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RESEARCH PAPER

Banach Algebra of Bounded Complex Radon Measures on Homogeneous Space

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Abstract
Let H be a compact subgroup of a locally compact group G. In this paper we define a convolution on $M(G/H)$, the space of all bounded complex Radon measures on the homogeneous space G/H. Then we prove that the measure space $M(G/H)$ with the newly well-defined convolution is a non-unital Banach algebra that possesses an approximate identity. Finally, it is shown that this Banach algebra is not involutive and also $L^1(G/H)$ with the new convolution is a two-sided ideal of it.

Keywords Complex Radon measure · Homogeneous spaces · Convolution · Banach algebra

Mathematics Subject Classification Primary 43A15 · Secondary 43A85

1 Introduction and Preliminaries

Let G be a locally compact group, and let $M(G)$ be the space of all bounded complex Radon measures on it. The convolution of any two measures μ_1 and μ_2 in $M(G)$ is defined by

$$\mu_1 * \mu_2(f) = \int_G \int_G f(xy)\mathrm{d}\mu_1(x)\mathrm{d}\mu_2(y), \quad (f \in C_c(G)). \tag{1.1}$$

It is well-known that $(M(G), *)$ is a unital Banach algebra, it is called the measure algebra and plays a key role in harmonic analysis, (See, e.g., Deitmar and Echterhoof 2009 and Fell and Doran 1988). Now let H be a compact subgroup of locally compact group G, and the homogeneous space G/H is a Hausdorff space on which G acts transitively by left. We should clear that H is not normal subgroup necessarily, so G/H does not possess a group structure but it will be a locally compact Hausdorff space. Let $M(G/H)$ denote the space of all bounded complex Radon measures on G/H. Compared with the measure algebra $M(G)$, it is worthwhile to investigate the existence of a convolution on $M(G/H)$ which makes it into a Banach algebra. Farashahi (2018) studied this problem in the case that H is a closed subgroup of a compact group G; However, the theory of homogeneous spaces in which H is a compact subgroup of a locally compact group G has many applications in physics and engineering. For example, if the Euclidian group $E(2)$ acts transitively on $\mathbb{R}^2$, then the isotropy subgroup of origin is the orthogonal group $O(2)$. In that sequel, the homogeneous space $E(2)/O(2)$ provides definition of X-ray transform that is used in many areas such as radio astronomy, positron emission tomography, crystallography, etc (See, e.g., Deans 1983, Ch. 1 and Helgason 2011). Now, we review some preliminaries and results in homogeneous spaces theory. Let $\mathrm{d}y$ be the left invariant Haar measure of locally compact group G. The modular function Δ_G is a continuous homomorphism from G into the multiplicative group $\mathbb{R}^+$. Furthermore, for all $x \in G$

$$\int_G f(y)\mathrm{d}y = \Delta_G(x) \int_G f(yx)\mathrm{d}y$$

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where $f \in C_c(G)$, the space of continuous functions on G with compact support. A locally compact group G is called unimodular if $\triangle_G(x) = 1$, for all $x \in G$. A compact group G is always unimodular. Assume that H is a closed subgroup of the locally compact group G, it is known that $C_c(G/H)$ consists of all $P_H f$ functions, where $f \in C_c(G)$ and

$$P_H f(xH) = \int_H f(xh)\mathrm{d}h \quad (x \in G).$$

Moreover, $P_H : C_c(G) \to C_c(G/H)$ is a bounded linear operator which is not injective (see, e.g., Folland 1995, Ch. 2, Section. 6). Suppose that μ is a Radon measure on G/H. For all $x \in G$ we define the translation of μ through x, by $\mu_x(E) = \mu(xE)$, where E is a Borel subset of G/H. Then μ is said to be G-invariant if $\mu_x = \mu$, for all $x \in G$. If H is compact, G/H admits a $G-$invariant Radon measure (See, e.g., Folland 1995, Corollary 2. 51). μ is said to be strongly quasi-invariant, if there is a continuous function $\lambda : G \times G/H \to (0, +\infty)$ which satisfies

$$\mathrm{d}\mu_x(yH) = \lambda(x, yH)\mathrm{d}\mu(yH).$$

If the function $\lambda(x, .)$ is reduced to a constant for each $x \in G$, then μ is called relatively invariant under G. We consider a rho-function for the pair (G, H) as a continuous function $\rho : G \to (0, +\infty)$ for which $\rho(xh) = \triangle_H(h)\triangle_G(h)^{-1}\ \rho(x)$, for each $x \in G$ and $h \in H$. It is well known that (G, H) admits a rho-function and for every rho-function ρ there is a strongly quasi-invariant measure μ on G/H such that

$$\int_G f(x)\mathrm{d}x = \int_{G/H} P_H f(xH)\mathrm{d}\mu(xH) \quad (f \in C_c(G)),$$

where in this case, $P_H f(xH) = \int_H \frac{f(xh)}{\rho(xh)}\mathrm{d}h$ and this equation is called quotient integral formula. This measure μ also satisfies

$$\frac{\mathrm{d}\mu_x}{\mathrm{d}\mu}(yH) = \frac{\rho(xy)}{\rho(y)} \quad (x, y \in G).$$

Let μ be a strongly quasi invariant measure on G/H which is associated with the rho-function ρ for the pair (G, H) . The mapping $T_H : L^1(G) \to L^1(G/H)$ is defined almost everywhere by

$$T_H f(xH) = \int_H \frac{f(xh)}{\rho(xh)}\mathrm{d}h \quad (f \in L^1(G))$$

is a surjective bounded linear operator with $\| T_H \| \leq 1$ (see Reiter and Stegeman 2000, Subsection 3.4) and also T_H satisfies the generalized Mackey–Bruhat formula,

$$\int_G f(x)\mathrm{d}x = \int_{G/H} T_H f(xH)\mathrm{d}\mu(xH) \quad (f \in L^1(G)), \tag{1.2}$$

which is also known as the quotient integral formula. Two useful operators left translation and right translation, denoted by L and R respectively, plays key role in the next section. The left translation of $\varphi \in C_c(G/H)$ by $x \in G$ is defined by $L_x(\varphi)(yH) = \varphi(x^{-1}yH)$. In a similar way, the left translation operator is defined for the integrable function on a homogeneous space G/H as follows:

$$L_x(\varphi)(yH) = \varphi(x^{-1}yH) \quad (\mu - \text{almost all}\, yH \in G/H),$$

where $\varphi \in L^p(G/H)$, $1 \leq p \leq \infty$. The mapping $x \mapsto L_x(\varphi)$ is continuous and also $\|L_x(\varphi)\|_p = \left(\frac{\rho(x)}{\rho(e)}\right)^{1/p} \|\varphi\|_p$. The right translation is defined in the same manner [for more details see Kamyabi-Gol and Tavalaei (2009)]. Now, let H be a compact subgroup and put

