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Beurling density on expansible locally compact groups and systems of translations

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ABSTRACT

In this paper, we investigate some properties of the expansive automorphisms on expansible locally compact groups. We define the upper and lower Beurling densities on expansible locally compact groups. Using this definition, we show that if for some $1 < p' < \infty$, the finite union of translates $\cup_{k=1}^{n} T_p(f_k, \Gamma_k)$ is a p'-Bessel sequence, then the upper Beurling density is finite, and if $\cup_{k=1}^{n} T_p(f_k, \Gamma_k)$ is a (C_q)-system, then the upper Beurling density cannot be finite. In particular, we conclude that there exists no p'-Bessel (C_q)-system in $L^p(G)$ of the form $\cup_{k=1}^{n} T_p(f_k, \Gamma_k)$, where G is an expansible group.

1. Introduction

For any $x \in \mathbb{R}^d$, and $h > 0$, let $Q_h(x)$ denote the closed cube centred at x of side length h. Suppose that $\Lambda = (\lambda_i)_{i \in I}$ is a (uniformly) discrete subset of $\mathbb{R}^d$, that is, there is a separation constant $\delta > 0$ such that $|\lambda_i - \lambda_j| \geq \delta > 0$, for $i \neq j$. Then, the upper Beurling density of Λ is defined as

$$D^+(\Lambda) = \limsup_{h \to \infty} \sup_{x \in \mathbb{R}^d} \frac{\#(\Lambda \cap Q_h(x))}{h^d}, \tag{1.1}$$

and the lower Beurling density is defined as

$$D^-(\Lambda) = \liminf_{h \to \infty} \inf_{x \in \mathbb{R}^d} \frac{\#(\Lambda \cap Q_h(x))}{h^d}. \tag{1.2}$$

If $D^+(\Lambda) = D^-(\Lambda)$, then Λ is said to have (uniform) density $D(\Lambda)$.

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In 2008, Gröchenig, Kutynoik and Seip extended the definition of the Beurling density on locally compact abelian groups [10]. Their results apply to compactly generated abelian groups. The main obstacle for not defining the Beurling densities similar to (1.1) and (1.2), was the absence of a natural substitution of rescaling. Therefore, they used a different approach. The main innovation of this paper, which compensates for the lack of rescaling, is that we use expansive automorphisms to define the upper and lower Beurling densities on expansible locally compact groups which are not necessarily abelian. In order to do so, we need some preliminaries and therefore we consider some properties of expansive automorphisms. Among them, it is shown that if the expansive neighborhood is compact, then the expansive property is independent of choice of the expansive neighborhood. The expansive automorphism has been defined by many authors, including [2,15].

Let G be a locally compact group and let $L^p(G)$ for $1 \leq p < \infty$ denote the usual Lebesgue space with respect to a fixed left Haar measure μ. For a subset (discrete) Γ of G and $g \in L^p(G)$, the translates of g along with Γ is defined as $T(g,\Gamma) = \{T_\gamma g\}_{\gamma\in\Gamma}$ in which $T_\gamma g(x) = g(\gamma^{-1}x)$.

In [4], it is shown that for any finite collection of nonzero elements $\{g_k\}_{k=1}^r \subseteq L^2(\mathbb{R}^d)$ and the disjoint unions $\Gamma = \cup_{k=1}^r \Gamma_k$, the necessary condition for $\cup_{k=1}^r T(g_k,\Gamma_k)$ to possess an upper frame bound in $L^2(\mathbb{R}^d)$, is that the upper Beurling density be finite. Also, in [4] it is shown that if $\cup_{k=1}^r T(g_k,\Gamma_k)$ has a lower frame bound for $L^2(\mathbb{R}^d)$, then $D^+(\Gamma) = \infty$. In [12], these results are extended to (C_q)-systems and p-Bessel sequences from $L^2(\mathbb{R}^d)$ to the reflexive Banach space $L^p(\mathbb{R}^d)$ $(1 < p < \infty)$. For the readers' convenience and in order to affirm the notations, we recall briefly on the concepts of (C_q)-system and p-Bessel sequence.

Let X be a separable Banach space with the dual space X^*, and $1 < p < \infty$. A family $\{f_k\}_1^\infty \subseteq X$ is a p-Bessel sequence for X^* if there exists constant $B > 0$ such that

$$\sum_1^\infty |(h, f_k)|^p \leq B\|h\|^p, \qquad \text{for all } h \in X^*.$$

Any 2-Bessel sequence is known as a Bessel sequence.

Now, assume that X is a separable reflexive Banach space, and $1 < q < \infty$ is a fixed number. The sequence $\{f_k\}_1^\infty \subseteq X$ is said to be a (C_q)-system in X with constant $C > 0$ if for any $f \in X$ and $\varepsilon > 0$, there exists a generalized linear combination $g = \sum_k a_k f_k$ in X such that

$$\|f - g\|_X < \varepsilon \;\text{ and }\; \left(\sum |a_n|^q\right)^{\frac{1}{q}} \leq C\|f\|_X,$$

where $C = (C_q)$ is a positive constant, not depending on f.

In this paper, we aim to extend the concepts of Beurling densities on expansible groups and investigate the necessary conditions for $\cup_{k=1}^r T(g_k,\Gamma_k)$ to be a p'-Bessel sequence $(1 < p' < \infty)$, and to be a C_q-system for $L^p(G)$, in which G is an expansible group.

The structure of this paper is as follows. In section 2, we recall the definition of an expansive automorphism and investigate some of its properties which are needed in the sequel. In section 3, we use an expansive automorphism α to define the upper and lower Beurling densities and prove one of our main theorems which states some equivalent conditions for the upper Beurling density to be finite (Theorem 3.7). This theorem plays a crucial role in assisting us achieve other main results in section 4. Finally, in section 4, we prove the other main result, namely, Theorem 4.5, which states that for an expansible group G if $\cup_{k=1}^m T_p(f_k,\Gamma_k)$ is a p'-Bessel sequence for some $1 < p' < \infty$, then the upper Beurling density is finite. Also, if $\cup_{k=1}^m T_p(f_k,\Gamma_k)$ is a (C_q)-system, then the upper Beurling density is infinite. In particular, we conclude that there exists no p'-Bessel (C_q)-system in $L^p(G)$ of the form $\cup_{k=1}^m T_p(f_k,\Gamma_k)$.

