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On the Uniformly Discrete and Minimal System for a Banach Subspace in $L_p(\mathbb{R}^d)$

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Abstract

In this note, it is shown that if $(f_i, g_i)_{i=1}^\infty \in L_p(\mathbb{R}^d) \times L_q(\mathbb{R}^d)$ is a Schauder frame for a closed subspace X of $L_p(\mathbb{R}^d)$, then X embeds almost isometrically into l_p . Also, the same conclusion holds, if for $f \in L_p(\mathbb{R}^d)$, the translations f by $\{x_i : x_i \in \mathbb{R}^d\}$ is a bounded minimal system for X . A basis (frame) for the Banach space $L_p[0, 1]^2$, $1 \leq p < \infty$ is constructed.

Keywords Schauder frame · Minimal system · Almost isometrically · wavelet

Mathematics Subject Classification Primary 43A15 · Secondary 43A85 · 65T60

1 Introduction

Frames are generalizations of orthonormal bases in Hilbert spaces. In 1952, Duffin and Schaffer presented some problems in non-harmonic Fourier series, and frame for Hilbert spaces. The main property of frames which makes them useful is their redundancy. Many properties of frames make them useful in various applications in mathematics, science and engineering. For a conclusive and comprehensive survey on various types of frames, one may refer to Casazza (2000), Hardin (1981) and the references therein. In 1991, Gröcheing introduced a more general concept of frame for Banach spaces called Banach

frame. Schauder frames for Banach spaces were introduced by Han and Larson (2000) and were further studied in Kaushik et al. (2013), Liu (2009), Liu and Zheng (2010), Vashisht (2012). In 1999, Casazza et al. (1999), presented various definitions of frames for Banach spaces including that of Schauder frames. In 2010, Liu and Zheng gave a characterization of Schauder frames which are close to the Schauder bases, which generalized some result based on Liu and Zheng (2010).

On the other hand, the isometry theory of Banach spaces was born and developed in connection with other areas of the Banach spaces theory. The important goal of this direction is to generalize the isometric theory to other classes of spaces the extension method for L_p -isometries discovered in the 70s by Plotkin (1969, 1970, 1971, 1972, 1976) and independently, by Rudin (1976) and Hardin (1981). The problem of how to check whether a given Banach space is isometric to a subspace of L_p was posed by Levy (1937). A well-known fact is that a Banach space embeds isometrically in a Hilbert space if and only if its norm satisfies the Parallelogram law (Fréchet 1935; Jordan and Von-Neumann 1935). It is worthwhile to note that a Hilbert space cannot be isomorphic to a subspace of l_p when $p \neq 2$ (Lindenstrauss and Tzafriri 1977).

In this paper, we show that if $(f_i, g_i)_{i=1}^\infty \in L_p(\mathbb{R}^d) \times L_q(\mathbb{R}^d)$ is a Schauder frame for the closed subspace X of $L_p(\mathbb{R}^d)$, then X embeds almost isometrically into l_p . The same conclusion holds if $f \in L_p(\mathbb{R}^d)$, the set $\{L_x f : x \in \mathbb{R}^d\}$ is a bounded minimal system for X .

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2 Preliminaries and Notation

Let X be a Banach space and X^* be its topological dual space. A sequence $\{x_n\}_{n=1}^\infty \subseteq X$ is called a Schauder basis of X if for every $x \in X$ there is a unique sequence of scalars $\{a_n\}_{n=1}^\infty$ such that $x = \sum_{n=1}^\infty a_n x_n$. A sequence $\{x_n\}_{n=1}^\infty$ which is a Schauder basis of its closed linear span is called a basic sequence. It is known that $\{x_n\}_{n=1}^\infty$ is a Schauder basis if and only if $\text{span}\{x_n\}_{n=1}^\infty$ is dense in X , and there exists a constant C such that for all $m, n \in \mathbb{N}$ with $m < n$, and all $\{a_i\}_{i=1}^n \subseteq \mathbb{R}$, $\|\sum_{i=1}^m a_i x_i\| \leq C \|\sum_{i=1}^n a_i x_i\|$. The smallest such C is called the basic constant of $\{x_n\}_{n=1}^\infty$. (See more details in Johnson and Lindenstrauss (2001), Lindenstrauss and Tzafriri (1977), Odell et al. (2011), Sari (2003)). A basis is called a monotone basis if the basic constant is 1.

A Schauder frame or simply a frame for a Banach space X is a sequence $(x_n, x_n^*)_{n=1}^\infty \subseteq X \times X^*$ such that for all $x \in X$, $x = \sum_{n=1}^\infty x_n^*(x) x_n$. Note that any basis for X is a frame for X , and moreover in this case, $\sup\{\|\sum_{i=1}^n x_i^*(x) x_i\|, n \in \mathbb{N}, \|x\| = 1\} < \infty$, is known as the frame constant.

3 Main Results

We start with the definition of a uniformly discrete set.

A set $\{x_i : i \in I\} \subseteq \mathbb{R}^d$ is called uniformly discrete if $\inf\{|x_i - x_j| : i \neq j, i, j \in \mathbb{N}\} > 0$, in which $|\cdot|$ denotes the usual distance in \mathbb{R}^d . Now, let $\{x_i : i \in \mathbb{N}\} \subseteq \mathbb{R}^d$ be a uniformly discrete set and $f \in L_p(\mathbb{R}^d)$, the set $\{L_{x_i} f : i \in \mathbb{N}\}$, is called the sequences of uniformly discrete of translate of f , in which $L_{x_i} f(x) = f(x - x_i)$.

Lemma 1 If $\{x_i\} \subseteq \mathbb{R}^d$ is a uniformly discrete set and $f \in L_1(\mathbb{R}^d)$ then

$$\sum_{i=1}^\infty \|L_{x_i} f|_{[0,1]^d}\|_1 \leq \|f\|_1.$$

Proof By using the invariant property of the Lebesgue measure on \mathbb{R}^d , we get,

$$\begin{aligned} \sum_{i=1}^\infty \|L_{x_i} f|_{[0,1]^d}\|_1 &= \sum_{i=1}^\infty \int_{[0,1]^d} |L_{x_i} f(x)| dx \\ &= \sum_{i=1}^\infty \int_{x_i + [0,1]^d} |f(x)| dx \\ &= \int_{\bigcup_{i=1}^\infty x_i + [0,1]^d} |f(x)| dx \leq \int_{\mathbb{R}^d} |f(x)| dx = \|f\|_1. \end{aligned}$$

□

In the following proposition, we show that left translations of $f \in L_p(\mathbb{R}^d) \cap L_1(\mathbb{R}^d)$ by a uniformly discrete set $\{x_i\} \subseteq \mathbb{R}^d$ is not a frame for $L_p(\mathbb{R}^d)$.