$$C_c(G{:}H) := \{f \in C_c(G) : R_h f = f, \forall h \in H\},$$

where R_h denotes the right translation through h. Let μ be a G-invariant Radon measure on G/H. One can prove that

$$C_c(G{:}H) = \{\varphi_{\pi_H} := \varphi \circ \pi_H : \varphi \in C_c(G/H)\}$$

and it is a left ideal of the algebra $C_c(G)$. Moreover the operator P_H is an algebraic isometric isomorphism between $C_c(G{:}H)$ and $C_c(G/H)$. Furthermore, $P_H(\varphi_{\pi_H}) = \varphi$, for all $\varphi \in C_c(G/H)$. These results can be extended, by approximation, to $T_H : L^1(G{:}H) \to L^1(G/H)$, where

$$L^1(G{:}H) := \{f \in L^1(G) : R_h f = f, \quad \forall h \in H\},$$

(See, e.g., Reiter and Stegeman 2000, P. 98) and also (see, e.g., Farashahi 2015, 2013; Kamyabi-Gol and Tavalaei 2009). Therein T_H is an algebraic isometrically isomorphism. By using this isomorphism one can define a well-defined convolution on $L^1(G/H)$. Let λ be a strongly quasi-invariant measure on G/H that arises from the rho-function ρ, then

$$\begin{aligned}\varphi * \psi(xH) =& T_H(\varphi_{\pi_H} * \psi_{\pi_H})(xH)\\ =& \int_{G/H}\int_H \varphi(yH)\psi(hy^{-1}xH)\frac{\rho(hy^{-1}x)}{\rho(x)}\mathrm{d}h\mathrm{d}\lambda(yH),\end{aligned}$$

(See, e.g., Farashahi 2013). Now, let $M(G)$ be the space of all bounded complex Radon measures on locally compact group G and H be a compact subgroup of G. Also assume that $\mu \in M(G)$. One can define $\sigma_\mu \in M(G/H)$ by

$$\int_{G/H} \varphi(xH)\mathrm{d}\sigma_\mu(xH) = \int_G \varphi_{\pi_H}(x)\mathrm{d}\mu(x) \quad (\varphi \in C_c(G/H)). \tag{1.3}$$

In other words $\sigma_\mu(\varphi) = \mu(\varphi_{\pi_H})$, for all φ in $C_c(G/H)$. Since $\|\sigma_\mu\| \leqslant \|\mu\|$, the linear map $\mu \mapsto \sigma_\mu$ is continuous and it can be shown that this map is surjective (see, e.g., Reiter and Stegeman 2000, P. 233).

2 The Main Results

Let us denote the space of all bounded complex Radon measures on locally compact Hausdorff space *G/H* by *M*(*G/H*) . In this section we establish some results to define a well-defined convolution on *M*(*G/H*) which makes it into a Banach algebra, then we introduce an approximate identity for it; After that the relationship between two Banach algebras *M*(*G/H*) and $L^1(G/H)$ is described, the last result asserts that $L^1(G/H)$ can be regarded as a Banach subalgebra of *M*(*G/H*) . From now on, we consider *H* as a compact subgroup of locally compact group *G*. We first introduce an important closed left ideal of *M*(*G*) in what follows. Let

$$M(G{:}H) = \{\mu \in M(G) : \mu(R_h f) = \mu(f); \forall f \in C_c(G), h \in H\},$$

where R_h denotes the right translation through *h*.

Proposition 2.1 *Let H be a compact subgroup of a locally compact group G and μ be a left Haar measure on G. Then*

$$M(G{:}H) = \{\mu_f : f \in L^1(G{:}H)\},$$

where $\mathrm{d}\mu_f(x) = f(x)\mathrm{d}\mu(x)$.

Proof For any $f \in L^1(G{:}H)$, it is clear that $\mu_f(x) = f(x)\mathrm{d}\mu(x) \in M(G)$ and also for all $g \in C_c(G)$ and $h \in H$ we have

$$\begin{aligned}\mu_f(R_h g) &= \int_G R_h g(x)\mathrm{d}\mu_f(x)\\ &= \int_G g(xh)f(x)\mathrm{d}\mu(x)\\ &= \int_G g(x)f(xh^{-1})\mathrm{d}\mu(xh^{-1})\\ &= \int_G g(x)\mathrm{d}\mu_f(x)\\ &= \mu_f(g).\end{aligned}$$

Note that since H is compact, $\Delta_G|_H = 1$. Now let $\mu_f \in M(G{:}H)$ for some $f \in L^1(G)$, so $\mu_f(R_h g) = \mu_f(g)$ for all $g \in C_c(G)$. In other words, $\int_G R_h g(x)\mathrm{d}\mu_f(x) = \int_G g(x)\mathrm{d}\mu_f(x)$. Since

$$\int_G g(xh)f(x)\mathrm{d}\mu(x) = \int_G g(x)f(x)\mathrm{d}\mu(x),$$

we have

$$\begin{aligned}\int_G g(x)f(xh^{-1})\mathrm{d}\mu(xh^{-1}) &= \int_G g(x)f(xh^{-1})\mathrm{d}\mu(xh^{-1})\\ &= \int_G g(xh)f(x)\mathrm{d}\mu(xh)\\ &= \int_G g(x)f(x)\mathrm{d}\mu(x),\end{aligned}$$

for all $g \in C_c(G)$. Therefore, $\int_G g(x)(f(xh) - f(x))\mathrm{d}\mu(x) = 0$ for all $g \in C_c(G)$ and $h \in H$. Then by Urysohn's Lemma to take suitable $g \in C_c(G)$ we get $f(xh) = f(x)$, for all $x \in G$ and $h \in H$. Thus $f \in L^1(G{:}H)$. □

Proposition 2.2 *Let H be a compact subgroup of a locally compact group G. Then M*(*G* : *H*) *is a closed left ideal of M*(*G*) *. Moreover,*

$$M(G{:}H) = \{\sigma_{P_H} := \sigma \circ P_H : \sigma \in M(G/H)\}.$$

Proof Let $\mu_1, \mu_2 \in M(G{:}H)$, then for all $f \in C_c(G)$ and $h \in H$ we have

$$\begin{aligned}\mu_1 * \mu_2(R_h f) &= \int_G R_h f(g)\mathrm{d}(\mu_1 * \mu_2)(g)\\ &= \int_G\int_G R_h f(xy)\mathrm{d}\mu_1(x)\mathrm{d}\mu_2(y)\\ &= \int_G \mu_1(R_h(L_{x^{-1}}f))(y)\mathrm{d}\mu_2(y)\\ &= \int_G \mu_1(L_{x^{-1}}f)(y)\mathrm{d}\mu_2(y)\\ &= \int_G\int_G f(xy)\mathrm{d}\mu_1(x)\mathrm{d}\mu_2(y)\\ &= \mu_1 * \mu_2(f).\end{aligned}$$

