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Proper efficiency, scalarization and transformation in multi-objective optimization: unified approaches

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ABSTRACT

In this paper, we investigate the relationships between proper efficiency and the solutions of a general scalarization problem in multi-objective optimization. We provide some conditions under which the solutions of the dealt with scalar program are properly efficient and vice versa. We also show that, under some conditions, if the considered general scalar problem is unbounded, then the original multi-objective problem does not have any properly efficient solution. In another part of the work, we investigate a general transformation of the objective functions which preserves proper efficiency. We show that several important results existing in the literature are direct consequences of the results of the present paper.

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1. Introduction

Consider a general multi-objective optimization problem,

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & x \in X, \end{aligned} \tag{1}$$

where $X \subseteq \mathbb{R}^n$ is the feasible set, and $f : X \to \mathbb{R}^p$ with $p \geq 2$ is the objective function. Multi-objective optimization problems arise naturally in many applications in engineering, management, economics, finance, etc. Indeed, each decision making or optimization problem with more than one criterion or objective can be cast as a multi-objective optimization problem.

Efficient solutions of problem (1) are defined as members of X for which it is impossible to improve some objective(s) without deteriorating (at least) another one [1,2]. Mathematically, $\bar{x} \in X$ is an efficient solution of problem (1) if there is no $x \in X$ with $f_i(x) \leq f_i(\bar{x})$ for every $i = 1, 2, \ldots, p$ and $f_j(x) < f_j(\bar{x})$ for some $j \in \{1, 2, \ldots, p\}$. Proper efficiency, an important solution concept in multi-objective

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optimization, has been proposed in order to eliminate efficient solutions with unbounded trade-offs [2,3].

Scalarization is one of the most common approaches to handle multi-objective optimization problems [1,2,4]. A scalarization method is to solve a single-objective problem (corresponding to (1)) whose optimal solutions can be (weakly, properly) efficient for (1). In addition, the single-objective programs derived from scalarization techniques are employed as subproblems in iterative methods which generate an approximation of the efficient set [5], and also in interactive approaches which try to produce the most preferred solution [2].

An important question concerning scalarization problems is about the connection between their solutions and proper efficiency, as well as their ability to generate properly efficient solutions. Scalarization methods are not only strong tools to generate (properly) efficient solutions, but also provide valuable information about (the quality of) these solutions. In this study, we consider a general (unified) scalarization program, and provide some sufficient conditions under which the optimal solutions of the dealt with scalarization problem are properly efficient. We list some well-known scalarization techniques which satisfy the given sufficient conditions. Furthermore, we focus on the parametric scalarization tools, and give sufficient conditions under which a parametric scalarization method is able to generate all properly efficient solutions. We investigate the unbounded case separately, and establish that under some conditions the unboundedness of the considered general scalarization problem implies the emptiness of the set of properly efficient solutions.

Another part of the current study is devoted to investigation of a general transformation which maps objective functions preserving proper efficiency. Transformation of objective functions has been mainly proposed for normalization of objectives with different units [6]. In addition, it has been exploited to facilitate handling multi-objective problems, for instance by convexification [7,8]. In this paper, we give sufficient conditions under which the set of properly efficient solutions of the original and transformed problems are the same. We show that several important results existing in the literature are direct consequences of the results of the present paper.

The rest of the paper is organized as follows. We review terminologies and notations in Section 2. Section 3 is devoted to the scalarization methods. A unified transformation for multi-objective problems is studied in Section 4. Section 5 contains a short conclusion.

2. Terminologies and notations

The p-dimensional Euclidean space is denoted by $\mathbb{R}^p$. Vectors are considered to be column vectors and the superscript T denotes the transpose operation. We use both notations $x^T y$ and $\langle x, y \rangle$ to denote the standard inner product of two vectors $x, y \in \mathbb{R}^p$. Throughout the paper, we employ the Euclidean norm defined as $\|x\| = \sqrt{x^T x}$. We denote the i-th component (resp. row) of a given vector (resp. matrix) y by y_i. We use e and e^i to denote the vector of ones and the i-th unit coordinate vector, respectively. For a set $Y \subseteq \mathbb{R}^p$, we use the notations $\text{int}(Y)$, $\text{cl}(Y)$, and $\text{co}(Y)$ for the interior, the closure, and the convex hull of Y, respectively. Furthermore, $\text{cone}(Y)$ and $\text{Pos}(Y)$ are the cone and the convex cone generated by Y, respectively. Recall that $\text{cone}(Y) = \{\lambda y : \lambda \geq 0,\ y \in Y\}$ and $\text{Pos}(Y) = \text{cone}(\text{co}(Y))$.

The nonnegative orthant is defined by

$$\mathbb{R}^p_+ = \left\{ y \in \mathbb{R}^p : y_i \geq 0, i = 1, 2, \ldots, p \right\}.$$

The notations $\leqq$, $\leq$ and $<$ stand for the following orders on $\mathbb{R}^p$ with $p \geq 2$,

$$x \leqq y \Longleftrightarrow y - x \in \mathbb{R}^p_+,$$
$$x \leq y \Longleftrightarrow y - x \in \mathbb{R}^p_+ \setminus \{0\},$$
$$x < y \Longleftrightarrow y - x \in \text{int}(\mathbb{R}^p_+).$$

Likewise, matrix inequalities are understood componentwise.

According to Rademacher's theorem, every locally Lipschitz function on $\mathbb{R}^n$ is almost everywhere differentiable in the sense of Lebesgue measure [9]. Let $g : \mathbb{R}^p \to \mathbb{R}^q$ be a locally Lipschitz function. The generalized Jacobian of g at $\bar{x}$, denoted by $\partial g(\bar{x})$, is defined by

$$\partial g(\bar{x}) = \text{co} \left\{ \lim_{\nu \to +\infty} \nabla g(x_\nu) : x_\nu \to \bar{x}, x_\nu \notin X_f \right\},$$

where X_f is the set of points at which g is not differentiable, and $\nabla g(x_\nu)$ is the $q \times p$ Jacobian matrix of g at x_ν. If g is continuously differentiable at $\bar{y}$, then $\partial g(\bar{y}) = \{\nabla g(\bar{y})\}$. See [9] for more information about the generalized Jacobian. Hereafter, for a measurable set $E \subseteq \mathbb{R}$, $\mu(E)$ denotes its Lebesgue measure.

The point $y^I \in \mathbb{R}^p$ in which $y^I_i = \min_{x \in X} f_i(x)$ for every $i = 1, 2, \ldots, p$, is called the ideal point of (1), and a point $y^U \in \mathbb{R}^p$ with $y^U < y^I$ is said to be a utopia point.

Several concepts for proper efficiency have been introduced in the literature. In what follows, we list some definitions which will be used in the sequel. For a comprehensive study of proper efficiency, the reader is referred to [10].