Proposition 1 Let $f \in L_p(\mathbb{R}^d) \cap L_1(\mathbb{R}^d)$, where $1 < p < \infty$ and $\{x_i\} \subseteq \mathbb{R}^d$ be a uniformly discrete set and $\{L_{x_i} f\}_{i=1}^\infty$ be the set of translates of f by $\{x_i\}$. Then, there is no sequence $\{g_i\}_{i=1}^\infty \subseteq L_q(\mathbb{R}^d)$, such that $(L_{x_i} f, g_i)_{i=1}^\infty$ is a frame for $L_p(\mathbb{R}^d)$, where q is the conjugate exponent of p .

Proof Assume that $(L_{x_i} f, g_i)_{i=1}^\infty \subseteq L_p(\mathbb{R}^d) \times L_q(\mathbb{R}^d)$ is a frame. That is for any $g \in L_p(\mathbb{R}^d)$, $\sum_{i=1}^\infty (g, g_i) L_{x_i} f$ converges to g in $L_p(\mathbb{R}^d)$.

So $\lim_{i \rightarrow \infty} \|(g, g_i) L_{x_i} f\|_p = 0$. Now, using the fact $\|L_{x_i} f\|_p = \|f\|_p$, we have

$$\lim_{i \rightarrow \infty} |(g, g_i)| \|L_{x_i} f\|_p = \|f\|_p \cdot \lim_{i \rightarrow \infty} |(g, g_i)| = 0.$$

This implies that $g_i \rightarrow 0$ in the w^* -topology, thus $\{g_i\}_{i=1}^\infty$ is bounded.

Put $K = \sup \|g_i\|_q$, choose $n_0 \in \mathbb{N}$ such that, $\sum_{i=n_0+1}^\infty \|L_{x_i} f|_{[0,1]^d}\|_1 < \frac{1}{4K}$. Choose $h : \mathbb{R}^d \rightarrow \mathbb{R}^d$, so that $|h| = \chi_{[0,1]^d}$ and, $|(h, g_i)| < \frac{1}{4n_0 \|f\|_1}$ for $i \leq n_0$, then $\|h\|_p = \|h\|_1 = 1$. Also $h = \sum_{i=1}^\infty (h, g_i) L_{x_i} f$, (the series converging in $L_p(\mathbb{R}^d)$) and also $h|_{[0,1]^d} = \sum_{i=1}^\infty \langle h, g_i \rangle L_{x_i} f|_{[0,1]^d}$. The series is converging in $L_1[0,1]^d$, so by Minkowski inequality and Lemma 1 we have,

$$\begin{aligned} 1 = \|h\|_1 &\leq \sum_{i=1}^\infty |(h, g_i)| \|L_{x_i} f|_{[0,1]^d}\|_1 \\ &\leq \sum_{i=1}^{n_0} |(h, g_i)| \|f\|_1 + \sum_{i=n_0+1}^\infty \|g_i\|_q \|L_{x_i} f|_{[0,1]^d}\|_1 \\ &\leq \frac{n_0}{4n_0 \|f\|_1} \|f\|_1 + \sup_i \|g_i\|_q \frac{1}{4K} = \frac{1}{2}. \end{aligned}$$

which is a contradiction. □

Corollary 1 Let $(L_{x_i} f, g_i)_{i=1}^\infty$ be a frame for $L_p(\mathbb{R}^d)$, $\frac{1}{p} + \frac{1}{q} = 1$, such that $\{L_{x_i} f\}_{i=1}^\infty$ is a sequence of uniformly discrete translations of $f \in L_p(\mathbb{R}^d)$. Then for all bounded measurable sets B of positive measure, we have: $\sum_{i=1}^\infty \|g_i|_B\|_1^q = \infty$.

Proof Suppose that B is a bounded measurable set of positive measure and $\sum_{i=1}^\infty \|g_i|_B\|_1^q < \infty$. Let $k \in L_\infty(B)$, such that $\|k\|_\infty = 1$. Hence by frame condition $k = \sum_{i=1}^\infty (k, g_i) L_{x_i} f|_B$, the series converging in $L_1(B)$. Since $m(B) < \infty$, we have:

$$\begin{aligned}
 m(B) &= \|k\|_1 \leq \sum_{i=1}^n |(k, g_i)| \|L_{x_i} f|_B\|_1 \\
 &\quad + \sum_{i=n+1}^{\infty} |(k, g_i|_B)| \|L_{x_i} f|_B\|_1 \\
 &\leq \sum_{i=1}^n |(k, g_i)| \|L_{x_i} f\|_1 + \sum_{i=n+1}^{\infty} \|k\|_{\infty} \cdot \|g_i|_B\|_1 \cdot \|L_{x_i} f|_B\|_1 \\
 &\leq \sum_{i=1}^n |(k, g_i)| \|L_{x_i} f\|_1 \\
 &\quad + \left(\sum_{i=n+1}^{\infty} \|g_i|_B\|_1^q \right)^{\frac{1}{q}} \left(\sum_{i=n+1}^{\infty} \|L_{x_i} f|_B\|_1^p \right)^{\frac{1}{p}}.
 \end{aligned}$$

Now, since $\sum_{i=1}^{\infty} \|g_i|_B\|_1^q < \infty$, we can choose n , such that the second term does not exceed of $\frac{m(B)}{4}$ and given this n , choose k to make the first term also less than $\frac{m(B)}{4}$. Thus $m(B) < \frac{1}{2}m(B)$; a contradiction. \square

Lemma 2 Let $f \in L_p(\mathbb{R}^d)$ and $Q = [-m, m]^d$ where $m \in \mathbb{N}$, then for any $\varepsilon > 0$ there exists m_0 such that for any $m > m_0$, $\|f|_{\mathbb{R}^d - Q}\|_p < \varepsilon \|f\|_p$.

Proof From elementary real analysis, proof is clear (Folland 1999). \square

In the following proposition, we show that for a bounded biorthogonal system $(L_{x_n} f, g_n)_{n=1}^{\infty}$ in $L_p(\mathbb{R}^d)$, the sequence $\{x_n\}$ is uniformly discrete.

Proposition 2 Let $f \in L_p(\mathbb{R}^d)$, $1 \leq p < \infty$, and let $(f_n, g_n)_{n=1}^{\infty}$ be a bounded biorthogonal system in $L_p(\mathbb{R}^d)$, where $f_n(x) = L_{x_n} f(x) = f(x - x_n)$ for some $\{x_n\}_{n=1}^{\infty} \subseteq \mathbb{R}^d$. Then, $\{x_n\}_{n=1}^{\infty}$ is uniformly discrete.

Proof Suppose that $\{x_n\}_{n=1}^{\infty}$ is not uniformly discrete. So, there are subsequences $\{i_n\}_{n=1}^{\infty}$ and $\{j_n\}_{n=1}^{\infty}$ of natural numbers such that for each n , $x_{i_n} \neq x_{j_n}$ and $\lim_{n \rightarrow \infty} \|x_{i_n} - x_{j_n}\| = 0$.