Therefore $\mu_1 * \mu_2 \in M(G{:}H)$. A similar calculation shows that $M(G : H)$ is a left ideal of $M(G)$. Furthermore, let μ be limit of the net $\{\mu_\alpha\}_{\alpha\in\Lambda}$ in $M(G : H)$, then $\mu(R_h f) = \lim \mu_\alpha(R_h f)$ for all $f \in C_c(G)$ and $h \in H$. But $\mu_\alpha(R_h f) = \mu_\alpha(f)$ and this implies that $\mu(R_h f) = \mu(f)$. It remains to prove the equality in this Proposition. Let $\sigma \in M(G/H)$ then $\sigma \circ P_H$ is a bounded linear functional on $C_c(G)$, since

$$\begin{aligned}|\sigma \circ P_H(f)| &= |\sigma(P_H(f))|\\ &= |\int_{G/H} P_H f(xH)\mathrm{d}\sigma(xH)|\\ &= \int_{G/H} |\int_H f(xh)\mathrm{d}h|\mathrm{d}|\sigma|(xH)\\ &\leqslant \|f\|_\infty\|\sigma\|.\end{aligned}$$

Thus $\|\sigma \circ P_H\| \leq \|\sigma\| < \infty$), so that the mapping $\sigma \circ P_H$ is a bounded linear functional on $C_c(G)$. Furthermore, for all $f \in C_c(G)$ and $h \in H$ we have

$$\sigma \circ P_H(R_h f) = \int_{G/H}\int_H R_h f(x\eta)\mathrm{d}\eta \mathrm{d}\sigma(xH) = \int_{G/H}\int_H f(x\eta h)\mathrm{d}\eta \mathrm{d}\sigma(xH) = \int_{G/H}\int_H f(x\eta)\mathrm{d}\eta \mathrm{d}\sigma(xH) = \sigma \circ P_H(f).$$

Thus $\sigma_{P_H} = \sigma \circ P_H \in M(G{:}H)$ for all $\sigma \in M(G/H)$. To show the reverse inclusion let μ be in $M(G)$ such that $\mu(R_h f) = \mu(f)$ for all f in $C_c(G)$ and h in H. Then by (1.3) there exists $\sigma \in M(G/H)$ such that for all f in $C_c(G)$ we have

$$\mu(f) = \int_G f(x)\mathrm{d}\mu(x) = \int_G R_h f(x)\mathrm{d}\mu(x) = \int_{G/H}\int_H f(x\eta h)\mathrm{d}\eta \mathrm{d}\sigma(xH) = \int_{G/H}\int_H f(x\eta)\mathrm{d}\eta \mathrm{d}\sigma(xH) = \sigma \circ P_H(f).$$

So $\mu = \sigma \circ P_H$ and the proof is complete. □

Now, consider the map $R_H : M(G) \to M(G/H)$ given by

$$R_H\mu(\varphi) := \mu(\varphi_{\pi_H}) = \int_G \varphi_{\pi_H}(x)\mathrm{d}\mu(x), \quad (\varphi \in C_c(G/H)). \tag{2.1}$$

Let $\varphi = \psi \in C_c(G/H)$ then $\varphi_{\pi_H} = \varphi \circ \pi_H = \psi \circ \pi_H = \psi_{\pi_H}$. Hence $\mu(\varphi_{\pi_H}) = \mu(\psi_{\pi_H})$ and this implies that $R_H\mu(\varphi) = R_H\mu(\psi)$. From the definition we can easily deduce that $R_H\mu$ is a positive linear functional on $C_c(G/H)$. So by the Riesz representation theorem there exists a unique Radon measure $\sigma \in M(G/H)$ such that

$$R_H\mu(\varphi) = \int_{G/H} \varphi(xH)\mathrm{d}\sigma(xH) = \sigma(\varphi). \tag{2.2}$$

Then $R_H\mu = \sigma \in M(G/H)$. Also based on definition (2.1) it is clear that $R_H(\mu_1) = R_H(\mu_2)$ if $\mu_1 = \mu_2$. So R_H is a well-defined map. To show that the mapping R_H is linear, consider an arbitrary scalar α and the elements μ_1 and μ_2 in $M(G)$. Then for any φ in $C_c(G/H)$ we have

$$R_H(\mu_1 + \mu_2)(\varphi) = (\mu_1 + \mu_2)(\varphi_{\pi_H}) = \mu_1(\varphi_{\pi_H}) + \mu_2(\varphi_{\pi_H}) = R_H\mu_1(\varphi) + R_H\mu_2(\varphi) = (R_H\mu_1 + R_H\mu_2)(\varphi),$$

thus R_H is linear. We shall show that R_H is a bounded operator. To do this, if we consider any φ in $C_c(G/H)$ then we have

$$|R_H\mu(\varphi)| = |\int_G \varphi_{\pi_H}(x)\mathrm{d}\mu(x)| \leqslant \int_G |\varphi(xH)|\mathrm{d}|\mu|(x) \leqslant \int_G \|\varphi\|_\infty \mathrm{d}|\mu|(x) \leqslant \|\mu\|\|\varphi\|_\infty.$$

So $\|R_H\mu\| \leqslant \|\mu\| < \infty$ and this implies $\|R_H\| \leqslant 1$. For surjectivity, let $\sigma \in M(G/H)$ and define μ on $C_c(G)$ by

$$\mu(f) := \sigma(P_H f) \quad (f \in C_c(G)). \tag{2.3}$$

Suppose $\varphi \in C_c(G/H)$, using Proposition 2.2, for $\mu \in M(G{:}H)$. Then by the definition of R_H, we have

$$R_H\mu(\varphi) = \mu(\varphi_{\pi_H}) = \sigma(P_H(\varphi_{\pi_H})) = \int_{G/H} P_H(\varphi_{\pi_H})(xH)\mathrm{d}\sigma(xH) = \int_{G/H} \varphi(xH)\mathrm{d}\sigma(xH) = \sigma(\varphi),$$

this proves surjectivity.

Remark 2.3 The operator R_H is an extension of the mapping $T_H : L^1(G) \to L^1(G/H)$ given by $T_H f(xH) = \int_H f(xh)\mathrm{d}h$, for all $x \in G$.

The next two Propositions play a central role for making $M(G/H)$ into a Banach algebra.