Definition 2.1 ([3]): A feasible solution $\bar{x} \in X$ is called a properly efficient solution of (1) in the Geoffrion's sense, if it is efficient and there exist a real number $M > 0$ such that for all $i \in \{1, 2, \ldots, p\}$ and $x \in X$ with $f_i(x) < f_i(\bar{x})$, there exists an index $j \in \{1, 2, \ldots, p\}$ with $f_j(x) > f_j(\bar{x})$ and

$$\frac{f_i(\bar{x}) - f_i(x)}{f_j(x) - f_j(\bar{x})} \leq M.$$

Definition 2.2 ([11]): A feasible solution $\bar{x} \in X$ is called a properly efficient solution of (1) in the Benson's sense, if

$$\mathrm{cl}\Big(\mathrm{cone}\big(f(X) + \mathbb{R}^p_+ - f(\bar{x})\big)\Big) \cap (-\mathbb{R}^p_+) = \{0\}.$$

Definition 2.3 ([12]): A feasible solution $\bar{x} \in X$ is called a properly efficient solution of (1) in the Henig's sense if there exists a closed convex pointed cone C with $R^p_+ \setminus \{0\} \subseteq \mathrm{int}(C)$ and

$$f(X) \cap \big(f(\bar{x}) - C\big) = \big\{f(\bar{x})\big\}.$$

Definitions 2.1–2.3 for multi-objective problem (1) with natural ordering cone $\mathbb{R}^p_+$ are equivalent; see [11, Theorem 3.2] and [12, Theorem 2.1]. We remark that having different definitions for the proper efficiency turns out to be of value. In fact, a result can be easily derived from a given definition, while the proof of the same result with other definitions may be less obvious.

Let C^* denote the dual cone of C, that is, $C^* = \{\xi : \langle \xi, y\rangle \geq 0, \forall y \in C\}$. For $\delta \in \mathbb{R}_+$, we define the convex cone

$$C_\delta = \left\{y \in \mathbb{R}^p : \langle e^i + \delta e, y\rangle \geq 0, i = 1, \dots, p\right\}.$$

Remark 2.1: For a convex cone C with $R^p_+ \setminus \{0\} \subseteq \mathrm{int}(C)$, there exists some $\delta > 0$ such that $R^p_+ \subseteq C_\delta \subseteq C$; see [13, p.1259].

An effective tool to analyse the scalarization methods and also proper efficiency is cone approximation [14,15]. Indeed, the cone C_δ can be obtained invoking Henig's procedure addressed in [14]. Henig considered a general ordering cone and an arbitrary norm. Here, we explain his procedure for the natural ordering cone and a special norm defined as $|||y||| = (1 + p\delta)^{-1} \sum_{i=1}^p |y_i|$, where $\delta > 0$ is a given scalar. This norm depends on δ. We denote it by $|||\cdot|||$ to avoid confusion with Euclidean norm. Define $K_\delta = \{y \in \mathbb{R}^p : y + \delta|||y|||B \subseteq \mathbb{R}^p_+\}$, where B is the unit ball in $\mathbb{R}^p$. We have

$$\begin{aligned} K_\delta &= \mathrm{Pos}\left\{y : |||y||| = 1, y \in K_\delta\right\} = \mathrm{Pos}\left\{y : e^T y = 1 + p\delta, y \geqq \delta e\right\} \\ &= \mathrm{Pos}\left\{e^1 + \delta e, \dots, e^p + \delta e\right\}. \end{aligned}$$

It can be seen that $C_\delta = K^*_\delta$.

3. Scalarization and proper efficiency: A general umbrella

Consider a general problem

$$\begin{array}{ll} \min & g(f(x)) \\ \text{s.t.} & x \in X, \end{array} \tag{2}$$

where $f(X) \subseteq Y \subseteq \mathbb{R}^p$ and $g : Y \to \mathbb{R}^q$ are given. When $q = 1$, problem (2) is corresponding to a scalarization approach. In this case, optimality and (proper) efficiency for (2) coincide. Throughout the paper, optimal solutions are understood in the global sense.

Definition 3.1 ([16]): Let $Y \subseteq \mathbb{R}^p$. A function $g : Y \to \mathbb{R}^q$ is called properly increasing on Y if there exists some $\delta > 0$ such that

$$y_1, y_2 \in Y, \ \ y_1 - y_2 \in C_\delta \setminus \{0\} \ \Rightarrow \ g(y_1) \geq g(y_2). \tag{3}$$

In fact, Definition 3.1 in [16] is given for $Y = \mathbb{R}^p$ and any ordering cone C with $R^p_+ \setminus \{0\} \subseteq \text{int}(C)$. We consider the ordering cone C_δ in the definition, due to Remark 2.1. The next theorem states that each efficient solution of (2) is a properly efficient solution of (1) when $g : Y \to \mathbb{R}^q$ is properly increasing. This theorem can be derived from some results in [16]. Here, we provide a short proof to keep the paper self-contained.

Theorem 3.1: *Let $f(X) \subseteq Y \subseteq \mathbb{R}^p$. If $g : Y \to \mathbb{R}^q$ is properly increasing on Y, then each efficient solution of (2) is a properly efficient solution of (1).*

Proof: Assume that $\bar{x}$ is an efficient solution of (2). By Definition 3.1, there exists some $\delta > 0$ such that (3) holds. Hence, $f(X) \cap (f(\bar{x}) - C_\delta) = \{f(\bar{x})\}$ (Otherwise, one gets $g(f(\hat{x})) \leq g(f(\bar{x}))$ for some $\hat{x} \in X$, which contradicts the efficiency of $\bar{x}$ for (2)). So, $\bar{x}$ is a properly efficient solution of (1). ■

Theorem 3.1 is an interesting result which can be employed in scalarization methods as well. However, it is not straightforward to check if a given function is properly increasing or not. In Proposition 3.1 and Theorem 3.2 below, we provide some necessary and sufficient conditions under which a given function is properly increasing on a given set.

Proposition 3.1: *Assume that the locally Lipschitz function $g : Y \to \mathbb{R}^q$ is properly increasing on open set $Y \subseteq \mathbb{R}^p$. Then, for each $y \in Y$ and each $\xi \in \partial g(y)$ one has*

$$\xi_i^T \in C_\delta^*, \quad \forall\, i = 1, 2, \ldots, q,$$

for some $\delta > 0$.

Proof: As g is properly increasing on Y, there exists some $\delta > 0$ such that (3) holds. Assume g is differentiable at $\bar{y} \in Y$. For any non-zero vector $d \in C_\delta$ and

$t > 0$ sufficiently small, we have $g(\bar{y} - td) \leq g(\bar{y})$. Hence,

$$0 \geqq \lim_{t \downarrow 0} \frac{g(\bar{y} - td) - g(\bar{y})}{t} = -\nabla g(\bar{y})^{\mathrm{T}} d.$$

Hence, $[\nabla g(\bar{y})]_i^T \in C_\delta^*$ for each i. By the definition of generalized Jacobian, it is seen that for each $y \in Y$ and each $\xi \in \partial g(y)$ one has

$$\xi_i^T \in C_\delta^*, \quad \forall\, i = 1, 2, \ldots, q,$$

and the proof is complete. ■

The converse of Proposition 3.1 does not hold necessarily (consider zero function). In the next theorem, we give some sufficient conditions under which a function is properly increasing. Recall that, for a measurable set $E \subseteq \mathbb{R}$, $\mu(E)$ denotes its Lebesgue measure.