Then,

$$\|g_{i_n}\| \geq \frac{|(g_{i_n}, f_{i_n} - f_{j_n})|}{\|f_{i_n} - f_{j_n}\|_p} = \frac{1}{\|f_{i_n} - f_{j_n}\|_p}.$$

On the other hand,

$$\|f_{i_n} - f_{j_n}\|_p = \|L_{(x_{i_n} - x_{j_n})} f - f\|_p$$

So, $\|f_{i_n} - f_{j_n}\|_p \rightarrow 0$ as $n \rightarrow \infty$ and this is a contradiction. \square

Proposition 3 Let $\{x_n\}_{n=1}^{\infty} \subseteq \mathbb{R}^d$ be uniformly discrete, where $1 \leq p < \infty$, $f \in L_p(\mathbb{R}^d)$. Then for every cube $Q = \prod_{i=1}^d [a_i, b_i]$, $\sum_{i=1}^{\infty} \|L_{x_i} f|_Q\|_p^p < \infty$.

Proof Choose $\varepsilon_0 > 0$, such that $\|x_i - x_j\|_p > \varepsilon_0$ for all $i \neq j$, and $\lambda = (\lambda_1, \dots, \lambda_d) \in \mathbb{Z}^d$. Consider

$$Q_{\lambda} = \prod_{i=1}^d [a_i + (\lambda_i - 1)\varepsilon_0, a_i + \lambda_i \varepsilon_0].$$

Note that, $Q = \bigcup_{\lambda \in \mathbb{Z}^d} Q_{\lambda}$ for a finite numbers of λ , and

$$\begin{aligned}
 \sum_{i=1}^{\infty} \|L_{x_i} f|_{Q_{\lambda}}\|_p &= \sum_{i=1}^{\infty} \int_{Q_{\lambda}} |L_{x_i} f(y)|^p dy \\
 &= \sum_{i=1}^{\infty} \int_{Q_{\lambda}} |f(y - x_i)|^p dy \\
 &= \sum_{i=1}^{\infty} \int_{Q_{\lambda} + x_i} |f(y)|^p dy \\
 &= \int_{\bigcup_{i=1}^{\infty} Q_{\lambda} + x_i} |f(y)|^p dy \leq \|f\|_p^p.
 \end{aligned}$$

Thus by using of triangular inequality for integrals,

$$\sum_{i=1}^{\infty} \|L_{x_i} f|_Q\|_p \leq l \sum_{i=1}^{\infty} \|L_{x_i} f|_{Q_{\lambda}}\|_p \leq l \|f\|_p.$$

for some scalar l . \square

In the following theorem, it is shown that the closed subspace X of $L_p(\mathbb{R}^d)$ embeds almost isometrically into l_p whenever $(f_i, g_i)_{i=1}^{\infty}$ is a frame for X with some special properties. We say that a Banach space X embeds almost isometrically into l_p if for any all $\varepsilon > 0$, there exists $T : X \rightarrow l_p$ with $(1 + \varepsilon)^{-1} \leq \|Tf\| \leq (1 + \varepsilon)$ for all $f \in X$ and $\|f\| = 1$.

Theorem 1 Let $(f_i, g_i)_{i=1}^{\infty} \subseteq L_p(\mathbb{R}^d) \times L_q(\mathbb{R}^d)$ be a frame for a closed subspace X of $L_p(\mathbb{R}^d)$ where $1 \leq p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, with the property that for every $\varepsilon > 0$ and every bounded cube $Q \subseteq \mathbb{R}^d$, there exists $n_0 \in \mathbb{N}$ such that,

$$\left\| \sum_{i=m+1}^{\infty} (f_i, g_i) f_i|_Q \right\|_p \leq \varepsilon \|f\|_p, \quad m \geq n_0, \quad f \in X.$$

Then X embeds almost isometrically into l_p .

Proof Let C be the frame constant for $(f_i, g_i)_{i=1}^{\infty}$. Thus for all $f \in X$ and $n \in \mathbb{N}$,

$$\left\| \sum_{i=1}^n (f_i, g_i) f_i \right\|_p \leq C \|f\|_p. \quad (1)$$

Let $\varepsilon > 0$ be given, by Lemma 2, there exists m_0 such that for any $m \geq m_0$, for cube $Q = [-m, m]^d$, $m \in \mathbb{N}$

$$\|f|_{\mathbb{R}^d - Q}\|_p \leq \varepsilon \|f\|_p. \quad (2)$$

By our assumption, there exists increasing sequences m_k , n_k of natural numbers such that for $f \in X$, $n \geq n_k$

$$\left\| \sum_{i=n+1}^{\infty} (g_i, f) f_i \Big|_{Q_{k-1}} \right\|_p \leq \varepsilon 2^{-k} \|f\|_p, \quad (3)$$

where $Q_k = [-m_k, m_k]^d$ and $Q_0 = Q_{-1} = \emptyset$. Let $V_0 = \emptyset$, $V_k = Q_k - Q_{k-1}$ for $k \in \mathbb{N}$. By (Folland 1999, Theorem 2.26) choose a partition P_k of V_k into cubes, $k \geq 1$ so that for all $f \in X$

$$\|f|_{V_k} - \sum_{E \in P_k} \frac{\chi_E}{m(E)} \int_E f(x) dx\|_p \leq \varepsilon 2^{-k} \|f\|_p. \quad (4)$$

Consider $T : X \rightarrow l_p(\bigcup_{r=1}^{\infty} P_r)$ defined by

$$Tf = \left\{ \int_E f(x) dx \right\}_{E \in \bigcup_{r=1}^{\infty} P_r}.$$

It is easy to see that $\|Tf\|_p \geq \|f\|_p$. Hence for any $\varepsilon > 0$, $\|Tf\|_p > \frac{1}{1+\varepsilon}$ where $\|f\|_p = 1$.

We are going to show that $\|Tf\|_p < 1 + \varepsilon$ when $\|f\|_p = 1$. To show this, for $f \in X$, by the frame condition and (2), we have