Proposition 2.4 *Let H be a compact subgroup of a locally compact group G. Then $R_H\,|_{M(G:H)}$, the restriction of R_H to $M(G : H)$, is a bijective mapping and also it is an isometry.*

Proof Since R_H is surjective, it is enough to show that it is injective. Let $\mu \in M(G{:}H)$ and $R_H(\mu) = 0$. Then there exists $\sigma \in M(G/H)$ such that $\mu = \sigma_{P_H} = \sigma \circ P_H$ and $R_H(\mu) = 0$ implies that for all $\varphi \in C_c(G/H)$, $\sigma(\varphi) = \sigma_{P_H}(\varphi_{\pi_H}) = 0$. So that $\mu = 0$ and therefore R_H is injective.

Let σ_{P_H} be in $M(G : H)$, then for all φ in $C_c(G/H)$, on one hand

$$|R_H(\sigma_{P_H})\varphi| = |\sigma_{P_H}(\varphi_{\pi_H})| \leqslant \|\sigma_{P_H}\|\|\varphi_{\pi_H}\|_\infty,$$

so $\|R_H\sigma_{P_H}\| \leqslant \|\sigma_{P_H}\|$ and on the other hand,

$$\begin{aligned}|\sigma_{P_H}(\varphi_{\pi_H})| &= |R_H\sigma_{P_H}(\varphi)| \\ &\leqslant \|R_H\sigma_{P_H}\|\|\varphi\| \\ &= \|R_H\sigma_{P_H}\|\|P_H(\varphi_{\pi_H})\| \\ &\leqslant \|R_H\sigma_{P_H}\|\|P_H\|\|\varphi_{\pi_H}\| \\ &\leqslant \|R_H\sigma_{P_H}\|\|\varphi_{\pi_H}\|.\end{aligned}$$

so $\|\sigma_{P_H}\| \leqslant \|R_H\sigma_{P_H}\|$. Hence the proof is complete. □

The remarkable equality $R_H(\delta_x) = \delta_{xH}$ is obtained by using the following equalities:

$$\begin{aligned}R_H(\delta_x)(\varphi) &= \delta_x(\varphi_{\pi_H}) \\ &= \int_G \varphi_{\pi_H}(y)\mathrm{d}\delta_x(y) \\ &= \varphi_{\pi_H}(x) \\ &= \varphi(xH) \\ &= \int_{G/H} \varphi(yH)\mathrm{d}\delta_{xH}(yH) \\ &= \delta_{xH}(\varphi),\end{aligned}$$

for all $x \in G$. Note that for all φ in $C_c(G/H)$ and $x \in G$ we get

$$\delta_{xH}(\varphi) = \int_{G/H} \varphi(yH)\mathrm{d}\delta_{xH}(yH) = \varphi(x\eta H) = \varphi(xH).$$

where $\eta \in H$. Now, we are able to define a convolution on $M(G/H)$.

Definition 2.5 Let H be a compact subgroup of a locally compact group G. The mapping $* : M(G/H) \times M(G/H) \to M(G/H)$ given by

$$\sigma_1 * \sigma_2(\varphi) := R_H(\sigma_{1_{P_H}} * \sigma_{2_{P_H}})(\varphi) \quad (\varphi \in C_c(G/H)), \tag{2.4}$$

is a well-defined convolution on $M(G/H)$.

To show that $*$ is well defined, let $\sigma_1, \sigma_2, \sigma'_1, \sigma'_2 \in M(G/H)$ and $(\sigma_1, \sigma_2) = (\sigma'_1, \sigma'_2)$. Using surjectivity of R_H, there exists $\sigma_{1_{P_H}}, \sigma_{2_{P_H}}, \sigma'_{1_{P_H}}, \sigma'_{2_{P_H}} \in M(G{:}H)$ such that

$$R_H(\sigma_{1_{P_H}}) = \sigma_1, R_H(\sigma_{2_{P_H}}) = \sigma_2, R_H(\sigma'_{1_{P_H}}) = \sigma'_1, R_H(\sigma'_{2H}) = \sigma'_2.$$

Therefore the injectivity of R_H implies that $(\sigma_{1_{P_H}}, \sigma_{2_{P_H}}) = (\sigma'_{1_{P_H}}, \sigma'_{2_{P_H}})$. Thus

$$\sigma_{2_{P_H}} * \sigma_{2_{P_H}} = \sigma'_{1_{P_H}} * \sigma'_{2_{P_H}},$$

since the convolution on $M(G)$ is well-defined. Then $R_H(\sigma_{1_{P_H}} * \sigma_{2_{P_H}}) = R_H(\sigma'_{1_{P_H}} * \sigma'_{2_{P_H}})$. Finally, by (2.4), $\sigma_1 * \sigma_2 = \sigma'_1 * \sigma'_2$. Consequently, convolution $*$ is well-defined. Using Proposition 2.4 and Definition 2.5 we deduce the following result.

Corollary 2.6 *The bijective mapping $R_H|_{M(G:H)}$ in Proposition 2.4 is an algebraic isometric isomorphism.*

Now some remarks are in orders.

Remark 2.7 With the notations as above, we have:

(i) $(\sigma_1 * \sigma_2)_{P_H} = \sigma_{1_{P_H}} * \sigma_{2_{P_H}}$, because $R_H(\sigma_{1_{P_H}} * \sigma_{2_{P_H}}) = R_H((\sigma_1 * \sigma_2)_{P_H})$ and R_H is one to one on $M(G : H)$.

(ii) One can simplify (2.4) as follows:

$$\begin{aligned}\sigma_1 * \sigma_2(\varphi) &= R_H(\sigma_{1_{P_H}} * \sigma_{2_{P_H}})(\varphi) \\ &= \sigma_{1_{P_H}} * \sigma_{2_{P_H}}(\varphi_{\pi_H}) \\ &= \int_G\int_G \varphi_{\pi_H}(xy)\mathrm{d}\sigma_{1_{P_H}}(x)\mathrm{d}\sigma_{2_{P_H}}(y) \\ &= \int_G\int_G \varphi(xyH)\mathrm{d}\sigma_{1_{P_H}}(x)\mathrm{d}\sigma_{2_{P_H}}(y),\end{aligned}$$

for all $\varphi \in C_c(G/H)$.