Theorem 3.2: *Let $Y \subseteq \mathbb{R}^p$ be an open convex set and let $g : Y \to \mathbb{R}^q$ be locally Lipschitz. If there exists some $\delta > 0$ such that*

(i) $\forall\, y \in Y, \forall\, \xi \in \partial g(y) : \xi_i \in C_\delta^*, i = 1, 2, \ldots, q,$
(ii) $\forall\, y \in Y, \forall\, d \in C_\delta \setminus \{0\} : \mu(\{t : 0 \in \partial g(y + td)\}) = 0,$

then g is properly increasing on Y.

Proof: Let $y \in Y$ and $d \in C_\delta \setminus \{0\}$. By mean value theorem (see [17, Theorem 8]),

$$g(y + d) - g(y) = \sum_{i=1}^{q} \lambda_i \xi^i d,$$

where $\xi^i \in \partial g(z_i), z_i \in [y, y + d], \lambda_i \geq 0$ for $i = 1, \ldots, q$ and $\sum_{i=1}^q \lambda_i = 1$. Here, $[y_1, y_2]$ stands for the line segment joining y_1 and y_2 in $\mathbb{R}^p$. Now, by assumption (i), we have $g(y + d) \geqq g(y)$.

Now, we show that g is properly increasing on Y with respect to $C_{\delta/2}$. Suppose that $y \in Y$ and d is a non-zero vector in $C_{\delta/2}$. As proved above, $g(y + d) \geqq g(y)$. So, we need to show that $g(y + d) \neq g(y)$. By indirect proof, assume that $g(\bar{y} + \bar{d}) = g(\bar{y})$ for some $\bar{y} \in Y$ and $0 \neq \bar{d} \in C_{\delta/2}$. Define the function $\theta : [0, 1] \to \mathbb{R}^q$ by $\theta(t) = g(\bar{y} + t\bar{d})$. Since θ is componentwise nondecreasing on its domain and $\theta(0) = \theta(1)$, it is a constant function and consequently $\theta'(t) = 0$ for each $t \in (0, 1)$. As $\theta'(t) \subset \partial g(\bar{y} + t\bar{d})\bar{d}$ (see [9, Corollary 2.6.6]),

$$0 \in \left\{\xi^T \bar{d} : \xi \in \partial g(\bar{y} + t\bar{d})\right\}, \quad \forall\, t \in (0, 1).$$

As $\bar{d} \in C_{\delta/2} \setminus \{0\} \subseteq \mathrm{int}(C_\delta)$, $\xi^T \bar{d} = 0$ together with assumption (i) imply that $\xi = 0$. This contradicts assumption (ii) and the proof is complete. ■

Theorem 3.3: *Let $Y \subseteq \mathbb{R}^p$ be an open convex set and let $g : Y \to \mathbb{R}^q$ be locally Lipschitz. If there are two matrices $L, U > 0$ such that*

$$\forall y \in Y, \quad \forall \xi \in \partial g(y) : L \leqq \xi \leqq U,$$

then g is properly increasing on Y.

Proof: The assumption $L \leqq \xi \leqq U$ for all $y \in Y$ and $\xi \in \partial g(y)$ implies the existence of a $\delta > 0$ such that $\xi_i \in C^*_\delta$ for each $y \in Y$, $\xi \in \partial g(y)$, and $i = 1, 2, \dots, q$. So, all assumptions of Theorem 3.2 are fulfilled. Hence, g is properly increasing. ■

Theorem 3.3 may not hold when $L = 0$. The following example clarifies this.

Example 3.1: Consider the multi-objective problem

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & x \in \mathbb{R}, \end{array} \tag{4}$$

with $f(x) = (-\exp(x), -\exp(-x))^{\mathrm{T}}$. Here, $\exp(\cdot)$ is the exponential function. Let $g : -\text{int}(\mathbb{R}^2_+) \to \mathbb{R}$ be given by $g(y) = -y_1 y_2$. It is readily seen that $\nabla g(y) > 0$ for each $y < 0$, and $f(\mathbb{R}) \subseteq \{y : y < 0\}$. Furthermore,

$$\text{argmin} \left\{ g(f(x)) : x \in \mathbb{R} \right\} = \mathbb{R}.$$

Problem (4) does not admit any properly efficient solution. Hence, g cannot be properly increasing on Y.

In the following, we show that several important existing results concerning scalarization and proper efficiency are directly derived from Theorems 3.1 and 3.3. Indeed, the following results have been proved for each scalarization method in the literature separately. Here, we give a unified framework.

- *Weighted Sum method* [18]: The scalar program of this method is formulated as

$$\begin{array}{ll} \min & \lambda^T f(x) \\ \text{s.t.} & x \in X, \end{array} \tag{5}$$

  where $\lambda \in \text{int}(\mathbb{R}^p_+)$. Setting $g(y) := \lambda^T y$, we have $\nabla g(y) = \lambda$. By Theorem 3.3, it is easily seen that g is properly increasing on $\mathbb{R}^p$. Consequently, by Theorem 3.1, the optimal solutions of (5) are properly efficient for (1).

- *Conic scalarization method* [19]: The scalar problem of this method can be written as

$$\begin{aligned} \min \quad & \sum_{i=1}^{p} \lambda_i (f_i(x) - y_i^r) + \alpha \sum_{i=1}^{p} |f_i(x) - y_i^r| \\ \text{s.t.} \quad & x \in X, \end{aligned} \tag{6}$$

  where $0 \leq \alpha < \lambda_i$, and is an arbitrary reference point. By setting

$$g(y) := \sum_{i=1}^{p} \lambda_i (y_i - y_i^r) + \alpha \sum_{i=1}^{p} |y_i - y_i^r|,$$

  the function g is convex but not necessarily differentiable on $\mathbb{R}^p$. We have $\partial g(y) \subseteq \{\lambda + \alpha\eta : -e \leqq \eta \leqq e\}$. By Theorem 3.3, g is properly increasing on $\mathbb{R}^p$ for $0 \leq \alpha < \lambda_i, (i = 1, \ldots, p)$. Therefore, by Theorem 3.1, the optimal solutions of (6) are properly efficient for (1) when $0 \leq \alpha < \lambda_i, (i = 1, \ldots, p)$.
- *Modified weighted Tchebycheff method* [20]: This method is written as

$$\begin{aligned} \min \quad & \max_i \left\{ \lambda_i (f_i(x) - y_i^U) + \alpha e^T (f(x) - y^U) \right\} \\ \text{s.t.} \quad & x \in X, \end{aligned} \tag{7}$$

  where $\lambda \in \text{int}(\mathbb{R}^p_+)$, $\alpha > 0$, and y^U is a utopia point. By setting $g(y) := \max_i \{\lambda_i (y_i - y_i^U) + \alpha e^T (y - y^U)\}$, the function g is convex but not necessarily differentiable on $\mathbb{R}^p$. By Theorem 3.3, it is seen that g is properly increasing on $\mathbb{R}^p$. Therefore, by Theorem 3.1, each optimal solution of (7) is properly efficient for (1).