$$\begin{aligned} 1 &= \left\| \sum_{i=1}^{\infty} (g_i, f) f_i \right\|_p \\ &= \left\| \sum_{r=1}^{\infty} \sum_{i=n_{r-1}+1}^{n_r} (g_i, f) f_i \Big|_{Q_r} \right. \\ &\quad \left. + \sum_{r=1}^{\infty} \sum_{i=n_{r-1}+1}^{n_r} (g_i, f) f_i \Big|_{\mathbb{R}^d - Q_r} \right\|_p \\ &\leq \left\| \sum_{r=1}^{\infty} \sum_{i=n_{r-1}+1}^{n_r} (g_i, f) f_i \Big|_{Q_r} \right\|_p \\ &\quad + \left\| \sum_{r=1}^{\infty} \sum_{i=n_{r-1}+1}^{n_r} (g_i, f) f_i \Big|_{\mathbb{R}^d - Q_r} \right\|_p \\ &\leq \left\| \sum_{r=1}^{\infty} \sum_{i=n_{r-1}+1}^{n_r} (g_i, f) f_i \Big|_{Q_r} \right\|_p \\ &\quad + \sum_{r=1}^{\infty} \left\| \sum_{i=n_{r-1}+1}^{n_r} (g_i, f) f_i \Big|_{\mathbb{R}^d - Q_r} \right\|_p \\ &\leq \left\| \sum_{r=1}^{\infty} \sum_{i=n_{r-1}+1}^{n_r} (g_i, f) f_i \Big|_{Q_r} \right\|_p + \sum_{r=1}^{\infty} C \|f|_{\mathbb{R}^d - Q_r}\|_p \\ &\leq \left\| \sum_{r=1}^{\infty} \sum_{i=n_{r-1}+1}^{n_r} (g_i, f) f_i \Big|_{Q_r} \right\|_p + \sum_{r=1}^{\infty} C \varepsilon 2^{-r} \|f\|_p \\ &\leq \left\| \sum_{r=1}^{\infty} \sum_{i=n_{r-1}+1}^{n_r} (g_i, f) f_i \Big|_{Q_r} \right\|_p + 2C\varepsilon. \end{aligned}$$

Now, the fact, $Q_r = Q_{r-2} \cup (Q_r - Q_{r-2})$, implies that,

$$\begin{aligned} &\left\| \sum_{r=1}^{\infty} \sum_{i=n_{r-1}+1}^{n_r} (g_i, f) f_i \Big|_{Q_r} \right\|_p \\ &= \left\| \sum_{r=1}^{\infty} \sum_{i=n_{r-1}+1}^{n_r} (g_i, f) f_i \Big|_{Q_r - Q_{r-2}} \right. \\ &\quad \left. + \sum_{r=1}^{\infty} \sum_{i=n_{r-1}+1}^{n_r} (g_i, f) f_i \Big|_{Q_{r-2}} \right\|_p \\ &\leq \left\| \sum_{r=1}^{\infty} \sum_{i=n_{r-1}+1}^{n_r} (g_i, f) f_i \Big|_{Q_r - Q_{r-2}} \right\|_p \\ &\quad + \left\| \sum_{r=1}^{\infty} \sum_{i=n_{r-1}+1}^{n_r} (g_i, f) f_i \Big|_{Q_{r-2}} \right\|_p \\ &\leq \left\| \sum_{r=1}^{\infty} \sum_{i=n_{r-1}+1}^{n_r} (g_i, f) f_i \Big|_{Q_r - Q_{r-2}} \right\|_p \\ &\quad + \sum_{r=1}^{\infty} \left\| \sum_{i=n_{r-1}+1}^{n_r} (g_i, f) f_i \Big|_{Q_{r-2}} \right\|_p \\ &\leq \left\| \sum_{r=1}^{\infty} \sum_{i=n_{r-1}+1}^{n_r} (g_i, f) f_i \Big|_{Q_r - Q_{r-2}} \right\|_p + \sum_{r=1}^{\infty} \varepsilon 2^{-r+1}. \end{aligned}$$

Thus, $1 = \left\| \sum_{i=1}^{\infty} (g_i, f) f_i \right\|_p$

$$\leq \left\| \sum_{r=1}^{\infty} \sum_{i=n_{r-1}+1}^{n_r} (g_i, f) f_i \Big|_{Q_r - Q_{r-2}} \right\|_p + 2C\varepsilon + 2\varepsilon.$$

Since $Q_r - Q_{r-2} = V_r \cup V_{r-1}$ and $\{V_r\}_{r=1}^{\infty}$ are pairwise disjoint; (4) implies that

$$\begin{aligned}
 & \left\| \sum_{i=1}^{\infty} (g_i, f) f_i \right\|_p \\
 & \leq \left\| \sum_{r=1}^{\infty} \sum_{i=n_{r-1}+1}^{n_r} (g_i, f) f_i \Big|_{V_r - V_{r-1}} \right\|_p + 2C\varepsilon + 2\varepsilon \\
 & = \left\| \sum_{r=1}^{\infty} \left(\sum_{i=n_{r-1}+1}^{n_r} (g_i, f) f_i \Big|_{V_r} + \sum_{i=n_{r-1}+1}^{n_r} (g_i, f) f_i \Big|_{V_{r-1}} \right) \right\|_p \\
 & \quad + 2C\varepsilon + 2\varepsilon \\
 & = \left\| \sum_{r=1}^{\infty} \left(\sum_{i=n_{r-1}+1}^{n_r} (g_i, f) f_i \Big|_{V_r} + \sum_{i=n_r+1}^{n_{r+1}} (g_i, f) f_i \Big|_{V_r} \right) \right\|_p \\
 & \quad + 2C\varepsilon + 2\varepsilon \\
 & = \sum_{r=1}^{\infty} \left(\chi_{V_r} \sum_{i=n_{r-1}+1}^{n_{r+1}} (g_i, f) f_i \right) \Big\|_p + 2C\varepsilon + 2\varepsilon \\
 & \leq \left\| \sum_{r=1}^{\infty} \sum_{i=n_{r-1}+1}^{n_{r+1}} \left((g_i, f) f_i \Big|_{V_r} \right. \right. \\
 & \quad \left. \left. - \sum_{E \in P_r} \frac{\chi_E}{m(E)} \int_E (g_i, f) f_i(x) dx \right) \right\|_p \\
 & \quad + \left\| \sum_{r=1}^{\infty} \chi_{V_r} \sum_{i=n_{r-1}+1}^{n_{r+1}} \sum_{E \in P_r} \frac{\chi_E}{m(E)} \int_E (g_i, f) f_i(x) dx \right\|_p \\
 & = \left\| \sum_{r=1}^{\infty} \chi_{V_r} \sum_{i=n_{r-1}+1}^{n_{r+1}} \sum_{E \in P_r} \frac{\chi_E}{m(E)} \int_E (g_i, f) f_i(x) dx \right\|_p \\
 & \quad + \left\| \sum_{r=1}^{\infty} \chi_{V_r} \sum_{i=n_{r-1}+1}^{n_{r+1}} \left((g_i, f) f_i \Big|_{V_r} \right. \right. \\
 & \quad \left. \left. - \sum_{E \in P_r} \frac{\chi_E}{m(E)} \int_E (g_i, f) f_i(x) dx \right) \right\|_p \\
 & \leq \left\| \sum_{r=1}^{\infty} \chi_{V_r} \sum_{i=n_{r-1}+1}^{n_{r+1}} \sum_{E \in P_r} \frac{\chi_E}{m(E)} \int_E (g_i, f) f_i(x) dx \right\|_p \\
 & \quad + \sum_{r=1}^{\infty} \left\| \sum_{i=n_{r-1}+1}^{n_{r+1}} (g_i, f) f_i \Big|_{V_r} \right. \\
 & \quad \left. - \sum_{E \in P_r} \frac{\chi_E}{m(E)} \int_E \sum_{i=n_{r-1}+1}^{n_{r+1}} (g_i, f) f_i(x) dx \right\|_p \\
 & \leq \left\| \sum_{r=1}^{\infty} \chi_{V_r} \sum_{i=n_{r-1}+1}^{n_{r+1}} \sum_{E \in P_r} \frac{\chi_E}{m(E)} \int_E (g_i, f) f_i(x) dx \right\|_p \\
 & \quad + \sum_{r=1}^{\infty} \left\| \sum_{i=n_{r-1}+1}^{n_{r+1}} (g_i, f) f_i \Big|_{V_r} \right. \\
 & \quad \left. - \sum_{E \in P_r} \frac{\chi_E}{m(E)} \int_E \sum_{i=n_{r-1}+1}^{n_{r+1}} (g_i, f) f_i(x) dx \right\|_p \\
 & \leq \left\| \sum_{r=1}^{\infty} \chi_{V_r} \sum_{i=n_{r-1}+1}^{n_{r+1}} \sum_{E \in P_r} \frac{\chi_E}{m(E)} \int_E (g_i, f) f_i(x) dx \right\|_p \\
 & \quad + \sum_{r=1}^{\infty} \frac{\varepsilon}{2^r} \left\| \sum_{i=n_{r-1}+1}^{n_{r+1}} (g_i, f) f_i \right\|_p,
 \end{aligned}$$