2. Preliminaries and notations

In this section, we briefly review the concepts of expansive automorphisms and expansive groups. Then, we state and investigate some properties of expansive automorphisms. Among them, we show that the expansive property is independent of choice of the expansive neighborhood. Throughout the paper, G is always a non-discrete locally compact group with the identity e and with the left Haar measure μ.

Definition 2.1. A topological automorphism α on the locally compact group G is called *expansive* if there is a neighborhood U of the identity e such that for any $x \neq e$ in U, $\alpha^n(x) \notin U$, for some $n \in \mathbb{N}$. Such a U is called an expansive neighborhood for α.

Note that if U is an expansive neighborhood, then any neighborhood V of e with $V \subset U$ is also an expansive neighborhood. In particular, we may always choose a compact neighborhood as an expansive neighborhood.

Obviously, α is expansive if and only if $\bigcap_{n\geq 1} \alpha^{-n}(U) = \{e\}$.

In the sequel, we show that the expansive property is independent of choice of the neighborhood U.

Definition 2.2. The locally compact group G is called *expansible* if it admits an expansive automorphism.

Some examples of expansible groups include the Euclidean vector group $\mathbb{R}^d$ and the additive group Δ_P of the field of p-adic numbers, and the 3-dimensional p-adic Heisenberg group $\mathbb{Q}_p^3$, [15].

In the next lemma, we collect some properties of expansive automorphisms which are needed in the sequel.

Lemma 2.3. *Let G be an expansible group with an expansive automorphism α and U be a closed neighborhood of the identity. For every $n \in \mathbb{Z}$ we set $U_n = \bigcap_{k\geq n} \alpha^k(U)$. Then we have:*

(i) *$U_n \subset U_{n+1}$ and $\alpha(U_n) = U_{n+1}$ for any $n \in \mathbb{Z}$;*
(ii) *$\bigcup_{n\geq 1} U_n = G$;*
(iii) *Every U_n has non-void interior;*
(iv) *For every compact subset C of G there exists some $n_0 \in \mathbb{N}$ such that $C \subset \alpha^n(U)$, for any $n \geq n_0$.*

Proof. (i) Obviously, we have $U_n \subset U_{n+1}$. The second part follows from the following equalities.

$$\alpha(U_n) = \alpha(\bigcap_{k\geq n} \alpha^k(U)) = \bigcap_{k\geq n} \alpha^{k+1}(U) = U_{n+1}.$$

(ii) Since α is an expansive automorphism, for any $x \in G$, there exists $n_0 \in \mathbb{N}$ such that $x \in \alpha^n(U)$, for any $n \geq n_0$.
(iii) Using (ii), we have $G = \bigcup_{n\geq 1} U_n$, in which every U_n is closed. By Baire's category Theorem, some U_n have non-void interior. Now, (i) implies that every U_n has non-void interior.
(iv) Let C be a compact subset of G. Hence, there exist $x_1, \cdots, x_m$ in G such that $C \subset x_1 U \cup \cdots \cup x_m U$. Moreover, there exist some $n_0 \in \mathbb{N}$, such that $x_j \in \alpha^n(U)$ for all $j = 1, \cdots, m$ and $n \geq n_0$. Thus, $C \subseteq \alpha^n(U^2) \subseteq \alpha^m(U)$, for any $m \geq n \geq n_0$. This completes the proof. $\square$

Remark 2.4. Part (i) of Lemma 2.3 implies that for any neighborhood U of the identity element e of G, $U \subset \alpha(U)$.

The following lemma shows that the expansive property is independent of the choice of the expansive neighborhood.

Lemma 2.5. *Let $\alpha \in Aut(G)$ be an expansive automorphism with an expansive neighborhood U. If U is compact, then for any compact neighborhood V of the identity e and for any $x \neq e$, in V, there exists $n_0 \in \mathbb{N}$ such that $\alpha^n(x) \notin V$, for all $n \geq n_0$.*

Proof. Let V be any compact neighborhood of e. We investigate three possible situations between the sets U and V.
(i) $V \subset U$: In this case there exists $n_0 \in \mathbb{N}$ such that $\alpha^n(x) \notin U$. Since $V \subset U$, we obtain $\alpha^n(x) \notin V$, for any $n \geq n_0$.
(ii) $U \subset V$: Let $x \in V$ and $x \neq e$. By part (iv) of Lemma 2.3, there is some $n_0 \in \mathbb{N}$ such that $V \subset \alpha^n(U)$, for all $n \geq n_0$. Hence, $x \in \alpha^n(U)$ and so, $\alpha^{-n}(x) \in U$ for all $n \geq n_0$. By assumption, there exists $m_0 \in \mathbb{N}$ such that $\alpha^{m-n}(x) \notin U$, for each $m \geq m_0$ and $n \geq n_0$. So, $x \notin \alpha^{n-m}(U)$, for all $m \geq m_0$ and $n \geq n_0$. Since $V \subset \alpha^n(U)$, for all $n \geq n_0$, we have $\alpha^{-m}(V) \subset \alpha^{n-m}(U)$, for any $m \geq m_0$. Hence, $x \notin \alpha^{-m}(V)$, for all $m \geq m_0$.
(iii) $V \cap U \neq \emptyset$ and neither $V \subset U$ nor $U \subset V$; Put $W = V \cap U$. Since $W \subset U$, by part (i) there exists $n_0 \in \mathbb{N}$ such that $\alpha^n(x) \notin W$ for any $n \geq n_0$. On the other hand, $W \subset V$ hence by part (ii), the result follows. □

Let μ be a fixed left Haar measure on locally compact group G. For a topological automorphism α on G, let μ_α be defined by $\mu_\alpha(U) = \mu(\alpha(U))$, for any Borel subset U of G. Then μ_α is also a left Haar measure on G. So, by the uniqueness of the left Haar measure, there exists a (unique) positive real number Δ_α such that $\mu_\alpha(U) = \Delta_\alpha.\mu(U)$. The following calculation shows that $\Delta : \ Aut(G) \to (0, \infty)$, defined by $\alpha \mapsto \Delta_\alpha$, is a homomorphism.

$$\begin{aligned}
\Delta_{\alpha o \beta}\mu(U) &= \mu(\alpha o \beta(U)) \\
&= \mu(\alpha(\beta(U))) \\
&= \Delta_\alpha \mu(\beta(U)) \\
&= \Delta_\alpha \Delta_\beta \mu(U),
\end{aligned}$$

in which α, β are automorphisms on G.