(iii) let $\mu \in M(G)$ and $\sigma \in M(G/H)$, if we define $\mu * \sigma := R_H(\mu * \sigma_{P_H})$ then we have

$$\begin{aligned}\mu * \sigma(\varphi) &= R_H(\mu * \sigma_{P_H})(\varphi) \\ &= \mu * \sigma_{P_H}(\varphi_{\pi_H}) \\ &= \int_G\int_G \varphi_{\pi_H}(xy)\mathrm{d}\mu(x)\mathrm{d}\sigma_{P_H}(y) \\ &= \int_G\int_G \varphi_{\pi_H}(xy)\mathrm{d}\sigma_{P_H}(y)\mathrm{d}\mu(x) \\ &= \int_G\int_G (L_{x^{-1}}\varphi_{\pi_H})(y)\mathrm{d}\sigma_{P_H}(y)\mathrm{d}\mu(x) \\ &= \int_G\int_{G/H} P_H(L_{x^{-1}}\varphi_{\pi_H})(yH)\mathrm{d}\sigma(yH)\mathrm{d}\mu(x) \\ &= \int_G\int_{G/H}\int_H L_{x^{-1}}\varphi_{\pi_H}(yh)\mathrm{d}h\mathrm{d}\sigma(yH)\mathrm{d}\mu(x) \\ &= \int_G\int_{G/H} \varphi_{\pi_H}(xy)\mathrm{d}\sigma(yH)\mathrm{d}\mu(x) \\ &= \int_{G/H}\int_G \varphi(xyH)\mathrm{d}\mu(x)\mathrm{d}\sigma(yH),\end{aligned}$$

for all $\varphi \in C_c(G/H)$.

By using part (iii) of Remark 2.7 it is deduced that $M(G/H)$ is a left $M(G)$ module. In the next main theorem, it is shown that $(M(G/H), *)$ is a Banach algebra and has an approximate identity.

Theorem 2.8 *$(M(G/H), *)$ is a Banach algebra and also it possesses an approximate identity.*

Proof It is well known that $M(G/H)$ endowed with the total variation norm is a Banach space (See, e.g., Reiter and Stegeman 2000, P. 233). The fact that convolution on $M(G/H)$ is associative follows by applying *(ii)* of Remark 2.7 twice and associativity of $M(G)$. Let σ_1, σ_2 be in $M(G/H)$. Then by using surjectivity of R_H, there exists $\sigma_{1_{P_H}}$ and, $\sigma_{2_{P_H}}$ in $M_H(G)$ such that $R_H(\sigma_{1_{P_H}}) = \sigma_1$ and $R_H(\sigma_{2_{P_H}}) = \sigma_2$. Now Definition 2.5 and the fact that $M(G:H)$ is an normed algebra imply that:

$$\begin{aligned}\|\sigma_1 * \sigma_2\| &= \|\sigma_{1_{P_H}} * \sigma_{2_{P_H}}\| \leqslant \|\sigma_{1_{P_H}}\|\|\sigma_{2_{P_H}}\| \\ &\leqslant \|R_H\sigma_{1_{P_H}}\|\|R_H\sigma_{2_{P_H}}\| \\ &= \|\sigma_1\|\|\sigma_2\|.\end{aligned}$$

Note that R_H is an isometry. Thus $(M(G/H), *)$ is a normed Banach algebra. To introduce an approximate identity, let $\{\varphi_\alpha\}_{\alpha\in\Lambda}$ be an approximate identity for the Banach algebra $L^1(G/H)$, see Farashahi (2013). Put $\sigma_\alpha := R_H(\mu_{(\varphi_\alpha)_{\pi_H}}$, for all $\alpha \in \Lambda$ where μ is the left Haar measure on G. Then by surjectivity of R_H, for any σ in $M(G/H)$ there exists $\sigma_{P_H} \in M(G{:}H)$ such that $R_H(\sigma_{P_H}) = \sigma$. Hence we have

$$\begin{aligned}\|\sigma_\alpha * \sigma - \sigma\| &= \|R_H(\mu_{(\varphi_\alpha)_{\pi_H}}) * R_H(\sigma_{P_H}) - R_H(\sigma_{P_H})\| \\ &= \|\mu_{(\varphi_\alpha)_{\pi_H}} * \sigma_{P_H} - \sigma_{P_H}\|,\end{aligned}$$

but by Proposition 2.1 there exists $\psi_{\pi_H} \in C_c(G{:}H)$ such that $\sigma_{P_H} = \mu_{\psi_{\pi_H}}$ and also it can be seen by direct computation that $\mu_f * \mu_g - \mu_h = \mu_{f*g-h}$, for all f, g and h in $C_c(G)$, so

$$\begin{aligned}\|\sigma_\alpha * \sigma - \sigma\| &= \|\mu_{(\varphi_\alpha)_{\pi_H}} * \mu_{\psi_{\pi_H}} - \mu_{\psi_{\pi_H}}\| \\ &= \|\mu_{(\varphi_\alpha)_{\pi_H} * \psi_{\pi_H} - \psi_{\pi_H}}\|.\end{aligned}$$

On the other hand, the embedding of $L^1(G)$ into $M(G)$ is isometric, therefore

$$\begin{aligned}\|\sigma_\alpha * \sigma - \sigma\| &= \|(\varphi_\alpha)_{\pi_H} * \psi_{\pi_H} - \psi_{\pi_H}\| \\ &= \|P_H\Big((\varphi_\alpha)_{\pi_H} * \psi_{\pi_H} - \psi_{\pi_H}\Big)\| \\ &= \|P_H\Big((\varphi_\alpha)_{\pi_H} * \psi_{\pi_H}\Big) - P_H\big(\psi_{\pi_H}\big)\| \\ &= \|P_H\Big((\varphi_\alpha)_{\pi_H}\Big) * P_H\big(\psi_{\pi_H}\big) - P_H\big(\psi_{\pi_H}\big)\| \\ &= \|\varphi_\alpha * \psi - \psi\|,\end{aligned}$$

Since $\{\varphi_\alpha\}_{\alpha\in\Lambda}$ is an approximate identity, $\|\varphi_\alpha * \psi - \psi\|$ tends to 0 as $\alpha \to \infty$. Note that in the two last equalities, P_H is an isometry from $C_c(G{:}H)$ onto $C_c(G/H)$. This implies that $\|\sigma_\alpha * \sigma - \sigma\|$ goes to 0 when $\alpha \to \infty$. □

In the sequel, consider δ_e as the unit element of the unital Banach algebra $M(G)$. If we define the point mass measure $\delta_H := R_H(\delta_e)$, then for all φ in $C_c(G/H)$, by the definition of R_H we have

$$\begin{aligned}\delta_H(\varphi) = R_H(\delta_e)(\varphi) = \delta_e(\varphi_{\pi_H}) = \int_G \varphi_{\pi_H}(x)\mathrm{d}\delta_e(x) \\ = \varphi_{\pi_H}(e) = \varphi(H).\end{aligned} \tag{2.5}$$