and by (1), we have

$$\begin{aligned}
 & \left\| \sum_{i=1}^{\infty} (g_i, f) f_i \right\|_p \\
 & \leq \left\| \sum_{r=1}^{\infty} \chi_{V_r} \sum_{i=n_{r-1}+1}^{n_{r+1}} \sum_{E \in P_r} \frac{\chi_E}{m(E)} \int_E (g_i, f) f_i(x) dx \right\|_p \\
 & \quad + 2C\varepsilon + 2\varepsilon + \sum_{r=1}^{\infty} \frac{\varepsilon}{2^r} \cdot C \\
 & \leq \left(\sum_{r=1}^{\infty} \sum_{E \in P_r} \left| \int_E \sum_{i=n_{r-1}+1}^{n_{r+1}} (g_i, f) f_i(x) dx \right|^p \right)^{\frac{1}{p}} + 4C\varepsilon + 2\varepsilon.
 \end{aligned} \tag{5}$$

The fact that E is of finite measure, by (3) we have

$$\begin{aligned}
 & \left(\sum_{E \in P_r} \left| \int_E \sum_{i=n_{r-1}+1}^{\infty} (g_i, f) f_i(x) dx \right|^p \right)^{\frac{1}{p}} \leq \left\| \sum_{i=n_{r-1}+1}^{\infty} (g_i, f) f_i \Big|_{Q_r} \right\|_p \\
 & \leq \varepsilon 2^{-(r+1)}.
 \end{aligned} \tag{6}$$

If $r > 1$ then by (2),

$$\begin{aligned}
 & \left(\sum_{E \in P_r} \left| \int_E \sum_{i=1}^{n_{r-1}} (g_i, f) f_i(x) dx \right|^p \right)^{\frac{1}{p}} = \left\| \sum_{i=1}^{n_{r-1}} (g_i, f) f_i \Big|_{Q_r - Q_{r-1}} \right\|_p \\
 & \leq \left\| \sum_{i=1}^{n_{r-1}} (g_i, f) f_i \Big|_{\mathbb{R}^d - Q_{r-1}} \right\|_p \\
 & \leq \varepsilon 2^{-(r-1)} \left\| \sum_{i=1}^{n_{r-1}} (g_i, f) f_i \right\|_p \\
 & \leq C\varepsilon 2^{-(r-1)} \cdot \|f\|_p \\
 & = 2^{-r+1} C\varepsilon.
 \end{aligned} \tag{7}$$

Using (6), (7), (8) and triangular inequality we have,

$$\begin{aligned}
 1 &= \|f\|_p \\
 &\leq \left\| \sum_{r=1}^{\infty} \left(\sum_{i=1}^{n_{r-1}} (g_i, f) f_i + \sum_{i=n_{r-1}+1}^{n_r} (g_i, f) f_i \right) \right\|_p \\
 &\leq \left\| \sum_{r=1}^{\infty} \sum_{i=1}^{n_{r-1}} (g_i, f) f_i \right\|_p + \left\| \sum_{r=1}^{\infty} \sum_{i=n_{r-1}+1}^{n_r} (g_i, f) f_i \right\|_p \\
 &\leq \sum_{r=1}^{\infty} 2^{-r+1} c\varepsilon \\
 &\quad + \left(\sum_{r=1}^{\infty} \sum_{E \in P_r} \left| \int_E \sum_{i=n_{r-1}+1}^{n_r} (g_i, f) f_i(x) dx \right|^p \right)^{\frac{1}{p}} \\
 &\quad + 4c\varepsilon + 2\varepsilon + \sum_{r=1}^{\infty} \varepsilon 2^{-(r+1)} \\
 &\leq 2c\varepsilon + \left(\sum_{E \in \bigcup_{r=1}^{\infty} P_r} \left| \int_E f(x) dx \right|^p \right)^{\frac{1}{p}} + 4c\varepsilon + 2\varepsilon + \frac{\varepsilon}{2} \\
 &\leq \left(\sum_{E \in \bigcup_{r=1}^{\infty} P_r} \left| \int_E f(x) dx \right|^p \right)^{\frac{1}{p}} + 6c\varepsilon + \frac{5\varepsilon}{2} \\
 &= 1 + 6c\varepsilon + \frac{5\varepsilon}{2}.
 \end{aligned}$$

This completes the proof. \square

In the sequel, we review some definitions which are necessary our next discussions.

Let X be a Banach space and X^* be its topological dual space. The pair $(x_n, x_n^*)_{n=1}^{\infty} \subseteq X \times X^*$ is called a biorthogonal system if $x_m^*(x_n) = \delta_{m,n}$. A sequence $\{x_n\}_{n=1}^{\infty} \subseteq X$ is called a minimal system if there exists a sequence, $\{x_n^*\}_{n=1}^{\infty} \subseteq X^*$ such that $(x_n, x_n^*)_{n=1}^{\infty}$ is a biorthogonal system. A minimal system $\{x_n\}_{n=1}^{\infty} \subseteq X$ is said to be fundamental if $\text{span}\{x_n : n \in \mathbb{N}\}$ is dense in X . It is well-known that a sequence $\{x_n\}_{n=1}^{\infty}$ is minimal if and only if $x_n \notin \text{span}\{x_m : n \neq m, m \in \mathbb{N}\}$ for all $n \in \mathbb{N}$. A sequence $\{x_n\}_{n=1}^{\infty}$ is called semi-normalized if it is bounded and bounded away from zero, that is $\inf\{\|x_n\| : n \in \mathbb{N}\} > C$ for some $C > 0$. Note that any basis $\{x_n\}_{n=1}^{\infty}$ is a fundamental bounded minimal system for X and any $x \in X$ can be represented uniquely of the form $x = \sum_{n=1}^{\infty} x_n^*(x) x_n$. In this case, $\{x_n^*\}_{n=1}^{\infty} \subseteq X^*$ is a basic sequence in X^* , that is a basis for the closed linear span $\{x_n^*\}_{n=1}^{\infty}$. It is a basis for X^* if X is reflexive (Johnson and Lindenstrauss 2001).