In the following proposition, we provide a sufficient condition for Δ_α to be in $(1, \infty)$.

Proposition 2.6. *Suppose that $\alpha \in Aut(G)$ is expansive, then $\Delta_\alpha > 1$.*

Proof. In the light of Lemma 2.3 (iv), G cannot be compact. Let U be a compact neighborhood of the identity e. Then by part (ii) of Lemma 2.3, we have $G = \liminf_{n\to\infty} \alpha^n(U)$. Consequently, Fatou's Lemma yields.

$$\infty = \mu(G) \leq \liminf_{n\to\infty} \mu(\alpha^n(U)) = \liminf_{n\to\infty} \Delta_\alpha^n \mu(U).$$

Hence, $\Delta_\alpha > 1$. □

In order to see more properties on expansive automorphisms, we refer the reader to [2,15].

In the forthcoming definition, we follow H. Feichtinger and K. Gröchenig, in [8].

Definition 2.7. A family $\{\lambda_i\}_{i\in I}$ is called *V-separated* or *uniformly separated* if for some relatively compact neighborhood V of the identity e, the sets $\{\lambda_i V\}_{i\in I}$ are pairwise disjoint.

The family $\{\lambda_i\}_{i\in I}$ is called *relatively uniformly separated* if it is the finite union of uniformly separated families.

Proposition 2.8. *Suppose that $\Lambda = \{\lambda_i\}_{i\in I}$ is V-separated and $\alpha \in Aut(G)$. Then $\alpha(\Lambda)$ is $\alpha(V)$-separated. In particular, if Λ is V-separated and α is an expansive automorphism, then $\alpha(\Lambda)$ is V-separated.*

Proof. By assumption, we have $\lambda_i V \cap \lambda_j V = \emptyset$, for each $i \neq j$. Since α is an automorphism, $\alpha(\lambda_i V \cap \lambda_j V) = \emptyset$, for each $i \neq j$. This implies that $\alpha(\lambda_i)\alpha(V) \cap \alpha(\lambda_j)\alpha(V) = \emptyset$, for each $i \neq j$. $\square$

3. Beurling density on expansible locally compact groups

In this section, first we recall the definitions of the upper and lower Beurling densities in $\mathbb{R}^n$ for a discrete set Λ, then we give the corresponding definitions on some locally compact groups.

For $h > 0$ and x a point in $\mathbb{R}^d$, let $Q_h(x)$ denote the closed cube centered at x of side length h.

Then the *lower Beurling density* of a discrete set Λ is defined by

$$D^-(\Lambda) = \liminf_{h\to\infty} \inf_{x\in\mathbb{R}^d} \frac{\#(\Lambda \cap Q_h(x))}{h^d},$$

and the *upper Beurling density* is defined by

$$D^+(\Lambda) = \limsup_{h\to\infty} \sup_{x\in\mathbb{R}^d} \frac{\#(\Lambda \cap Q_h(x))}{h^d}.$$

The following definition is the main innovation of our approach which has substantial consequences. This approach can compensate for the lack of rescaling in the definition of the Beurling density on locally compact groups.

Definition 3.1. Assume that G is an expansible group with an expansive automorphism α, and with the left Haar measure μ. Let $\Lambda = \{\lambda_i\}_{i\in I} \subseteq G$ and U be a compact neighborhood of the identity e. For each natural number n, let $\nu^-(n)$ and $\nu^+(n)$ denote respectively the smallest and largest number of points of Λ that lie in any $x\alpha^n(U)$, for all $x \in G$. In other words,

$$\nu^-(n) = \inf_{x\in G} \#(\Lambda \cap x\alpha^n(U)), \text{ and } \nu^+(n) = \sup_{x\in G} \#(\Lambda \cap x\alpha^n(U)).$$

Note that we have $0 \leq \nu^-(n) \leq \nu^+(n) \leq \infty$, for each $n \in \mathbb{N}$. Then the *lower Beurling density* of the set Λ is defined as

$$D^-(\Lambda) = \liminf_{n\to\infty} \frac{\nu^-(n)}{\mu(\alpha^n(U))},$$

and the *upper Beurling density* is defined as

$$D^+(\Lambda) = \limsup_{n\to\infty} \frac{\nu^+(n)}{\mu(\alpha^n(U))}.$$

We have $0 \leq D^-(\Lambda) \leq D^+(\Lambda) \leq \infty$. If $D^+(\Lambda) = D^-(\Lambda)$, then Λ is said to have uniform Beurling density, $D(\Lambda) = D^+(\Lambda) = D^-(\Lambda)$. Note that if Λ is the disjoint union of $\Lambda_1, \ldots, \Lambda_r$, then we always have $\#(\Lambda \cup x\alpha^n(U)) = \sum_{k=1}^r \#(\Lambda_k \cup x\alpha^n(U))$ and therefore $\sum_{k=1}^r D^-(\Lambda_k) \leq D^-(\Lambda) \leq D^+(\Lambda) \leq \sum_{k=1}^r D^+(\Lambda_k)$.

Now, we give an example that demonstrates the definition of the upper and lower Beurling densities in $\mathbb{R}^d$, as a special case of Definition 3.1.