Although properly increasing notion provides a unified framework for obtaining some well-known results, it is not applicable for some methods. For instance, consider the following scalarization problem, known as compromise programming [21],

$$\begin{aligned} \min \quad & \left(\sum_{i=1}^{p} \lambda_i^s (f_i(x) - y_i^U)^s \right)^{1/s} \\ \text{s.t.} \quad & x \in X, \end{aligned} \tag{8}$$

where $\lambda \in \text{int}(\mathbb{R}^p_+)$, $s > 1$, and y^U is a utopia point. It is known that optimal solutions of (8) are properly efficient for (1) [21,22]. For instance, assume $p = s = 2$, $\lambda_1 = \lambda_2 = 1$ and $y^U = e$. Suppose that $Y = \text{int}(e + \mathbb{R}^2_+)$, $\delta > 0$ and $g(y) = \sqrt{y_1^2 + y_2^2}$. Since

$$\lim_{\alpha \to 0} \frac{1}{\alpha} \left(\begin{pmatrix} \cos(2\alpha) \\ \sin(2\alpha) \end{pmatrix} - \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} \right) = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

for $\bar{\alpha} > 0$ sufficiently small, $(\cos(2\bar{\alpha}) \;\; \sin(2\bar{\alpha}))^{\mathrm{T}} - (\cos\bar{\alpha} \;\; \sin\bar{\alpha})^{\mathrm{T}} \in \text{int}(C_\delta)$. On the other hand, for r sufficiently large, by setting $\hat{y} := r(\cos(2\bar{\alpha}) \;\; \sin(2\bar{\alpha}))^{\mathrm{T}}$ and

$\bar{y} := r(\cos\bar{\alpha} \quad \sin\bar{\alpha})^{\mathrm{T}}$, we have $\hat{y}, \bar{y} \in Y$ and $\hat{y} - \bar{y} \in C_\delta \setminus \{0\}$ while $g(\hat{y}) = g(\bar{y}) = r$. Therefore, g cannot be properly increasing on Y. In the following, we give a result which covers these cases.

Theorem 3.4: *Assume the following conditions*:

(i) *The properly efficient set of* (1) *is non-empty*;
(ii) Y *is open and convex satisfying* $\mathrm{cl}(f(X)) \subseteq Y$;
(iii) $g : Y \to \mathbb{R}^q$ *is locally Lipschitz on* Y *;*
(iv) $\xi > 0,\ \forall\, y \in Y, \forall \xi \in \partial g(y)$.

Then the set of efficient solutions of (2) *is a subset of the set of properly efficient solutions of* (1).

Proof: By indirect proof, suppose that $\bar{x}$ is an efficient solution of (2) while it is not properly efficient for (1). By Benson's proper efficiency definition, there are $\{x_\nu\} \subseteq X$, $\{d_\nu\} \subseteq \mathbb{R}^p_+$ and $\{t_\nu\} \subseteq \mathbb{R}_+$ such that

$$\lim_{\nu\to\infty} t_\nu(f(x_\nu) + d_\nu - f(\bar{x})) = -d, \tag{9}$$

where $0 \neq d \in \mathbb{R}^p_+$. If $f(x_\nu) - f(\bar{x}) = 0$ for some subsequence of $\{x_\nu\}$, then due to

$$t_\nu(f(x_\nu) - f(\bar{x})) \leqq t_\nu(f(x_\nu) + d_\nu - f(\bar{x})), \tag{10}$$

we get $0 \leqq -d$ which contradicts $0 \neq d \in \mathbb{R}^p_+$. So, without loss of generality, we assume $f(x_\nu) - f(\bar{x}) \neq 0$ for each ν. Hence, the sequence $\{(f(x_\nu) - f(\bar{x}))/\|f(x_\nu) - f(\bar{x})\|\}$ is well-defined and, by working with subsequences if it is necessary, we get

$$\lim_{\nu\to\infty} \frac{f(x_\nu) - f(\bar{x})}{\|f(x_\nu) - f(\bar{x})\|} = -\bar{d}, \tag{11}$$

for some $0 \neq \bar{d} \in \mathbb{R}^p$. Now, we show that $\bar{d} \in \mathbb{R}^p_+$. To this end, as $\{d_\nu\} \subseteq \mathbb{R}^p_+$, it is sufficient to prove that

$$\lim_{\nu\to\infty} \frac{f(x_\nu) + d_\nu - f(\bar{x})}{\|f(x_\nu) - f(\bar{x})\|} \leqq 0. \tag{12}$$

We have

$$\lim_{\nu\to\infty} \frac{f(x_\nu) + d_\nu - f(\bar{x})}{\|f(x_\nu) - f(\bar{x})\|} = \lim_{\nu\to\infty} \frac{t_\nu(f(x_\nu) + d_\nu - f(\bar{x}))}{t_\nu(\|f(x_\nu) - f(\bar{x})\|)}. \tag{13}$$

If $t_\nu\|f(x_\nu) - f(\bar{x})\|$ goes to $+\infty$ or a positive scalar (by choosing an appropriate subsequence if necessary), then by (9) and (13), the inequality (12) is derived.

Otherwise, we have $\lim_{\nu\to\infty} t_\nu \|f(x_\nu) - f(\bar{x})\| = 0$, and by (9),

$$\begin{aligned} -d &= \lim_{\nu\to\infty} t_\nu (f(x_\nu) + d_\nu - f(\bar{x})) \\ &= \lim_{\nu\to\infty} t_\nu (f(x_\nu) - f(\bar{x})) + \lim_{\nu\to\infty} t_\nu d_\nu = 0 + \lim_{\nu\to\infty} t_\nu d_\nu \geqq 0, \end{aligned}$$

which contradicts $0 \neq d \in \mathbb{R}^p_+$.

Therefore, one may assume

$$\lim_{\nu\to\infty} \frac{f(x_\nu) - f(\bar{x})}{\|f(x_\nu) - f(\bar{x})\|} = -\bar{d}, \tag{14}$$

for some $0 \neq \bar{d} \in \mathbb{R}^p_+$. By [17, Theorem 8] (taking (ii) and (iii) into account), for each ν, there are $\{y^1_\nu, \dots, y^q_\nu\} \subseteq Y$ and $\lambda^\nu \in \mathbb{R}^q_+$ such that

$$g(f(x_\nu)) - g(f(\bar{x})) = \sum_{k=1}^{q} \lambda^\nu_k \xi^k_\nu (f(x_\nu) - f(\bar{x})), \tag{15}$$

where $y^k_\nu \in [f(x_\nu), f(\bar{x})]$ and $\xi^k_\nu \in \partial g(y^k_\nu)$, $k = 1, \dots, q$, and $\sum_{k=1}^{q} \lambda^\nu_k = 1$. Suppose that $\{f(x_\nu)\}$ has a cluster point. Without loss of generality, due to (ii) and (iii), one may assume that $\lambda^\nu \to \lambda, f(x_\nu) \to \bar{y} \in Y, y^k_\nu \to y^k \in Y$ and $\xi^k_\nu \to \xi^k \in \partial g(y^k)$ for $k = 1, \dots, q$. By (iv), accompanying (14) and (15), we get

$$\lim_{\nu\to\infty} \frac{g(f(x_\nu)) - g(f(\bar{x}))}{\|f(x_\nu) - f(\bar{x})\|} = -\hat{d},$$

for some $\hat{d} \in \mathbb{R}^q_+$. Indeed, $\hat{d} > 0$ (because of (iv)). The preceding relation contradicts the efficiency of $\bar{x}$ for (2). Now we consider the case that $\{f(x_\nu)\}$ is unbounded. Let $\hat{x}$ be a properly efficient solution of (1). One can infer from (14),

$$\lim_{\nu\to\infty} \frac{f(x_\nu) - f(\hat{x})}{\|f(x_\nu) - f(\bar{x})\|} = -\bar{d}.$$