Let X be a subspace of $L_p(\mathbb{R}^d)$, $1 < p < \infty$ and $\{f_i\} \subseteq X$ be a semi-normalized fundamental minimal system. We prove that in this case with an additional condition, the subspace X embeds almost isometrically into l_p .

Theorem 2 Let X be a subspace of $L_p(\mathbb{R}^d)$, $1 < p < \infty$ and $\{f_i\}_{i=1}^{\infty} \subseteq X$ be a semi-normalized fundamental minimal system. Suppose that for each $\varepsilon > 0$ and each bounded cube $Q \subseteq \mathbb{R}^d$ there exists $n \in \mathbb{N}$ such that for all $m_2 \geq m_1 \geq m > n$, for all $f = \sum_{i=1}^{m_2} a_i f_i$ with $\|f\|_p = 1$ and scalars a_i , with $\|\sum_{i=m}^{m_1} a_i f_i|_Q\|_p \leq \varepsilon$. Then, X embeds almost isometrically into l_p .

Proof Let $\{f_i\}_{i=1}^{\infty}$ be a fundamental minimal system for $X \subseteq L_p(\mathbb{R}^d)$. Let $n_k, k \in \mathbb{N}$ be an increasing sequence and let $m' \geq m \geq n > n_k$, and cube, $Q_k = [-n_k, n_k]^d$, then by our assumption, if $\|f\|_p = 1, f = \sum_{i=1}^{m'} a_i f_i$ we have

$$\left\| \sum_{i=n}^m a_i f_i|_{Q_{k-1}} \right\|_p \leq \varepsilon 2^{-k}, \quad (9)$$

where we take $Q_0 = \emptyset$. Furthermore for every f of the form $f = \sum_{i=1}^{n_k} a_i f_i$, by using Lemma 2, we have,

$$\|f|_{\mathbb{R}^d - Q_k}\|_p \leq \varepsilon 2^{-k}. \quad (10)$$

Let $V_k = Q_k - Q_{k-1}$ for $k \in \mathbb{N}$, choose a partition P_k of V_k into cubes.

Now, for every f of the form $f = \sum_{i=1}^{n_{k+1}} a_i f_i$ we get,

$$\|f|_{V_k} - \sum_{E \in P_k} \frac{\chi_E}{m(E)} \int_E f(x) dx\|_p \leq \varepsilon 2^{-k}. \quad (11)$$

See (Folland 1999, Theorem 2.26). Let $f \in \text{span}\{f_i\}$ and $f = \sum_{i=1}^l a_i f_i$ with $\|f\|_p = 1$. By (10) and the method in the proof of Theorem 1, we have

$$\begin{aligned}
 1 &= \left\| \sum_{i=1}^l a_i f_i \right\|_p \\
 &= \left\| \sum_{r=1}^l \sum_{i=n_{r-1}+1}^{n_r} a_i f_i|_{Q_r} + \sum_{r=1}^l \sum_{i=n_{r-1}+1}^{n_r} a_i f_i|_{\mathbb{R}^d - Q_r} \right\|_p \\
 &\leq \left\| \sum_{r=1}^l \sum_{i=n_{r-1}+1}^{n_r} a_i f_i|_{Q_r} \right\|_p + \sum_{r=1}^l \left\| \sum_{i=n_{r-1}+1}^{n_r} a_i f_i|_{\mathbb{R}^d - Q_r} \right\|_p \\
 &\leq \left\| \sum_{r=1}^l \sum_{i=n_{r-1}+1}^{n_r} a_i f_i|_{Q_r} \right\|_p + \sum_{r=1}^l \varepsilon 2^{-r} \\
 &\leq \left\| \sum_{r=1}^l \sum_{i=n_{r-1}+1}^{n_r} a_i f_i|_{Q_r} \right\|_p + 2\varepsilon.
 \end{aligned}$$

Now, by considering $Q_r = Q_{r-2} \cup (Q_r - Q_{r-2})$ and $Q_r - Q_{r-2} = V_r \cup V_{r-1}$ we have,

$$\begin{aligned}
 & \left\| \sum_{r=1}^l \sum_{i=n_{r-1}+1}^{n_r} a_i f_i \right\|_{Q_r} \Big|_p \\
 &= \left\| \sum_{r=1}^l \sum_{i=n_{r-1}+1}^{n_r} a_i f_i \right\|_{Q_r - Q_{r-2}} \\
 &+ \left\| \sum_{r=1}^l \sum_{i=n_{r-1}+1}^{n_r} a_i f_i \right\|_{Q_{r-2}} \Big|_p \\
 &\leq \left\| \sum_{r=1}^l \sum_{i=n_{r-1}+1}^{n_r} a_i f_i \right\|_{Q_r - Q_{r-2}} \Big|_p + 2\varepsilon \\
 &= \left\| \sum_{r=1}^l \sum_{i=n_{r-1}+1}^{n_r} a_i f_i \right\|_{V_r \cup V_{r-1}} \Big|_p + 4\varepsilon \\
 &= \sum_{r=1}^l \chi_{V_r} \sum_{i=n_{r-1}+1}^{n_r} a_i f_i \Big|_p + 4\varepsilon.
 \end{aligned}$$

But, $\{V_r\}_{i=1}^\infty$ are pairwise disjoint, so by (11) we have

$$\leq \left(\sum_{r=1}^l \sum_{E \in P_r} \left| \int_E \sum_{i=n_{r-1}+1}^{n_r} a_i f_i(x) dx \right|^p \right)^{\frac{1}{p}} + 6\varepsilon. \quad (12)$$

If $r > 1$ then by (9) we have

$$\begin{aligned}
 & \left(\sum_{E \in P_r} \left| \int_E \sum_{i=1}^{n_{r-1}} a_i f_i(x) dx \right|^p \right)^{\frac{1}{p}} \\
 &= \left\| \sum_{i=1}^{n_{r-1}} a_i f_i \right\|_{Q_r - Q_{r-1}} \Big|_p \leq 2^{-r+1} \varepsilon.
 \end{aligned} \quad (13)$$

Now by (12), (13) and using triangular inequality we have,

$$\begin{aligned}
 1 &= \|f\|_p \\
 &\leq \left(\sum_{E \in \bigcup_{r=1}^l P_r} \left| \int_E f(x) dx \right|^p \right)^{\frac{1}{p}} + 6\varepsilon + \sum_{r=1}^l 2^{-r+1} \varepsilon \\
 &\leq \left(\sum_{E \in \bigcup_{r=1}^l P_r} \left| \int_E f(x) dx \right|^p \right)^{\frac{1}{p}} + 8\varepsilon \\
 &= 1 + 8\varepsilon.
 \end{aligned}$$

Thus $T : X \rightarrow l_p \left(\bigcup_{r=1}^l P_r \right)$ given by $f \mapsto \left(\int_E f(x) dx \right)_{E \in \bigcup_{r=1}^l P_r}$ is the desired embedding. \square

Proposition 4 Let $f \in L_p(\mathbb{R}^d) \cap L_1(\mathbb{R}^d)$ where $1 \leq p < \infty$ and let $\{x_i\}_{i=1}^\infty \subseteq \mathbb{R}^d$ be a sequence of uniformly discrete set, then $\{L_{x_i} f\}_{i=1}^\infty$ is not a fundamental bounded minimal system for $L_p(\mathbb{R}^d)$.