Example 3.2. Let $G = \mathbb{R}^d$, $U = Q_1(0)$ be the unit cube centered at 0 and Λ be a discrete set in $\mathbb{R}^d$. Consider the expansive automorphism $\alpha : \mathbb{R}^d \to \mathbb{R}^d$, given by $\alpha(x) = 2x$, for $x \in \mathbb{R}^d$. Then definitions of the upper and lower Beurling densities coincide with the discrete case of the upper and lower Beurling densities in $\mathbb{R}^d$ respectively, that is

$$\begin{aligned} D^+(\Lambda) &= \limsup_{n\to\infty} \frac{\sup_{x\in G} \#(\Lambda \cap x\alpha^n(U))}{\mu(\alpha^n(U))} \\ &= \limsup_{n\to\infty} \frac{\sup_{x\in\mathbb{R}^d} \#(\Lambda \cap Q_{2^n}(x))}{(2^n)^d}, \end{aligned}$$

and

$$\begin{aligned} D^-(\Lambda) &= \liminf_{n\to\infty} \frac{\inf_{x\in G} \#(\Lambda \cap x\alpha^n(U))}{\mu(\alpha^n(U))} \\ &= \liminf_{n\to\infty} \frac{\inf_{x\in\mathbb{R}^d} \#(\Lambda \cap Q_{2^n}(x))}{(2^n)^d}. \end{aligned}$$

Also, we provide some other examples, well beyond the case of $\mathbb{R}^d$.

Example 3.3. Consider the subgroup of the upper triangular 3×3 matrices of the form

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & -a \\ 0 & 0 & 1 \end{bmatrix},$$

where $a, b \in \mathbb{R}$. This group can be identified with $\mathbb{R}^2$ with the product

$$(a_1, b_1) \circ (a_2, b_2) = (a_1 + a_2, b_1 + b_2 - a_1a_2).$$

Then $G = (\mathbb{R}^2, \circ)$ is a locally compact abelian group with the identity $e = (0, 0)$ and $(a, b)^{-1} = (-a, -b-a^2)$. The Haar measure on this group is the Lebesgue measure. The mapping $\alpha : G \to G$ defined by $\alpha(a, b) = (2a, 4b)$, is an expansive automorphism on G. Let U be a compact neighborhood of the identity, and Λ be a sequence in G. Then the upper Beurling density of Λ is defined as

$$D^+(\Lambda) = \limsup_{n\to\infty} \frac{\sup_{(a,b)\in G} \#(\Lambda \cap (a, b)\alpha^n(U))}{\mu(\alpha^n(U))},$$

and the lower Beurling density is defined as

$$D^-(\Lambda) = \liminf_{n\to\infty} \frac{\inf_{(a,b)\in G} \#(\Lambda \cap (a, b)\alpha^n(U))}{\mu(\alpha^n(U))}.$$

To work out the following examples we need to briefly discuss the group of p-adic integers. For further information concerning p-adic analysis, the reader is referred to [9,11].

Fix a prime number p. Any nonzero rational number r can be written uniquely as $r = p^m q$, where $m \in \mathbb{Z}$ and q is a rational number whose numerator and denominator are not divisible by p. We define the p-adic norm of r, denoted by $|r|_p$, to be p^{-m}, and we set $|0|_p = 0$. The p-adic norm satisfies

$$|r_1 + r_2|_p \le \max(|r_1|_p, |r_2|_p), \;\; |r_1r_2|_p = |r_1|_p|r_2|_p,$$

and induces a non-Archimedian metric

$$d_p(r_1, r_2) = |r_1 - r_2|_p,$$

on $\mathbb{Q}$, with respect to which the completion of $\mathbb{Q}$ yields a field that is called the field of p-adic numbers and is denoted by $\mathbb{Q}_p$. $\mathbb{Q}_p$ is a locally compact abelian group under addition, with the Haar measure $d\lambda$ for which we have $d\lambda(cx) = |c|_p d\lambda(x)$, for $x, c \in \mathbb{Q}_p$ (see p. 202 of [11]). The closed unit ball $\Delta_p = \{x \in \mathbb{Q}_p; |x|_p \leq 1\}$, is an additive subgroup of $\mathbb{Q}_p$, which is called the group of p-adic integers.

Example 3.4. For some prime number p, let $\mathbb{Q}_p$ denote the locally compact field of p-adic numbers. Then for every $t \in \mathbb{Q}_p \setminus \{0\}$, there is a topological automorphism α_t on the additive group Δ_p, defined by $\alpha_t(x) = tx$, for any $x \in \Delta_p$. Let $|.|_p$ denote the p-adic valuation of $\mathbb{Q}_p$. In view of $|\alpha_t(x)|_p = |t|_p|x|_p$, if $|t|_p > 1$, then α_t is expansive. Assume that U is compact a neighborhood of the identity in Δ_p, and Λ is a sequence in Δ_p. Then the upper Beurling density of Λ is defined as

$$D^+(\Lambda) = \limsup_{n\to\infty} \frac{\sup_{x\in\Delta_p} \#(\Lambda \cap x\alpha_t^n(U))}{\mu(\alpha_t^n(U))},$$

and the lower Beurling density is defined as

$$D^-(\Lambda) = \liminf_{n\to\infty} \frac{\inf_{x\in\Delta_p} \#(\Lambda \cap x\alpha_t^n(U))}{\mu(\alpha_t^n(U))}.$$

Example 3.5. Fix a prime number p. Let Δ_p be the group of p-adic integers. Consider the locally compact abelian group $G = \mathbb{R} \times \Delta_p$. Let $a := (1/p, 0, 0, ...) \in \Delta_p$. Then the mapping $\alpha : \mathbb{R} \times \Delta_p \to \mathbb{R} \times \Delta_p$ defined for $(x, v) \in \mathbb{R} \times \Delta_p$, by $\alpha(x, v) = (2x, av)$, is a topological automorphism on $\mathbb{R} \times \Delta_p$ (p. 434(e) of [11]) and α is expansive. Now, assume that U is a compact neighborhood of the identity, and let Λ be a sequence in $\mathbb{R} \times \Delta_p$. Then the upper Beurling density of Λ is defined as

$$D^+(\Lambda) = \limsup_{n\to\infty} \frac{\sup_{(x,v)\in\mathbb{R}\times\Delta_p} \#(\Lambda \cap (x, v)\alpha^n(U))}{\mu(\alpha^n(U))},$$

and the lower Beurling density is defined as

$$D^-(\Lambda) = \liminf_{n\to\infty} \frac{\inf_{(x,v)\in\mathbb{R}\times\Delta_p} \#(\Lambda \cap (x, v)\alpha^n(U))}{\mu(\alpha^n(U))}.$$

Now we provide an example for a nonabelian locally compact group.