Note that for all φ in $C_c(G/H)$ we have

$$\delta_H(\varphi) = \int_{G/H} \varphi(xH)\mathrm{d}\delta_H(xH) = \varphi(H).$$

Lemma 2.9 *Let H be a compact subgroup of a locally compact group G. δ_H is a right multiplicative identity in the algebra $M(G/H)$.*

Proof Suppose that φ is in $C_c(G/H)$ and $\sigma \in M(G/H)$. Then we have

$$\begin{aligned}\sigma * \delta_H(\varphi) &= R_H(\sigma_{P_H} * (\delta_H)_{P_H})(\varphi) \\ &= (\sigma_{P_H} * (\delta_H)_{P_H})(\varphi_{\pi_H}) \\ &= \int_G\int_G \varphi_{\pi_H}(st)\mathrm{d}\sigma_{P_H}(s)\mathrm{d}(\delta_H)_{P_H} \\ &= \int_G\int_G (L_{s^{-1}}\varphi_{\pi_H})(t)\mathrm{d}(\delta_H)_{P_H}(t)\mathrm{d}\sigma_{P_H}(s) \\ &= \int_G\int_{G/H} P_H(L_{s^{-1}}\varphi_{\pi_H})(xH)\mathrm{d}(\delta_H)(xH)\mathrm{d}\sigma_{P_H}(s) \\ &= \int_G\int_{G/H}\int_H L_{s^{-1}}\varphi_{\pi_H}(xh)\mathrm{d}h\mathrm{d}(\delta_H)(xH)\mathrm{d}\sigma_{P_H}(s) \\ &= \int_G\int_H L_{s^{-1}}\varphi_{\pi_H}(\eta h)\mathrm{d}hd_{\delta_H}(xH)\mathrm{d}\sigma_{P_H}(s),\end{aligned}$$

for some $\eta \in H$. Therefore, because of $\mathrm{d}h$ is invariant, we have

$$\begin{aligned}\sigma * \delta_H(\varphi) &= \int_G\int_H L_{s^{-1}}\varphi_{\pi_H}(\eta h)\mathrm{d}h\mathrm{d}\sigma_{P_H}(s) \\ &= \int_G\int_H \varphi_{\pi_H}(sh)\mathrm{d}h\mathrm{d}\sigma_{P_H}(s) \\ &= \int_G \varphi_{\pi_H}(s)\mathrm{d}\sigma_{P_H}(s) \\ &= \int_{G/H}\int_H \varphi_{\pi_H}(xh)\mathrm{d}h\mathrm{d}\sigma(xH) \\ &= \int_{G/H} \varphi(xH)\mathrm{d}\sigma(xH) \\ &= \sigma(\varphi).\end{aligned}$$

Thus, for all σ in $M(G/H)$ we have $\sigma * \delta_H = \sigma$. □

Corollary 2.10 *Let H be a compact subgroup of a locally compact group G. If σ is a two-sided identity in the algebra $M(G/H)$, then $\sigma = \delta_H$.*

Proof Since $\delta_H = \delta_H * \sigma = \sigma * \delta_H = \sigma$, the last equality is satisfied by considering Lemma 2.9. □

Generally, δ_H is not a left identity in the algebra $M(G/H)$. Hence $(M(G/H),*)$ fails to be a unital Banach algebra.

Corollary 2.11 *The Banach algebra $(M(G/H),*)$ is not an involutive algebra.*

Proof If $(M(G/H),*)$ is an involutive algebra, then $\sigma * \delta_H = \sigma$ for all $\sigma \in M(G/H)$. This implies that $(\sigma * (\delta_H)^*)^* = \delta_H * \sigma^* = \sigma^*$. Thus $\delta_H * \sigma = \sigma$, for all $\sigma \in M(G/H)$. So δ_H is a left identity too, a contradiction.□

Proposition 2.12 *Let H be a compact subgroup of a locally compact group G.*

(i) $\delta_{xH} * \delta_{yH} = \delta_{xyH}$ is satisfied for all $x, y \in G$ if and only if H is normal.
(ii) δ_H is an identity for the Banach algebra $(M(G/H),*)$ if and only if H is normal.

Proof (i) Let the equality $\delta_{xH} * \delta_{yH} = \delta_{xyH}$ hold for all $x, y \in G$. If H is not normal subgroup of G. Since ηxH and xH are two disjoint points in locally compact Hausdorff space G/H. Let K be a compact neighborhood of xH in G/H, then by Urysohn's Lemma for LCH Spaces, there exists ψ in $C_c(G/H)$ such that $\psi|_K = 1$ and absolutely $\psi(xH) = 1$. Now by lemma 2.47 in Folland (1995) there exist $\varphi := Pf$ for some f in $C_c(G)$, moreover $supp\varphi$ is compact, $\varphi(xH) = 1$ and $\varphi(\eta xH) = 0$. Hence, by the hypothesis, $\delta_{\eta H} * \delta_{xH}(\varphi) = \delta_{xH}(\varphi) = \varphi(xH) = 1$. On the other hand $\delta_{\eta H} * \delta_{xH}(\varphi) = \delta_{\eta xH}(\varphi) = \varphi(\eta xH) = 0$ and this is impossible. Thus H is normal. For the converse, let H be a normal subgroup of G. Considering Lemma 2.9 and Corollary 2.10, it is enough to show that δ_H is a left multiplicative identity in $M(G/H)$. Let φ be in $C_c(G/H)$ and σ be an arbitrary element in $M(G/H)$. Then

$$\begin{aligned}(\delta_H * \sigma)(\varphi) &= R_H((\delta_H)_{P_H} * \sigma_{P_H})(\varphi)\\ &= ((\delta_H)_{P_H} * \sigma_{P_H})(\varphi_{\pi_H})\\ &= \int_G\int_G \varphi_{\pi_H}(st)\mathrm{d}(\delta_H)_{P_H}(s)\mathrm{d}\sigma_{P_H}(t)\\ &= \int_G\int_G (R_t\varphi_{\pi_H})(s)\mathrm{d}(\delta_H)_{P_H}(s)\mathrm{d}\sigma_{P_H}(t)\\ &= \int_G\int_{G/H} P_H(R_t\varphi_{\pi_H})(xH)\mathrm{d}(\delta_H)(xH)\mathrm{d}\sigma_{P_H}(t)\\ &= \int_G\int_{G/H}\int_H R_t\varphi_{\pi_H}(xh)\mathrm{d}h\mathrm{d}(\delta_H)(xH)\mathrm{d}\sigma_{P_H}(t)\\ &= \int_G\int_H R_t\varphi_{\pi_H}(\eta h)\mathrm{d}h\mathrm{d}(\delta_H)(xH)\mathrm{d}\sigma_{P_H}(t),\end{aligned}$$