Proof Let $(L_{x_i} f, g_i)_{i=1}^\infty$ is a fundamental bounded biorthogonal system for $L_p(\mathbb{R}^d)$, and $\{g_i\}_{i=1}^\infty \subseteq L_q(\mathbb{R}^d)$ is bounded. Since $\sup_i \|L_{x_i} f\|_p \cdot \|g_i\|_q < \infty$ and $\|L_{x_i} f\|_p = \|f\|_p$ for $i \in \mathbb{N}$ then $k := \sup_i \|g_i\|_q < \infty$; choose $n_0 \in \mathbb{N}$

such that $\sum_{i=n_0+1}^\infty \|L_{x_i} f\|_{[0,1]^d} < \frac{1}{4k}$. Consider $h : \mathbb{R}^d \rightarrow \mathbb{R}^d$ so that $|h| = \chi_{[0,1]^d}$ and $|(h, g_i)| < \frac{1}{4n_0 \|f\|_1}$ for $i \leq n_0$. Thus $\|h\|_p = \|h\|_1 = 1$. For $\varepsilon > 0$ we can choose, $g = \sum_{i=1}^\infty a_i L_{x_i} f$ with $\|h - g\|_p < \varepsilon$. Now by Holder inequality,

$$\begin{aligned}
 \int_{\mathbb{R}^d} \chi_{[0,1]^d}(t) |g(t) - h(t)| dt &\leq \left(\int_{\mathbb{R}^d} |\chi_{[0,1]^d}(t)|^q dt \right)^{\frac{1}{q}} \\
 &+ \left(\int_{\mathbb{R}^d} |g(t) - h(t)|^p dt \right)^{\frac{1}{p}} \\
 &= \|g - h\|_p < \varepsilon.
 \end{aligned}$$

Thus $\|g|_{[0,1]^d} - h\|_1 < \varepsilon$. So,

$$\begin{aligned}
 1 - \varepsilon &\leq \|g|_{[0,1]^d}\|_1 \leq \sum_{i=1}^{n_0} |a_i| \|L_{x_i} f\|_{[0,1]^d} \Big|_1 \\
 &+ \sum_{i=n_0+1}^\infty \|L_{x_i} f\|_{[0,1]^d} \Big|_1,
 \end{aligned} \quad (14)$$

that is, $\|h\|_1 - \|g|_{[0,1]^d}\|_1 \leq \|g|_{[0,1]^d} - h\|_1 < \varepsilon$.

Then

$$\begin{aligned}
 1 - \varepsilon &\leq \|g|_{[0,1]^d}\|_1 \leq \sum_{i=1}^{n_0} |a_i| \|L_{x_i} f\|_{[0,1]^d} \Big|_1 \\
 &+ \sum_{i=n_0+1}^\infty |a_i| \|L_{x_i} f\|_{[0,1]^d} \Big|_1.
 \end{aligned}$$

For $i \leq n_0$, and by biorthogonal condition, we have

$$|a_i| = |g_i(g)| \leq |g_i(g - h)| + |g_i(g)| < k\varepsilon + \frac{1}{4n_0(\|f\|_1)}.$$

For $i > n_0$, since,

$$|a_i| = |g_i(g)| \leq \|g_i\|_q \cdot \|g\|_p \leq k \|g\|_p < k(1 + \varepsilon),$$

thus $|a_i| \leq k(1 + \varepsilon)$. Hence by (14) we have,

$$\begin{aligned}
 1 - \varepsilon &\leq n_0 \left(k\varepsilon + \frac{1}{4n_0 \|f\|_1} \right) \|f\|_1 + \sum_{i=n_0+1}^\infty k(1 + \varepsilon) \|L_{x_i} f\|_{[0,1]^d} \Big|_1 \\
 &\leq n_0 k\varepsilon \|f\|_1 + \frac{1}{4} + \frac{1}{4} (1 + \varepsilon) \\
 &\leq \frac{3}{4} < 1 - \varepsilon,
 \end{aligned}$$

which is a contradiction, if $\varepsilon < \frac{1}{4}$. \square

We now construct an example of a frame of $L_p[0, 1]^2$ for $1 \leq p < \infty$, which in fact is basis.

Example 1 Set $\phi = \chi_{[0,1]}$ and let $\psi(t) = \chi_{[0, \frac{1}{2}]} - \chi_{[\frac{1}{2}, 1]}$, which are the Haar scaling function and Haar wavelet, respectively. For $n = 0, 1, 2, \dots$ and $0 \leq k \leq 2^n - 1$.

Put $\Psi_{n,k}(t) = 2^{\frac{n}{2}} \psi(\frac{t-k}{2^n})$. Consider the Haar system $H =$

$\{\phi\} \cup \{\Psi_{n,k}\}_{n=0,1,2,\dots}$ which forms a basis for $0 \leq k \leq 2^n - 1$

$L_p[0, 1]$. Note that $\phi + \Psi_{0,0} = 2\chi_{[0, \frac{1}{2}]}$ and it is easy to see that $\chi_{[\frac{k}{2^n}, \frac{k+1}{2^n})}$ is a finite sum of the members of the Haar system H . So $\chi_{[\frac{k}{2^n}, \frac{k+1}{2^n})}$ is in $\text{span}(H)$ for $n = 0, 1, 2, \dots$ and $0 \leq k \leq 2^n - 1$. It is easy to check that,

$$\text{span}(H) = \text{span}\{\chi_{[\frac{k}{2^n}, \frac{k+1}{2^n})} : n = 0, 1, 2, \dots, \quad 0 \leq k \leq 2^n - 1\}. \quad (15)$$

It is known that the right-hand side of (15) is dense in $L_p[0, 1]$, so the Haar system H is complete in $L_p[0, 1]$ for $1 \leq p < \infty$. It is worthwhile to mention that $L_\infty[0, 1]$ is not separable, which implies that it does not possess a basis. Therefore, we focus on $1 \leq p < \infty$.