Example 3.6. Let $G = \mathbb{Q}_p \times \mathbb{Q}_p \times \mathbb{Q}_p$ be the 3-dimensional Heisenberg group whose binary operation is given by

$$(x, y, z)(x', y', z') = (x + x', y + y', z + z' + \frac{1}{2}(xy' - x'y)).$$

G is a totally disconnected locally compact group. The mapping $\alpha(x, y, z) = (p^{-1}x, p^{-1}y, p^{-2}z)$, for all $(x, y, z) \in G$, is an automorphism on G. It is not hard to see that α is expansive. Consider U as a compact neighborhood of the identity, and let Λ be a sequence in G. Then the upper Beurling density of Λ is defined as

$$D^+(\Lambda) = \limsup_{n\to\infty} \frac{\sup_{(x,y,z)\in\mathbb{Q}_p\times\mathbb{Q}_p\times\mathbb{Q}_p} \#(\Lambda \cap (x, y, z)\alpha^n(U))}{\mu(\alpha^n(U))},$$

and the lower Beurling density is defined as

$$D^-(\Lambda) = \liminf_{n\to\infty} \frac{\inf_{(x,y,z)\in\mathbb{Q}_p\times\mathbb{Q}_p\times\mathbb{Q}_p} \#(\Lambda \cap (x,y,z)\alpha^n(U))}{\mu(\alpha^n(U))}.$$

The following theorem is a generalization of [5, Lemma 2.3] in the setting of an expansible group G, which provides some equivalent ways to view the meaning of the finite upper Beurling density.

Theorem 3.7. *Let G be an expansible group with an expansive automorphism α, and the left Haar measure μ. Let $E = \{e_i\}_{i\in I} \subset G$, and U be a compact neighborhood of the identity. Then the following statements are equivalent.*

(i) $D^+(E) < \infty$.
(ii) *E is relatively uniformly separated.*
(iii) *There exists $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$,*

$$\sup_{x\in G} \#(E \cap xa^n(U)) < \infty.$$

Proof. (i $\Rightarrow$ iii): Let $\nu^+(n) = \sup_{x\in G} \#(E \cap x\alpha^n(U))$. Since $D^+(E) < \infty$, for every $\varepsilon > 0$ there exists M_ε such that for all $n > M_\varepsilon$ we have

$$\frac{\nu^+(n)}{\mu(\alpha^n(U))} - D^+(E) < \varepsilon.$$

This implies that $\nu^+(n) < \infty$.

(iii $\Rightarrow$ ii) Suppose that there exists $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$, $\sup_{x\in G} \#(E \cap xa^n(U)) < \infty$. In order to prove E is relatively uniformly separated by [8, Lemma 3.3], it is enough to show that for any relatively compact set W with non-void interior, we have $\sup_{i\in I} \#\{k|e_kW \cap e_iW \neq \emptyset\} < \infty$. Using Lemma 2.3, there exists $n_0 \in \mathbb{N}$ such that $\overline{W} \subseteq \alpha^n(U)$, for any $n \geq n_0$. Hence, it is sufficient to show that $\sup_{i\in I} \#\{k|e_k\alpha^n(U) \cap e_i\alpha^n(U) \neq \emptyset\} < \infty$. Suppose that there exists $i_0 \in I$ such that $\#\{k|e_k\alpha^n(U) \cap e_{i_0}\alpha^n(U) \neq \emptyset\} = \infty$. So, there exist infinitely many $k \in I$ such that $e_k\alpha^n(U) \cap e_{i_0}\alpha^n(U) \neq \emptyset$. Thus, there exists $m_0 \in \mathbb{N}$, such that for any $m \geq m_0$, $e_k \in e_{i_0}\alpha^m(U)$, for all such $k \in I$. On the other hand, $e_{i_0}\alpha^m(U) \subseteq \cup_{x\in G} x\alpha^n(U^\circ)$, then $e_{i_0}\alpha^m(U) \subseteq \cup_{s=1}^{\ell} x_s\alpha^n(U^\circ)$. By the assumption, the number of elements of E in each $x_s\alpha^n(U^)$ is finite, for $s = 1, ..., \ell$. Therefore, the number of elements of E that exists in $\cup_{s=1}^{\ell} x_s\alpha^n(U)$ is finite, while we have concluded that for infinitely many k, $e_k \in \cup_{s=1}^{\ell} x_s\alpha^n(U^\circ)$, which is a contradiction.

(ii $\Rightarrow$ i) Assume that E is relatively uniformly separated. So by the definition, we may partition I into disjoint sets $I_1, \cdots, I_r$ such that each $E_k = \{e_i\}_{i\in I_k}$ is U_k-uniformly separated, for $k = 1, \ldots, r$. Let $W = \cap_{k=1}^{r} U_k$. Thus, for any $x \in G$, the set xW contains at most one element of E_k, for $k = 1, \ldots, r$, and thus, at most r elements of E. Therefore,

$$\begin{aligned} D^+(E) &= \limsup_{n\to\infty} \sup_{x\in G} \frac{\#(E \cap x\alpha^n(W))}{\mu(\alpha^n(W))} \\ &\leq \limsup_{n\to\infty} \frac{r.\Delta_\alpha^n.\mu(W)}{\Delta_\alpha^n \mu(W)} \\ &= r < \infty. \quad \square \end{aligned}$$

Note that under the hypothesis of Theorem 3.7, we may consider the part (iii) of Theorem 3.7 as follows.

Remark 3.8. Let G be an expansible group with an expansive automorphism α. Let $E = \{e_i\}_{i\in I} \subset G$, and U be a compact neighborhood of the identity. Then the following statements are equivalent.

(i) There exists $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$, $\sup_{x\in G} \#(E \cap x a^n(U)) < \infty$.
(ii) For any $n \in \mathbb{N}$, we have $\sup_{x\in G} \#(E \cap x a^n(U)) < \infty$.

