-0.0224	0.0486
		-0.0109	0.0484	-0.0257	0.0449	-0.0164	0.0516
		-0.0699	0.0514	-0.0782	0.0668	-0.0900	0.0828
10	6	-0.0095	0.0436	-0.0247	0.0494	-0.0130	0.0448
		-0.0247	0.0518	-0.0262	0.0531	-0.0241	0.0506
		-0.0245	0.0534	-0.0157	0.0451	-0.0136	0.0469
		-0.0202	0.0455	-0.0127	0.0432	-0.0135	0.0506
		-0.0706	0.0508	-0.0797	0.0652	-0.0855	0.0757
20	4	-0.0186	0.0560	-0.0184	0.0427	-0.0133	0.0479
		-0.0125	0.0514	-0.0172	0.0474	-0.0853	0.0438
		-0.0231	0.0481	-0.0198	0.0433	-0.0185	0.0464
		-0.0159	0.0401	-0.0154	0.0493	-0.0164	0.0411
		-0.0712	0.0500	-0.0808	0.0628	-0.0857	0.0763
20	15	-0.0092	0.0510	-0.0130	0.0412	-0.0247	0.0410
		-0.0141	0.0400	-0.0109	0.0461	-0.0065	0.0421
		-0.0239	0.0426	-0.0204	0.0420	-0.0024	0.0441
		-0.0082	0.0396	-0.0230	0.0430	-0.0116	0.0406
		-0.0797	0.0442	-0.0850	0.0537	-0.0961	0.0633

4 Conclusions and further remarks

In this paper, based on independent SOSs coming from heterogeneous exponential populations under a linear trend model, the MLEs parameters were obtained on the basis of multiple SOS samples. The GLR tests were derived for testing homogeneity of the exponential populations. Some open problems were also mentioned. The results of this paper may be extended in some directions. For example, derivation of the uniformly most powerful scale-invariant test (if exist) is worth for further consideration.

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