Note that, in the two-dimensional case $L_p[0, 1]^2$, we will need one scaling function, but three wavelet functions: horizontal, vertical and diagonal wavelets. More precisely, let $\Phi = \chi_{[0,1]^2}$, then for $(x, y) \in \mathbb{R}^2$ we define

$$\Psi^1(x, y) = \phi(x)\psi(y),$$

$$\Psi^2(x, y) = \psi(x)\phi(y),$$

$$\Psi^3(x, y) = \psi(x)\psi(y).$$

$$\text{Put } \Psi_{n,k}^m(x, y) = 2^{\frac{n}{2}} \psi^m\left(\frac{x-2^{-n}k}{2^n}, \frac{y-2^{-n}k}{2^n}\right),$$

where $n = 0, 1, 2, \dots$, $0 \leq k \leq 2^n - 1$, $m = 1, 2, 3$. Consider the Haar system,

$$H = \{\Phi\} \cup \{\Psi_{n,k}^m\}_{n=0,1,2,\dots, \quad 0 \leq k \leq 2^n - 1, \quad m=1,2,3}.$$

We claim that H forms a basis for $L_p[0, 1]^2$. Indeed, we have

$$\Phi + \Psi_{0,0}^1 = \begin{cases} 2 & \text{if } (x, y) \in [0, 1] \times [0, \frac{1}{2}] \\ 0 & \text{if } (x, y) \in [0, 1] \times [\frac{1}{2}, 1], \end{cases}$$

and

$$\Phi + \Psi_{0,0}^2 = \begin{cases} 2 & \text{if } (x, y) \in [0, \frac{1}{2}] \times [0, 1] \\ 0 & \text{if } (x, y) \in [\frac{1}{2}, 1] \times [0, 1], \end{cases}$$

and

$$\Phi + \Psi_{0,0}^3 = \begin{cases} 2 & \text{if } (x, y) \in [0, \frac{1}{2}] \times [0, \frac{1}{2}] \\ 0 & \text{if } (x, y) \in [\frac{1}{2}, 1] \times [0, \frac{1}{2}] \\ 0 & \text{if } (x, y) \in [0, \frac{1}{2}] \times [\frac{1}{2}, 1] \\ 2 & \text{if } (x, y) \in [\frac{1}{2}, 1] \times [\frac{1}{2}, 1]. \end{cases}$$

Then, one can show that $\chi_{[\frac{k}{2^n}, \frac{k+1}{2^n})^2}$ is a finite sum of the members of the Haar system H , so $\chi_{[\frac{k}{2^n}, \frac{k+1}{2^n})^2}$ is in the

$\text{span}(H)$ for $n = 0, 1, 2, \dots$, $0 \leq k \leq 2^n - 1$. It is easy to check that,

$$\text{span}(H) = \text{span}\{\chi_{[\frac{k}{2^n}, \frac{k+1}{2^n})^2}\}_{n=0,1,2,\dots, \quad 0 \leq k \leq 2^n - 1} \quad (16)$$

Similarly as the one-dimensional case, the right-hand side of (16) is dense in $L_p[0, 1]^2$, and therefore, the Haar system H is complete in $L_p[0, 1]^2$ for $1 \leq p < \infty$.

Now, we show that H actually is a basis for $L_p[0, 1]^2$.

Enumerate the Haar system as ,

$$\{h_1, h_2, \dots\} = \{\Phi, \Psi_{0,0}^m, \Psi_{1,0}^m, \Psi_{1,1}^m, \dots\}. \quad (17)$$

Indeed, for a natural number $N > 1$ and scalars c_1, c_2, \dots, c_N , we consider the functions,

$$g_{N-1} = \sum_{n=1}^{N-1} c_n h_n \text{ and } g_N = \sum_{n=1}^N c_n h_n.$$

Note that g_{N-1} and g_N agree possibly on the square $S = [0, 1]^2$. Let $S_{11} = [0, \frac{1}{2}]^2$, $S_{12} = [0, \frac{1}{2}] \times [\frac{1}{2}, 1]$, $S_{21} = [\frac{1}{2}, 1] \times [0, \frac{1}{2}]$, $S_{22} = [\frac{1}{2}, 1]^2$.

Then, g_{N-1} takes a constant value c on S , and there are constants d_N^1 and d_N^2 such that :

$$g_N = \begin{cases} c + d_N^1 & \text{on } S_{11} \\ c - d_N^1 & \text{on } S_{22} \\ c + d_N^2 & \text{on } S_{12} \\ c - d_N^2 & \text{on } S_{21}. \end{cases}$$

Let m be the integer such that the area of S_{ij} , is 2^{-2m} for $1 \leq i, j \leq 2$. Then

$$\begin{aligned} & \int_S \left| \sum_{n=1}^N c_n h_n(t) \right|^p dt - \int_S \left| \sum_{n=1}^{N-1} c_n h_n(t) \right|^p dt \\ &= \int_S |g_N(t)|^p dt - \int_S |g_{N-1}(t)|^p dt \\ &= \int_{S_{11}} |c + d_N^1|^p dt + \int_{S_{12}} |c - d_N^2|^p dt + \int_{S_{22}} |c - d_N^1|^p dt \\ & \quad + \int_{S_{21}} |c + d_N^2|^p dt - \int_S |c|^p dt \\ &= \frac{|c + d_N^1|^p}{2^{2m+2}} + \frac{|c - d_N^2|^p}{2^{2m+2}} + \frac{|c - d_N^1|^p}{2^{2m+2}} + \frac{|c + d_N^2|^p}{2^{2m+2}} - \frac{|c|^p}{2^{2m}} \\ &= 2^{-2m-2} (|c + d_N^1|^p + |c - d_N^2|^p \\ & \quad + |c - d_N^1|^p + |c + d_N^2|^p - 4|c|^p). \end{aligned} \quad (18)$$

Note that quantity in equation (18) is nonnegative. Therefore,

$$\int_S \left| \sum_{n=1}^{N-1} c_n h_n(t) \right|^p dt \leq \int_S \left| \sum_{n=1}^N c_n h_n(t) \right|^p dt.$$

So

$$\begin{aligned} \left| \sum_{n=1}^{N-1} c_n h_n \right|_p &= \left(\int_S \left| \sum_{n=1}^{N-1} c_n h_n(t) \right|^p dt \right)^{\frac{1}{p}} \\ &\leq \left(\int_S \left| \sum_{n=1}^N c_n h_n(t) \right|^p dt \right)^{\frac{1}{p}} = \left\| \sum_{n=1}^N c_n h_n \right\|^p. \end{aligned}$$

Thus, $H = \{h_n\}$ is a monotone basis for $L_p[0, 1]^2$.

Remark 1 It is worthwhile to mention that for

$$H_1 = \{\chi_{[0,1]}\} \cup \{\psi_{n,k}\}_{n=0,1,2,\dots} \\ 0 \leq k \leq 2^n - 1$$

, the Haar system $H = H_1 \otimes H_1$ is a basis for $L_{p_1}[0, 1] \otimes L_{p_2}[0, 1]$, where $1 \leq p_1, p_2 < \infty$. See (Gelbaum and Gil de lamarid 1960).

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