Proof. The implication (ii $\Rightarrow$ i) is clear. So, we show the converse.
(i $\Rightarrow$ ii) For any $n < n_0$, $\alpha^n(U) \subseteq \alpha^{n_0}(U)$, so for any $n < n_0$

$$\sup_{x\in G} \#(E \cap x a^n(U)) < \sup_{x\in G} \#(E \cap x a^{n_0}(U)) < \infty.$$

The rest of the proof follows from (i). $\square$

4. Upper Beurling density and systems formed by translates of finite sets of elements in $L^p(G)$

In this section, we investigate a necessary condition for which a system of translates possesses a p'-Bessel sequence and a C_q-system in the Banach space $L^p(G)$, in which G is an expansible group.

Aldroubi, Sun and Tang, in 2001, introduced the concept of p-frame in $L^p(\mathbb{R})$ [1]. This concept is a generalization of the classical (Hilbert) frames [4,7] and can be extended to Banach spaces [3,6].

Let X be a separable Banach space, X^* be its topological dual, and $1 < p < \infty$. As mentioned earlier, a family $\{f_k\}_1^\infty \subset X$ is a *p-Bessel sequence* for X^*, if there exists constant $B > 0$ such that

$$\sum_{k=1}^{\infty} |(h, f_k)|^p \leq B\|h\|^p, \quad \text{for all } h \in X^*.$$

Let G be a locally compact group and $\Gamma \subseteq G$. For $f \in L^p(G)$, the translation operator T_γ on $L^p(G)$ is defined as

$$T_\gamma f(x) = f(\gamma^{-1}x), \quad \text{for all } x \in G.$$

If $\Gamma \subseteq G$, then the collection of translates of $f \in L^p(G)$ along with Γ is defined as $T_p(f, \Gamma) = \{T_\gamma f\}_{\gamma\in\Gamma}$.

We need the following lemma which is basic for deriving our results in this section.

Lemma 4.1. *Let G be an expansible group with an expansive automorphism α, and Γ be a sequence in G, and $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Assume that $f \in L^p(G)$, $\tilde{f} \in L^q(G)$, and $(f, \tilde{f}) \neq 0$. If Γ is not relatively uniformly separated, then for any $0 < \varepsilon < |(f, \tilde{f})|$, we have*

$$\sup_{y\in G} \#\{\gamma \in \Gamma : \ |(T_\gamma f, T_y \tilde{f})| > \varepsilon\} = \infty,$$

where $(f, \tilde{f})$ denotes the dual pairs of L^p and L^q.

Proof. Consider the function $x \mapsto (T_x f, \tilde{f})$, for $x \in G$. Since this function is continuous at e, so for any $0 < \varepsilon < |(f, \tilde{f})|$, there is a compact neighborhood $\alpha^n(U)$ of the identity e, for some $n > 0$ such that

$$\inf_{x\in\alpha^n(U)} |(T_x f, \tilde{f})| > \varepsilon.$$

Indeed, if we fix $0 < \varepsilon < |(f, \tilde{f})|$ and set $\varepsilon' = |(f, \tilde{f})| - \varepsilon$, then $|(T_x f, \tilde{f}) - (f, \tilde{f})| < \varepsilon'$. So, $|(f, \tilde{f})| - |(T_x f, \tilde{f})| < |(f, \tilde{f})| - \varepsilon$. Then we have $|(T_x f, \tilde{f})| > \varepsilon$ for all $x \in \alpha^n(U)$. The function $x \mapsto (T_x f, \tilde{f})$ is continuous, thus, it takes its minimum on $\alpha^n(U)$.

By Theorem 3.7, there exists $N_n \in \mathbb{N}$, and $y \in G$ such that the compact neighborhood $y\alpha^n(U)$ contains at least N_n elements of Γ. Since for any $\gamma \in y\alpha^n(U)$, we have $y^{-1}\gamma \in \alpha^n(U)$,

$$|(T_\gamma f, T_y \tilde{f})| = |(T_{y^{-1}\gamma} f, \tilde{f})| > \varepsilon.$$

So,

$$\#\{\gamma \in \Gamma : \ |(T_\gamma f, T_y \tilde{f})| > \varepsilon\} \geq \#(\Gamma \cap y\alpha^n(U)) \geq N_n.$$

Since $N_n \in \mathbb{N}$ is arbitrary, the conclusion follows. □

Keep in mind that, $T_p(g, \Gamma) = \{T_\gamma g\}_{\gamma \in \Gamma}$ for $g \in L^p(G)$, and $\Gamma \subseteq G$. In the next proposition we show that if $T_p(f, \Gamma)$ is a p'-Bessel sequence, then Γ is relatively uniformly separated.

Proposition 4.2. *Let G be an expansible group with an expansive automorphism α, f a nonzero function in $L^p(G)$, $(1 < p < \infty)$, and Γ a sequence in G. If $T_p(f, \Gamma)$ is a p'-Bessel sequence for some $1 < p' < \infty$, then Γ is relatively uniformly separated.*

Proof. Assume that Γ is not relatively uniformly separated. Choose ε such that $0 < \varepsilon < \|f\|_p$, and $N \in \mathbb{N}$ arbitrary. By Hahn-Banach Theorem, there exists an $\tilde{f} \in L^q(G)$ (the dual space of $L^p(G)$), with $\|\tilde{f}\|_q = 1$ such that $(f, \tilde{f}) = \|f\|_p > \varepsilon$. By Lemma 4.1, there exists $x \in G$ such that

$$\#\{\gamma \in \Gamma : \ |(T_\gamma f, T_x \tilde{f})| > \varepsilon\} \geq N.$$

Let $\Gamma_N = \{\gamma \in \Gamma : \ |(T_\gamma f, T_x \tilde{f})| > \varepsilon\}$. We have

$$\sum_{\gamma \in \Gamma} |(T_\gamma f, T_x \tilde{f})|^{p'} \geq \sum_{\gamma \in \Gamma_N} |(T_\gamma f, T_x \tilde{f})|^{p'} > N\varepsilon^{p'}.$$

Since $N \in \mathbb{N}$ is arbitrary and $\|T_x \tilde{f}\|_q = \|\tilde{f}\|_q$ is fixed, $T_p(f, \Gamma)$ is not p'-Bessel sequence, which leads to a contradiction. Thus, Γ is relatively uniformly separated. □

The following result is very useful.