This contradicts the proper efficiency of $\hat{x}$ for (1), and the proof is complete. ■

As checking the existence of properly efficient solution might be demanding in some cases, we replace this assumption (i.e. (i) in Theorem 3.4) with another condition in the next theorem.

Theorem 3.5: *Assume the following conditions:*

(i) *Y is an open convex set with $Y \subseteq a + \mathbb{R}^p_+$ for some $a \in \mathbb{R}^p$;*
(ii) *g is locally Lipschitz on Y;*
(iii) *$\xi > 0, \forall y \in Y, \forall \xi \in \partial g(y)$;*
(iv) *$\mathrm{cl}(f(X)) \subseteq Y$.*

Then each efficient solution of (2) *is a properly efficient solution for* (1).

Proof: The proof is same as that of Theorem 3.4. Only we need to show that the sequence $\{f(x_\nu)\}$ is bounded. If it is unbounded, then by choosing an appropriate subsequence, (i) and (iv) imply that

$$\lim_{\nu\to\infty} \frac{f(x_\nu) - f(\bar{x})}{\|f(x_\nu) - f(\bar{x})\|} = \lim_{\nu\to\infty} \frac{f(x_\nu)}{\|f(x_\nu)\|} = \hat{d},$$

for some $\hat{d} \in \mathbb{R}^p_+$, which contradicts (14). ■

In the following, we demonstrate that Theorem 3.5 implies that the optimal solutions of some common and uncommon scalar problems are properly efficient.

- *Compromise programming* [21]: As mentioned above, the scalar program of this method is written as

$$\begin{aligned} \min \quad & \left(\sum_{i=1}^{p} \lambda_i^s (f_i(x) - y_i^U)^s \right)^{1/s} \\ \text{s.t.} \quad & x \in X, \end{aligned} \tag{16}$$

  where $\lambda \in \operatorname{int}(\mathbb{R}^p_+)$, $s > 1$, and y^U is a utopia point. Suppose that $Y = \{y : y > y^U\}$, and $g : Y \to \mathbb{R}$ given as

$$g(y) := \left(\sum_{i=1}^{p} \lambda_i^s (y_i - y_i^U)^s \right)^{1/s}.$$

  All assumptions of Theorem 3.5 are fulfilled. Hence, the optimal solutions of (16) are properly efficient for (1).
- *Multiplicative scalarization:* Let $y^U \in \mathbb{R}^p$ be a utopia point for (1). Consider the following scalarization method

$$\begin{aligned} \min \quad & \prod_{i=1}^{p} (f_i(x) - y_i^U)^{\lambda_i} \\ \text{s.t.} \quad & x \in X, \end{aligned} \tag{17}$$

  where $\lambda > 0$ is a fixed vector. It is seen that the function $g : \operatorname{int}(y^U + \mathbb{R}^p_+) \to \mathbb{R}$ given by $g(y) = \prod_{i=1}^{p} (y_i - y_i^U)^{\lambda_i}$ fulfils all conditions of Theorem 3.5. So, each optimal solution of (17) is a properly efficient solution of (1).

- *Exponential scalarization:* By considering $g(y) := \sum_{i=1}^{p} \lambda_i e^{y_i}$ with $\lambda > 0$ (a fixed vector), we get the following scalarization

$$\begin{aligned} \min \quad & \sum_{i=1}^{p} \lambda_i e^{f_i(x)} \\ \text{s.t.} \quad & x \in X. \end{aligned} \tag{18}$$

  By setting $Y = \{y : y > y^U\}$, all assumptions of Theorem 3.5 are fulfilled. Therefore, the optimal solutions of (18) are properly efficient for (1).

A multi-objective optimization problem generally has numerous properly efficient solutions. One important question in this regard is ‘ under which condition(s) a parametric scalarization technique is able to generate each properly efficient solution? ’. A general parametric scalarization problem can be written as

$$(P_u): \quad \begin{aligned} \min \quad & g(f(x), u) \\ \text{s.t.} \quad & x \in X, \end{aligned} \tag{19}$$

where x is the decision variable and u is a parameter varying within an arbitrary set U. The set $Y \subseteq \mathbb{R}^p$ with $f(X) \subseteq Y$ and the function $g : Y \times U \to \mathbb{R} \cup \{+\infty\}$ are given. Theorem 3.6 below, gives sufficient conditions under which the parametric problem (19) generates all properly efficient solutions of (1).

Theorem 3.6: *Let $f(X) \subseteq Y$. Assume that for each $\bar{y} \in Y$ and each $\delta > 0$, there exists some $u \in U$ with*

$$\{y \in Y : g(y, u) < g(\bar{y}, u)\} \subseteq (\bar{y} - C_\delta). \tag{20}$$

If $\bar{x}$ is a properly efficient solution for (1), then $\bar{x}$ solves (19) for some $u \in U$.

Proof: Suppose that $\bar{x}$ is a properly efficient solution of (1). Due to the Henig's proper efficiency definition, invoking Remark 2.1, there is some $\delta > 0$ such that $f(X) \cap (f(\bar{x}) - C_\delta) = \{f(\bar{x})\}$. Set $\bar{y} := f(\bar{x})$. By the assumption, there exists some $\bar{u} \in U$ such that

$$\left\{x \in X : g(f(x), \bar{u}) < g(f(\bar{x}), \bar{u})\right\} \subseteq (f(\bar{x}) - C_\delta).$$

This implies

$$\bar{x} \in \operatorname{argmin}\{g(f(x), \bar{u}) : x \in X\},$$

and the proof is complete. ■

As an application of Theorem 3.6, we establish that conic scalarization method [19] produces all properly efficient solutions. It is enough to show that this method fulfils all assumptions of Theorem 3.6. As mentioned above, the scalar program of this method is written as

$$\begin{aligned} \min \quad & \sum_{i=1}^{p} \lambda_i (f_i(x) - y_i^r) + \alpha \sum_{i=1}^{p} |f_i(x) - y_i^r| \\ \text{s.t.} \quad & x \in X, \end{aligned}$$