for some $\eta \in H$. Since H is normal and $\mathrm{d}h$ is invariant, we have

$$\begin{aligned}(\delta_H * \sigma)(\varphi) &= \int_G\int_H R_t\varphi_{\pi_H}(h)\mathrm{d}h\mathrm{d}\sigma_{P_H}(t)\\ &= \int_G\int_H \varphi_{\pi_H}(ht)\mathrm{d}h\mathrm{d}\sigma_{P_H}(t)\\ &= \int_G\int_H \varphi(htH)\mathrm{d}h\mathrm{d}\sigma_{P_H}(t)\\ &= \int_G\int_H \varphi(hHt)\mathrm{d}h\mathrm{d}\sigma_{P_H}(t)\\ &= \int_G\int_H \varphi(tH)\mathrm{d}h\mathrm{d}\sigma_{P_H}(t)\\ &= \int_G \varphi_{\pi_H}(t)\mathrm{d}\sigma_{P_H}(t)\\ &= \int_G\int_H \varphi(tH)\mathrm{d}h\mathrm{d}\sigma_{P_H}(t)\\ &= \int_{G/H}\int_H \varphi_{\pi_H}(th)\mathrm{d}h\mathrm{d}\sigma(tH)\mathrm{d}\sigma_{P_H}(t)\\ &= \int_{G/H}\varphi(xH)\mathrm{d}\sigma(xH)\\ &= \sigma(\varphi).\end{aligned}$$

This implies that $\delta_H * \sigma = \sigma$, that is δ_H is an identity for $M(G/H)$. (ii) Assume that H is a normal subgroup of G, the proof to show that $M(G/H)$ has an identity is the same as the proof of the converse part in *(i)*. Conversely, suppose that σ is the two-sided identity in $M(G/H)$ and assume that H is not normal. Then there exists some $\eta \in H$ and $x \in G$ such that $\eta xH \neq xH$. Now take a φ in $C_c(G/H)$ with $\varphi(xH) = 1$ and $\varphi(\eta xH) = 0$, this is possible by Urysohn's Lemma. By *(i)* above $\delta_{\eta H} * \delta_{xH}(\varphi) = \delta_{xH}(\varphi) = \varphi(xH) = 1$, and on the other hand $\delta_{\eta H} * \delta_{xH}(\varphi) = \delta_{\eta xH}(\varphi) = \varphi(\eta xH) = 0$, a contradiction. □

Proposition 2.13 *Let H be a compact subgroup of a locally compact group G and also let λ be a strongly quasi-invariant measure on G/H. Fix φ in $L^1(G/H,\lambda)$. Then $\psi \mapsto \int_{G/H}\psi(xH)\varphi(xH)\mathrm{d}\lambda(xH)$, $\psi \in C_c(G/H)$, defines a bounded measure on G/H. Denoting this measure by λ_φ, the mapping $\varphi \mapsto \lambda_\varphi$ is an isometric injection on $L^1(G/H,\lambda)$ into M(G/H) .*

Proof Let φ be a non zero element of $C_c(G/H)$. For all xH in G/H, set $\varphi_n(xH) := (|\varphi(xH)|/\|\varphi\|_\infty)^{1/n} sgn(\overline{\varphi(xH)})$ for all $n \geqslant 1$. Clearly, $\varphi_n\varphi \geqslant 0$, $\|\varphi_n\|_\infty \leqslant 1$, and also $\varphi_n\varphi \uparrow |\varphi|$ as n goes to ∞. Hence by using the monotone convergence Theorem we have

$$\begin{aligned}\int_{G/H}|\varphi|\mathrm{d}\lambda(xH) = \lim\int_{G/H}\varphi_n\varphi\mathrm{d}\lambda(xH) &\leqslant \int_{G/H}\|\varphi_n\|_\infty\varphi\mathrm{d}\lambda(xH)\\ &\leqslant \int_{G/H}\varphi\mathrm{d}\lambda(xH) = \|\lambda_\varphi\|.\end{aligned}$$

The inverse is obvious. Therefore, $\|\lambda_\varphi\| = \|\varphi\|_1$ for all φ in $C_c(G/H)$. The general case then follows by approximation for all φ in $L^1(G/H)$. □

Remark 2.14 The asserted inclusion in Proposition 2.13 is reduced to an injection of the algebra $L^1(G)$ into $M(G)$ in the case of $H = \{e\}$.

The relation between some of the induced measures on G/H and G, in Proposition 2.13 and Remark 2.14 respectively, has been proved in the following Lemma.

Lemma 2.15 *Let H be a compact subgroup of a locally compact group G. Fix μ as a left Haar measure on G and λ as a strongly quasi-invariant measure on G/H with associated rho-function ρ, then for all $\varphi \in C_c(G/H)$ we have $R_H(\mu_{(\varphi_{\pi_H})}) = \lambda_\varphi$, where $\mathrm{d}\lambda_\varphi(xH) = \varphi(xH)\mathrm{d}\lambda(xH)$ and $\mathrm{d}\mu_{(\varphi_{\pi_H})}(x) = \varphi_{\pi_H}(x)\mathrm{d}\mu(x)$.*

Proof Let $\psi \in C_c(G/H)$, then we have

$$\begin{aligned} R_H(\mu_{(\varphi_{\pi_H})})(\psi) &= \mu_{(\varphi_{\pi_H})}(\psi_{\pi_H}) \\ &= \int_G \psi_{\pi_H}(x)\mathrm{d}\mu_{(\varphi_{\pi_H})}(x) \\ &= \int_G \psi_{\pi_H}(x)\varphi_{\pi_H}(x)\mathrm{d}\mu(x) \\ &= \int_{G/H}\int_H \frac{\psi_{\pi_H}(xh)\varphi_{\pi_H}(xh)}{\rho(xh)}\mathrm{d}h\mathrm{d}\lambda(xH) \\ &= \int_{G/H}\varphi(xH)\int_H \frac{\psi_{\pi_H}(xh)}{\rho(xh)}\mathrm{d}h\mathrm{d}\lambda(xH) \\ &= \int_{G/H} P_H(\psi_{\pi_H})(xH)\mathrm{d}\lambda_\varphi(xH) \\ &= \lambda_\varphi(P_H(\psi_{\pi_H})) \\ &= (\lambda_\varphi)(\psi). \end{aligned}$$

□

Consider the notations as in Proposition 2.13. Put $\Lambda := \{\lambda_\varphi : \varphi \in L^1(G/H)\}$.