Proposition 4.3. *Let G be an expansible group with an expansive automorphism α. Let $f \in L^p(G)$, for $1 < p < \infty$, and Γ be a relatively uniformly separated sequence in G. Then for any compact neighborhood U of the identity e and $x \in G$,*

(i) $\sum_{\gamma \in \Gamma} \|\chi_{x\alpha^{-n}(U)} T_\gamma f\|_p^p < \infty$, *for sufficiently large $n \in \mathbb{N}$.*
(ii) $\sum_{\gamma \in \Gamma} \|\chi_{x\alpha^{-n}(U)} T_\gamma f\|_p^p \to 0$, *as $n \to \infty$.*

Proof. Let $x \in G$ be arbitrary and fixed. If Γ is relatively uniformly separated, then the translation $x^{-1}\Gamma = \{x^{-1}\gamma : \ \gamma \in \Gamma\}$ of Γ is also relatively uniformly separated. Moreover,

$$\begin{aligned} \sum_{\gamma \in \Gamma} \|\chi_{x\alpha^{-n}(U)} T_\gamma f\|_p^p &= \sum_{\gamma \in \Gamma} \|\chi_{\alpha^{-n}(U)} T_{x^{-1}\gamma} f\|_p^p \\ &= \sum_{\gamma \in x^{-1}\Gamma} \|\chi_{\alpha^{-n}(U)} T_\gamma f\|_p^p. \end{aligned}$$

So, it suffices to show that for any relatively uniformly separated sequence Γ in G,

(i) $\sum_{\gamma \in \Gamma} \|\chi_{\alpha^{-n}(U)} T_\gamma f\|_p^p < \infty$, for sufficiently large $n \in \mathbb{N}$.
(ii) $\sum_{\gamma \in \Gamma} \|\chi_{\alpha^{-n}(U)} T_\gamma f\|_p^p \to 0$, as $n \to \infty$.

(i) By definition, there are finite U_k-separated sets Γ_k for $1 \le k \le m$, such that $\Gamma = \cup_{k=1}^m \Gamma_k$. By properties of the expansive automorphism, there is $n \in \mathbb{N}$ such that $\alpha^{-n}(U) \subset U_k$, for all $k = 1, ..., m$. Hence, the sets $\{\gamma^{-1}\alpha^{-n}(U)\}_{\gamma\in\Gamma_k}$ are disjoint. Then,

$$\begin{aligned}\sum_{\gamma\in\Gamma} \|\chi_{\alpha^{-n}(U)} T_\gamma f\|_p^p &= \sum_{k=1}^m \sum_{\gamma\in\Gamma_k} \|\chi_{\alpha^{-n}(U)} T_\gamma f\|_p^p \\ &= \sum_{k=1}^m \sum_{\gamma\in\Gamma_k} \int_{\alpha^{-n}(U)} |T_\gamma f(y)|^p d\mu(y) \\ &= \sum_{k=1}^m \sum_{\gamma\in\Gamma_k} \int_{\gamma^{-1}\alpha^{-n}(U)} |f(y)|^p d\mu(y) \\ &\le \sum_{k=1}^m \|f\|_p^p = m.\|f\|_p^p < \infty.\end{aligned}$$

(ii) For any $k = 1, ..., m$,

$$\chi_{\Gamma_k^{-1}\alpha^{-n}(U)}(y)|f(y)|^p \to 0, a.e. \text{ as } n \to \infty,$$

where $\Gamma_k^{-1}\alpha^{-n}(U) = \bigcup_{\gamma\in\Gamma_k} \gamma^{-1}\alpha^{-n}(U)$.
Since $\chi_{\Gamma_k^{-1}\alpha^{-n}(U)}(y)|f(y)|^p \le |f(y)|^p$, the Lebesgue Dominated Convergence Theorem implies that,

$$\begin{aligned}\lim_{n\to\infty} \sum_{\gamma\in\Gamma_k} \int_{\gamma^{-1}\alpha^{-n}(U)} |f(y)|^p d\mu(y) &= \lim_{n\to\infty} \int_{\Gamma_k^{-1}\alpha^{-n}(U)} |f(y)|^p d\mu(y) \\ &= \lim_{n\to\infty} \int_G \chi_{\Gamma_k^{-1}\alpha^{-n}(U)} |f(y)|^p d\mu(y) \\ &= 0.\end{aligned}$$

Then,

$$\begin{aligned}\lim_{n\to\infty} \sum_{\gamma\in\Gamma} \|\chi_{\alpha^{-n}(U)} T_\gamma f\|_p^p &= \lim_{n\to\infty} \sum_{k=1}^m \sum_{\gamma\in\Gamma_k} \int_{\gamma^{-1}\alpha^{-n}(U)} |f(y)|^p d\mu(y) \\ &= \sum_{k=1}^m \lim_{n\to\infty} \sum_{\gamma\in\Gamma_k} \int_{\gamma^{-1}\alpha^{-n}(U)} |f(y)|^p d\mu(y) \\ &= 0. \quad \square\end{aligned}$$

In 2007, S. Nitzan and A. Olevskii introduced the concept of (C_q)-system in Hilbert spaces [13,14]. In 2012, B. Liu and R. Liu extended this useful definition to reflexive Banach spaces, as follows [12].

Let X be a separable reflexive Banach space and $1 < q < \infty$ be a fixed number. We say that a sequence $\{f_k\}_1^\infty \subset X$ is a (C_q)-*system* in X with constant $C > 0$ (complete with ℓ_q control over the coefficients) if for any $f \in X$ and $\varepsilon > 0$, there exists a linear combination $g = \sum a_k f_k$ such that

$$\|f - g\|_X < \varepsilon \ \text{ and } \ (\sum |a_n|^q)^{\frac{1}{q}} \le C\|f\|_X,$$

where $C = C(q)$ is a positive constant not depending on f.

The following lemma which was proven in [12], expresses the relation between a (C_q)-system and the lower p-frame bounds.