Here,

$$U = \left\{ (\alpha, \lambda, y^r) : y^r \in \mathbb{R}^p,\ \lambda \in \operatorname{int}(\mathbb{R}^p_+),\ 0 \le \alpha < \lambda_i\ (i = 1, \ldots, p) \right\}$$

and $g : \mathbb{R}^p \times U \to \mathbb{R} \cup \{+\infty\}$ is given by

$$g(y, (\alpha, \lambda, y^r)) = \sum_{i=1}^{p} \lambda_i (y_i - y_i^r) + \alpha \sum_{i=1}^{p} |y_i - y_i^r|.$$

Now, we check the assumptions of Theorem 3.6. Let $\bar{y} \in \mathbb{R}^p$ and $\delta > 0$ be given. We show that for $\lambda = e$, $\alpha \in (1/(2\delta + 1), 1)$ and $y^r = \bar{y}$ we have

$$\left\{ y \in \mathbb{R}^p : \sum_{i=1}^{p} (y_i - \bar{y}_i) + \alpha \sum_{i=1}^{p} |y_i - \bar{y}_i| < 0 \right\} \subseteq (\bar{y} - C_\delta),$$

or equivalently

$$\left\{ y \in \mathbb{R}^p : \sum_{i=1}^{p} y_i + \alpha \sum_{i=1}^{p} |y_i| < 0 \right\} \subseteq -C_\delta. \tag{21}$$

It is not difficult to see that, for each $\alpha > 0$,

$$\begin{aligned} & \left\{ y \in \mathbb{R}^p : \sum_{i=1}^{p} y_i + \alpha \sum_{i=1}^{p} |y_i| < 0 \right\} \\ & \quad = \left\{ y \in \mathbb{R}^p : \sum_{i=1}^{p} y_i + \alpha \sum_{i=1}^{p} \beta_i y_i < 0,\ \forall\, \beta \in \{-1, 1\}^p \right\}. \end{aligned}$$

To establish the inclusion (21), it is sufficient to show that for $j = 1, \ldots, p$, and $y \in \mathbb{R}^p$ with $\sum_{i=1}^{p} y_i + \alpha \sum_{i=1}^{p} |y_i| < 0$, we have $(e^j + \delta e)^{\mathrm{T}} y \le 0$.

Suppose that $\sum_{i=1}^{p} y_i + (1/(2\delta+1)) \sum_{i=1}^{p} |y_i| < 0$. Let $j \in \{1, 2, \ldots, p\}$ be arbitrary. Consider $\bar{\beta} \in \{-1, 1\}^p$ with

$$\bar{\beta}_i = \begin{cases} -1, & i \neq j \\ 1, & i = j \end{cases}$$

By our discussion, $\sum_{i=1}^{p} y_i + (1/(2\delta+1)) \sum_{i=1}^{p} \bar{\beta}_i y_i < 0$. As

$$\begin{aligned} e^j + \delta e &= \frac{1}{2}(2\delta+1)\left(\sum_{i=1}^{p} e^i + \frac{1}{2\delta+1}\left(e^j - \sum_{\substack{i=1 \\ i \neq j}}^{p} e^i\right)\right) \\ &= \frac{1}{2}(2\delta+1)\left(\sum_{i=1}^{p} e^i + \frac{1}{2\delta+1}\left(\sum_{i=1}^{p} \bar{\beta}_i e^i\right)\right), \end{aligned}$$

we have $(e^j + \delta e)^{\mathrm{T}} y < 0$. Now, to prove (21), assume $\alpha \in (1/(2\delta+1), 1)$ and $\sum_{i=1}^{p} y_i + \alpha \sum_{i=1}^{p} |y_i| < 0$. As $1/(2\delta+1) < \alpha$, we get $\sum_{i=1}^{p} y_i + (1/(2\delta+1)) \sum_{i=1}^{p} |y_i| < 0$. So, due to the above discussion, $(e^j + \delta e)^{\mathrm{T}} y \leq 0$ for each $j = 1, 2, \ldots, p$. This leads to $y \in -C_\delta$ and (21) is proved.

In the same way, one can show that the modified weighted Tchebycheff method satisfies the conditions of Theorem 3.6, and so, by choosing λ and α appropriately, this technique is also able to generate all properly efficient solutions.

By Theorem 3.1, one can obtain a properly efficient solution. However, scalarization methods can be applied to recognize the emptiness of the set of properly efficient solutions. Consider the following scalarization problem:

$$\begin{array}{ll} \min & g(f(x)) \\ \text{s.t.} & x \in X, \\ & f(x) \leqq \epsilon, \end{array} \tag{22}$$

in which $Y \subseteq \mathbb{R}^p$ is a given set containing $f(X)$. Furthermore, $g : Y \to \mathbb{R} \cup \{\infty\}$ is a lower semi-continuous function and $\epsilon \in \mathbb{R}^p$.

Theorem 3.7: *If problem (22) is unbounded, then multi-objective problem (1) does not have any properly efficient solution.*

Proof: By indirect proof assume that $\bar{x}$ is a properly efficient solution of (1). Due to the Henig proper efficiency, there exists a convex pointed cone $C \subseteq \mathbb{R}^p$ with $\mathbb{R}^p_+ \setminus \{0\} \subseteq \operatorname{int}(C)$ and $(f(X) - f(\bar{x})) \cap (-C) = \{0\}$. We show that the set $\{y \in f(X) : y \leqq \epsilon\}$ is bounded. If not, there exist a nonnegative sequence $\{t_\nu\}$ and a sequence $\{d_\nu\} \subseteq \mathbb{R}^p_+$ such that $\|d_\nu\| = 1$ for each ν, and $t_\nu \to +\infty$ and $\{\epsilon - t_\nu d_\nu\} \subseteq f(X)$. Without loss of generality, we may assume $d_\nu \to \bar{d} \in \mathbb{R}^p_+ \setminus$

$\{0\}$. As $\mathbb{R}^p_+ \setminus \{0\} \subseteq \text{int}(C)$ and $(1/t_\nu)(\epsilon - t_\nu d_\nu - f(\bar{x})) \to -\bar{d}$, for ν sufficiently large, we have

$$\frac{1}{t_\nu}(\epsilon - t_\nu d_\nu - f(\bar{x})) \in -C \setminus \{0\},$$

which contradicts $(f(X) - f(\bar{x})) \cap (-C) = \{0\}$. Therefore, the set $\{y \in f(X) : y \leqq \epsilon\}$ is bounded. So, minimum of the lower semi-continuous function g on $cl(\{y : y \in f(X), y \leqq \epsilon\})$ is finite. This implies the finiteness of the optimal value of problem (22) and completes the proof. ■

As an application, we apply Theorem 3.7 to Benson's problem [23] written as

$$\begin{aligned} \min \quad & \sum_{i=1}^{p} f_i(x) \\ \text{s.t.} \quad & f(x) \leqq f(\bar{x}), \\ & x \in X, \end{aligned} \tag{23}$$

where $\bar{x} \in X$. Benson [23] showed that, under convexity, if problem (23) is unbounded, then problem (1) does not have any properly efficient solution. Soleimani-damaneh and Zamani [24] established this result without convexity. In addition, Zamani [25] proved it for a general ordering cone. It is readily seen that the above-mentioned results reported in [23,24] follow from Theorem 3.7.