Theorem 2.16 *Let λ be a strongly quasi-invariant measure on G/H arises from the rho-function ρ. The Banach algebra $(L^1(G/H), \lambda)$ is a two-sided ideal of the Banach algebra $M(G/H)$.*

Proof By Proposition 2.13 there is a one to one corresponding between $L^1(G/H)$ and the range of the injection $\varphi \mapsto \lambda_\varphi$, so it is enough to show that Λ is a two-sided ideal of $M(G/H)$. To do this it is enough to show that $\lambda_\varphi * \sigma << \lambda$ for all $\lambda_\varphi \in \Lambda$ and $\sigma \in M(G/H)$, since Λ consists precisely of those $\sigma \in M(G/H)$ such that $\sigma << \lambda$. Now, by Definition 2.5 we have $\lambda_\varphi * \sigma = R_H((\lambda_\varphi)_{P_H} * \sigma_{P_H})$, but $(\lambda_\varphi)_{P_H} * \sigma_{P_H} = (\lambda_\varphi * \sigma)_{P_H} \in M(G{:}H)$ and so by Proposition 2.1 it equals to $\mu_{\psi_{\pi_H}}$ for some ψ in $L^1(G/H)$. Thus $R_H((\lambda_\varphi)_{P_H} * \sigma_{P_H}) = R_H(\mu_{\psi_{\pi_H}}) = \lambda_\psi \in \Lambda$. Therefore, $\lambda_\varphi * \sigma = \lambda_\psi << \lambda$ and the proof is complete. □

Fix a strongly quasi-invariant measure λ on G/H arises from the rho-function ρ. Here and in the rest of sequel we set

$$C_c^\rho(G{:}H) = \{\varphi^\rho_{\pi_H} := \varphi \circ \pi_H \cdot \rho^{1/p} : \varphi \in C_c(G/H)\},$$

and also take $L^p(G{:}H) = \overline{C_c^\rho(G{:}H)}^{\|\cdot\|_p}$, for all $1 \le p < \infty$. In a similar calculation in Farashahi (2013), Kamyabi-Gol and Tavalaei (2009) and Reiter and Stegeman (2000), one can see that $C_c^\rho(G{:}H)$ is a left ideal of the algebra $C_c(G)$. Then $T_H^p : L^p(G) \to L^p(G/H)$ defined by $T_H^p(f)(xH) = \int_H \frac{f(xh)}{\rho(xh)^{1/p}}\mathrm{d}h$ is a surjective and bounded operator with $\|T_H^p\| \le 1$. Consider the surjectivity of $T_H^p : L^p(G{:}H) \to L^p(G/H)$, for all $1 \le p < \infty$. By using Proposition 3. 39 in Folland (1995), we know that $\varphi_{\pi_H} * \psi_{\pi_H}$ belongs to $L^p(G{:}H)$. Then one can define:

$$\begin{aligned} \varphi * \psi(xH) &= T_H^p(\varphi_{\pi_H} * \psi_{\pi_H})(xH) \\ &= \int_{G/H}\int_H \varphi(yH)\psi(hy^{-1}xH)\left(\frac{\rho(hy^{-1}x)}{\rho(x)}\right)^{1/p}\mathrm{d}h\mathrm{d}\lambda(yH), \end{aligned}$$

for all $\varphi \in L^1(G/H)$ and $\psi \in L^p(G/H)$. Let $\sigma \in M(G/H)$ and $\varphi \in L^p(G/H)$, there exist $\sigma_{P_H} \in M(G{:}H)$ and $\varphi_{\pi_H} \in L^p(G{:}H)$ such that $T_H^p(\varphi_{\pi_H})$ and $R_H(\sigma_{P_H}) = \sigma$. Then we define the function $\sigma * \varphi$ in a natural way as follows:

$$\sigma * \varphi(xH) = T_H^p(\sigma_{P_H} * \varphi_{\pi_H})(xH).$$

But we have the following calculation:

$$\begin{aligned} \sigma * \varphi(xH) &= T_H^p(\sigma_{P_H} * \varphi_{\pi_H})(xH) \\ &= \int_H \frac{(\sigma_{P_H} * \varphi_{\pi_H})(x\eta)}{\rho(x\eta)^{1/p}}\mathrm{d}\eta \\ &= \int_H \frac{\int_G \varphi_{\pi_H}(y^{-1}x\eta)\mathrm{d}\sigma_{P_H}(y)}{\rho(x\eta)^{1/p}}\mathrm{d}\eta \\ &= \int_H \frac{1}{\rho(x)^{1/p}}\int_{G/H}\int_H \varphi_{\pi_H}(h^{-1}y^{-1}x\eta)\mathrm{d}h\mathrm{d}\sigma(yH)\mathrm{d}\eta \\ &= \int_{G/H}\int_H\int_H \varphi(h^{-1}y^{-1}xH)\left(\frac{\rho(h^{-1}y^{-1}x)}{\rho(x)}\right)^{1/p}\mathrm{d}h\mathrm{d}h\mathrm{d}\sigma(yH) \\ &= \int_{G/H}\int_H \varphi(hy^{-1}xH)\left(\frac{\rho(hy^{-1}x)}{\rho(x)}\right)^{1/p}\mathrm{d}h\mathrm{d}\sigma(yH), \end{aligned}$$

and in a similar calculation we get

$$\varphi * \sigma = \int_{G/H}\Delta(y^{-1})\int_H \varphi(xhy^{-1}H)\left(\frac{\rho(xhy^{-1})}{\rho(x)}\right)^{1/p}\mathrm{d}h\mathrm{d}\sigma(yH).$$

Proposition 2.17 *Suppose* $1 \leq p < \infty$ *and let* $\sigma \in M(G/H)$ *and also* $\varphi \in L^p(G/H)$. *Then* $\sigma * \varphi \in L^p(G/H)$ *and* $\|\sigma * \varphi\|_p \leq \|\sigma\| \|\varphi\|_p$

Proof Fix $\sigma \in M(G/H)$ and $\varphi \in L^p(G/H)$. Considering definition of T_H^p, the function $\sigma * \varphi$ belongs to $L^p(G/H)$. Using the fact that the mapping T_H^p and R_H are isometric on $L^p(G:H)$ and $M(G:H)$, respectively, we get

$$\begin{aligned}\|T_H^p(\sigma_{P_H} * \varphi_{\pi_H})\|_p &= \|\sigma_{P_H} * \varphi_{\pi_H}\|_p \leq \|\sigma_{P_H}\| \|\varphi_{\pi_H}\|_p \\ &= \|R_H(\sigma_{P_H})\| \|T_H^p(\varphi_{\pi_H})\|_p.\end{aligned}$$

This implies $\|\sigma * \varphi\|_p \leq \|\sigma\| \|\varphi\|_p$. □

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