Lemma 4.4. *Let X be a separable reflexive Banach space with the dual space X^*, and $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. A system $\{f_n\} \subset X$ is a (C_q)-system in X with constant $K > 0$ if and only if*

$$\frac{1}{K}\|h\| \leq (\sum_{n=1}^{\infty} |(h, f_n)|^p)^{\frac{1}{p}}, \quad \textit{for all} \quad h \in X^*.$$

Now, we can prove the main result of this section.

Theorem 4.5. *Let G be an expansible group with an expansive automorphism α and $\Gamma \subseteq G$. Suppose that $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. For $m \in \mathbb{N}$, choose $f_1, f_2, ..., f_m$ in $L^p(G)$ and arbitrary disjoint sequences $\Gamma_1, \Gamma_2, ..., \Gamma_m$ in G, such that $\Gamma = \cup_{k=1}^m \Gamma_k$.*

(i) *If for some $1 < p' < \infty$, $\bigcup_{k=1}^m T_p(f_k, \Gamma_k)$ is a p'-Bessel sequence, then $D^+(\Gamma) < \infty$.*
(ii) *If $\bigcup_{k=1}^m T_p(f_k, \Gamma_k)$ is a (C_q)-system, then $D^+(\Gamma) = \infty$.*

In particular, there exists no p'-Bessel (C_q)-system in $L^p(G)$ of the form $\bigcup_{k=1}^m T_p(f_k, \Gamma_k)$.

Proof. (i) Suppose that for some $1 < p' < \infty$, $\bigcup_{k=1}^m T_p(f_k, \Gamma_k)$ is a p'-Bessel sequence. Which is equivalent so that, $T_p(f_k, \Gamma_k)$ is a p'-Bessel sequence in $L^q(G)$, for each $k = 1, \cdots, m$. Then, by Proposition 4.2, each Γ_k is relatively uniformly separated. By Theorem 3.7, Γ_k has finite upper Beurling density for each $1 \leq k \leq m$, i.e., $D^+(\Gamma_k) < \infty$. Let U be a compact neighborhood of e in G. Then by definition

$$\begin{aligned}
\nu_\Gamma^+(n) &= \sup_{x \in G} \#(\Gamma \cap x\alpha^n(U)) \\
&= \sup_{x \in G} \#(\bigcup_{k=1}^m (\Gamma_k \cap x\alpha^n(U))) \\
&\leq \sum_{k=1}^m \sup_{x \in G} \#(\Gamma_k \cap x\alpha^n(U)) \\
&= \sum_{k=1}^m \nu_{\Gamma_k}^+(n).
\end{aligned}$$

It follows that

$$\begin{aligned}
D^+(\Gamma) &= \limsup_{n \to \infty} \frac{\nu_\Gamma^+(n)}{\mu(\alpha^n(U))} \\
&\leq \limsup_{n \to \infty} \frac{\sum_{k=1}^m \nu_{\Gamma_k}^+(n)}{\mu(\alpha^n(U))} \\
&\leq \sum_{k=1}^m \limsup_{n \to \infty} \frac{\nu_{\Gamma_k}^+(n)}{\mu(\alpha^n(U))} \\
&= \sum_{k=1}^m D^+(\Gamma_k) < \infty.
\end{aligned}$$

Thus, Γ has finite upper Beurling density.

(ii) By assumption, Γ is the disjoint union of sequences Γ_k, for $k = 1, \cdots, m$. For each $k = 1, \cdots, m$, $D^+(\Gamma_k) < D^+(\Gamma)$. Hence, if Γ has finite upper Beurling density, then each Γ_k has finite upper Beurling density. Therefore, Theorem 3.7 implies that each Γ_k is relatively uniformly separated. Hence, for each k, we can write Γ_k as a union of subsequences Δ_{kj}, for $j = 1, \ldots, s_k$, each of which is U_{kj}-separated. Now, we can find $n \in \mathbb{N}$ such that $\alpha^{-n}(U) \subset U_{kj}$, for each $k = 1, \cdots, m$, and $j = 1, \ldots, s_k$, for any compact neighborhood U of e. Consider the disjoint sets $\{\gamma^{-1}\alpha^{-n}(U)\}_{\gamma\in\Delta_{kj}}$ and define $\beta_{kj} = \cup_{\gamma\in\Delta_{kj}}(\gamma^{-1}\alpha^{-n}(U))$. Then, we have

$$\begin{aligned}
\sum_{k=1}^{m}\sum_{\gamma\in\Gamma_k} |(\chi_{\alpha^{-n}(U)}, T_\gamma f_k)|^p &= \sum_{k=1}^{m}\sum_{j=1}^{s_k}\sum_{\gamma\in\Delta_{kj}} |(\chi_{\alpha^{-n}(U)}, \chi_{\alpha^{-n}(U)} T_\gamma f_k)|^p \\
&\le \sum_{k=1}^{m}\sum_{j=1}^{s_k}\sum_{\gamma\in\Delta_{kj}} \|\chi_{\alpha^{-n}(U)}\|_q^p \|\chi_{\alpha^{-n}(U)} T_\gamma f_k\|_p^p \\
&= \|\chi_{\alpha^{-n}(U)}\|_q^p \sum_{k=1}^{m}\sum_{j=1}^{s_k}\sum_{\gamma\in\Delta_{kj}} \|\chi_{\alpha^{-n}(U)} T_\gamma f_k\|_p^p \\
&= \|\chi_{\alpha^{-n}(U)}\|_q^p \sum_{k=1}^{m}\sum_{j=1}^{s_k}\int_{\beta_{kj}} |f_k(x)|^p dx.
\end{aligned}$$

However, for each fixed k and j, the function $\chi_{\beta_{kj}}(x)|f_k(x)|^p$ converges to zero pointwise a.e. as $n \to \infty$, and is dominated by the integrable function $|f_k(x)|^p$. Thus, by the Lebesgue Dominated Converges Theorem, $\lim_{n\to\infty} \int_{\beta_{kj}} |f_k(x)|^p dx = 0$. Then, it is not hard to see that $\bigcup_{k=1}^{m} T_p(f_k, \Gamma_k)$ is not a (C_q)-system, by Lemma 4.4. This completes the proof. □

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