Another scalarization technique, to which one can apply Theorem 3.7, is Gerstewitz/Pascoletti–Serafini scalarization. It is known that a variety of important scalarization techniques can be modelled as special cases of Gerstewitz/Pascoletti–Serafini scalarization [4,26]. This method is formulated as

$$\begin{aligned} \min \quad & t \\ \text{s.t.} \quad & f(x) \leqq a + tr, \\ & x \in X, \end{aligned} \tag{24}$$

where $a \in \mathbb{R}^p$ and $r \in \mathbb{R}^p_+$. If (24) is unbounded, then

$$\begin{aligned} \min \quad & g(f(x)) \\ \text{s.t.} \quad & f(x) \leqq \epsilon, \\ & x \in X, \end{aligned} \tag{25}$$

with $g(y) = \min\{t : y \leqq a + tr\}$ is unbounded for some $\epsilon \in \mathbb{R}^p$. The considered g is lower semi-continuous. Hence, by Theorem 3.7, one can infer that if Gerstewitz/Pascoletti–Serafini scalarization with $a \in \mathbb{R}^p$ and $r \in \mathbb{R}^p_+$ is unbounded, then problem (1) does not have any properly efficient solution. This result has been derived in [25, Proposition 2.2] as well.

4. Proper efficiency and transformation

In this section, we investigate the relationship between the multi-objective problem (1) and its objective-transformed correspondence regarding the proper efficiency. Transforming the objective functions has been mainly proposed for convexifying the problem [7,8,27,28] and normalizing the objectives with different units [6].

Let $Y \subseteq \mathbb{R}^p$ with $f(X) \subseteq Y$ and a vector-valued function $\phi : Y \to \mathbb{R}^q$ be given. An objective-transformed version of (1), invoking ϕ, can be written as

$$\begin{array}{ll} \min & \phi(f(x)) \\ \text{s.t.} & x \in X. \end{array} \tag{26}$$

In the sequel, we provide some sufficient conditions under which properly efficient solution sets of problems (1) and (26) coincide. It is known when ϕ is $\mathbb{R}^p_+$-transformation on Y, then the efficient solutions of problems (1) and (26) are the same [29, p. 296]. Recall that a function $\phi : Y \to Z \subseteq \mathbb{R}^p$ is called $\mathbb{R}^p_+$-transformation on Y if it is bijective and

$$\bar{y} \leqq \hat{y} \Leftrightarrow \phi(\bar{y}) \leqq \phi(\hat{y}), \quad \forall\, \bar{y}, \hat{y} \in Y.$$

By mean value theorem [17, Theorem 8], if $\phi : Y \to \phi(Y)$ is bijective and the generalized Jacobian of ϕ and ϕ^{-1} are positive on open sets Y and $\phi(Y)$, respectively, then it is $\mathbb{R}^p_+$-transformation on Y.

By the following example, we show that $\mathbb{R}^p_+$-transformation property of ϕ on Y is not sufficient for coincidence of the properly efficient solutions of problems (1) and (26).

Example 4.1: Consider the multi-objective problem

$$\begin{array}{ll} \min & \begin{bmatrix} x^2 \\ x \end{bmatrix} \\ \text{s.t.} & x \leq 0. \end{array}$$

Let $\phi : \mathbb{R}_+ \times (-\mathbb{R}_+) \to \mathbb{R}^2$ be given by $\phi(y) = (\sqrt{y_1}, y_2)^{\mathrm{T}}$. It can be seen that ϕ is $\mathbb{R}^2_+$-transformation on $\mathbb{R}_+ \times (-\mathbb{R}_+)$. The point $\bar{x} = 0$ is not properly efficient, because there does not exist $\lambda \in \mathrm{int}(\mathbb{R}^2_+)$ such that $\bar{x} \in \mathrm{argmin}_{x \leq 0}\{\lambda_1 x^2 + \lambda_2 x\}$; see [1, Theorem 3.13]. Notice that the considered multi-objective problem is convex. Nevertheless, $\bar{x}$ is properly efficient for the problem transformed by ϕ. This follows from the fact that each efficient solution of a linear multi-objective optimization problem is properly efficient [1].

Theorem 4.1: *Assume the following conditions:*

(i) *Y is an open convex set satisfying* $\mathrm{cl}(f(X)) \subseteq Y$*;*

(ii) *Either the properly efficient set of* (1) *is non-empty or* $Y \subseteq a + \mathbb{R}^p_+$ *for some* $a \in \mathbb{R}^p$;
(iii) $\phi : Y \to \mathbb{R}^q$ *is locally Lipschitz on* Y;
(iv) $\ker(M) \cap \mathbb{R}^p_+ = \{0\}, \ \forall\, y \in Y, \forall M \in \partial\phi(y)$;
(v) $M \geqq 0, \ \forall\, y \in Y, \forall M \in \partial\phi(y)$.

Then the set of properly efficient solutions of (26) *is a subset of that of* (1).

Proof: It is proved similar to Theorem 3.4. ■

In general, Theorem 4.1 does not hold without condition (*ii*). The following example clarifies this point.

Example 4.2: Consider the multi-objective problem

$$\begin{array}{ll} \min & \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix} \\ \text{s.t.} & x \leq 1, \end{array}$$

with $f_1(x) = x$ and

$$f_2(x) = \begin{cases} -x, & -1 \leq x \leq 1 \\ 1, & x \leq -1 \end{cases}$$

Let $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ given by $\phi(y) = [\,\exp(y_1), \exp(y_2)\,]^T$. The original and the transformed problems have the same efficient solutions. Figure 1 illustrates that all efficient points of the transformed problem are properly efficient while the original problem does not have any properly efficient solution.

Note that in the same line one can establish Theorem 4.1 when $f(X)$ is Lipschitz arc-wise (arc-wise $\mathbb{R}^p_+$-convex) connected while Y is not necessarily convex.

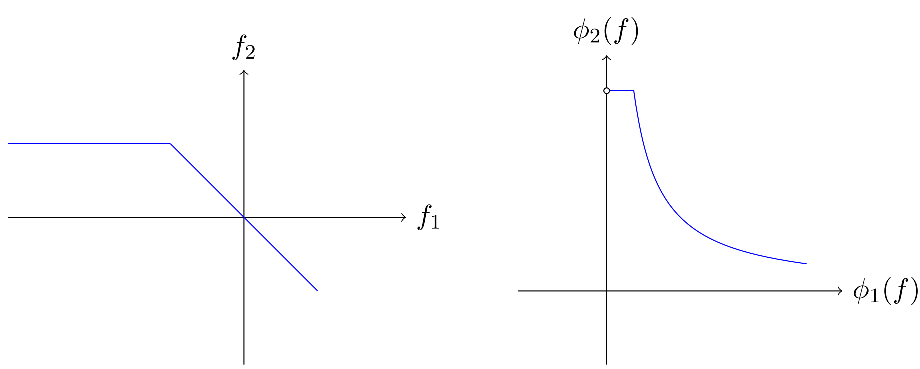

Figure 1. $f(X)$ and $\phi(f(X))$.

Definition 4.1 ([29]): A set $Y \subseteq \mathbb{R}^p$ is called

(i) Lipschitz arc-wise connected if for each $y_1, y_2 \in Y$, there exists a Lipschitz function $\gamma : [0,1] \to Y$ such that $\gamma(0) = y_1$ and $\gamma(1) = y_2$.
(ii) arc-wise $\mathbb{R}^p_+$-convex connected if for each $y_1, y_2 \in Y$, there exists a convex function $\gamma : [0,1] \to Y$ such that $\gamma(0) = y_1$ and $\gamma(1) = y_2$.

Remark 4.1: Since convex functions on compact subsets of Euclidean spaces are Lipschitz [9], arc-wise $\mathbb{R}^p_+$-convex connectivity implies Lipschitz arc-wise connectivity.

Corollary 4.1 below, addresses the result of Theorem 4.1 for differentiable case.

Corollary 4.1: *Assume that the properly efficient set of (1) is non-empty, and Y is an open convex set satisfying $\mathrm{cl}(f(X)) \subseteq Y$. Furthermore, assume that $\phi : Y \to \mathbb{R}^q$ is continuously differentiable on Y. If $\nabla\phi(y) \geqq 0$ and $\{d \in \mathbb{R}^p_+ \setminus \{0\} : \nabla\phi(y)d = 0\} = \emptyset$ for each $y \in Y$, then the set of properly efficient solutions of (26) is a subset of that of (1).*

In the next corollary, we give other sufficient conditions under which the properly efficient sets of problems (1) and (26) are the same.

Corollary 4.2: *Assume the following conditions:*

(i) *$Y_1, Y_2 \subseteq \mathbb{R}^p$ are open convex sets;*
(ii) *Either the properly efficient set of (1) is non-empty or $Y_1 \subseteq a + \mathbb{R}^p_+$ for some $a \in \mathbb{R}^p$;*
(iii) *$\phi : Y_1 \to Y_2$ and its inverse, ϕ^{-1}, are locally Lipschitz;*
(iv) *$\mathrm{cl}(f(X)) \subseteq Y_1$ and $\mathrm{cl}(\phi(f(X))) \subseteq Y_2$;*
(v) *$\ker(M) \cap \mathbb{R}^p_+ = \{0\}, \ \forall\ y \in Y_1, \forall M \in \partial\phi(y); \ M \geqq 0, \ \forall\ y \in Y, \forall\ M \in \partial\phi(y)$;*
(vi) *$\ker(M) \cap \mathbb{R}^p_+ = \{0\}, \ \forall\ y \in Y_2, \forall M \in \partial\phi^{-1}(y); \ M \geqq 0, \ \forall\ y \in Y, \forall M \in \partial\phi^{-1}(y)$.*

Then the properly efficient solutions of (1) and (26) are the same.

Proof: It follows from Theorem 4.1. ■

In [27], Zarepisheh et al. have proved, given integer $l > 0$, the set of properly efficient solutions of (1) coincides with that of the following problem

$$\begin{array}{ll} \min & \begin{bmatrix} f_1(x)^l \\ \vdots \\ f_p(x)^l \end{bmatrix} \\ \text{s.t.} & x \in X, \end{array}$$

provided that $y^I > 0$. This result follows from Corollary 4.2. It is enough to consider $Y_1 = Y_2 = \{y : y > 0\}$ and $\phi(f(x)) = (f_1(x)^l, \ldots, f_p(x)^l)^T$.

Hirschberger [29, Theorem 5.2] showed that both problems (1) and (26) share the same properly efficient solutions provided that the following conditions hold:

(a) $f(X), \phi(f(X)) \subseteq \mathbb{R}^p$ are closed and arc-wise $\mathbb{R}^p_+$-convex;
(b) Y_1 and Y_2 are open sets with $f(X) \subseteq Y_1$ and $\phi(f(X)) \subseteq Y_2$;
(c) $\phi : Y_1 \to Y_2$ is a diffeomorphism (both ϕ and ϕ^{-1} are bijective and differentiable);
(d) $\phi : Y_1 \to Y_2$ is $\mathbb{R}^p_+$-transformation;
(e) The properly efficient set of (1) is non-empty.

Since ϕ is $\mathbb{R}^p_+$-transformation, for given $\bar{y} \in Y$ and $d \in \mathbb{R}^p_+ \setminus \{0\}$,

$$\nabla\phi(\bar{y})d = \lim_{t\to 0} \frac{\phi(\bar{y} + td) - \phi(\bar{y})}{t} \geqq 0.$$

As $d \in \mathbb{R}^p_+ \setminus \{0\}$ is arbitrary, we have $\nabla\phi(\bar{y}) \geqq 0$. In addition, ϕ is a diffeomorphism. Thus, $\nabla\phi(\bar{y})$ is invertible and $\ker(\nabla\phi(\bar{y})) \cap \mathbb{R}^p_+ = \{0\}$. Similarly, under these circumstances, one can derive condition (vi) of Corollary 4.2 as well. Since the efficient set of (1) is non-empty, $\mathbb{R}^p_+$-transformation property implies that the efficient set of (26) is also non-empty. Consequently, by [29, Proposition 4.1], the properly efficient set of (26) will be non-empty. As mentioned earlier, Corollary 4.2 holds under arc-wise $\mathbb{R}^p_+$-convex connectivity as well. So, Hirschberger's result follows from Corollary 4.2 when ϕ and ϕ^{-1} are locally Lipschitz on their domain.

In another paper, Zarepisheh and Pardalos [28] investigated some special classes of transformations. They considered the transformed problem

$$\begin{aligned} \min \quad & \begin{bmatrix} g_1(f_1(x)) \\ \vdots \\ g_p(f_p(x)) \end{bmatrix} \\ \text{s.t.} \quad & x \in X, \end{aligned} \tag{27}$$

in which $g_i : [\inf_{x\in X} f_i(x), \sup_{x\in X} f_i(x)] \to \mathbb{R}$ for $i = 1, 2, \ldots, p$. This transformation is a special case of transformation ϕ investigated in Corollary 4.2. They established if $y^I \in \mathbb{R}^p$ exists and the following conditions are satisfied for each $i = 1, 2, \ldots, p$, then the properly efficient solutions of problems (1) and (27) are the same [28, Theorem 2]:

(I) g_i is continuous on $[\inf_{x\in X} f_i(x), \sup_{x\in X} f_i(x)]$;
(II) g_i is differentiable and g_i' is positive on $I_i := (\inf_{x\in X} f_i(x), \sup_{x\in X} f_i(x))$;
(III) Both g_i and g_i' are increasing on $(\inf_{x\in X} f_i(x), \sup_{x\in X} f_i(x))$.

This result is valid when the interval considered in (II) and (III) is replaced with closed interval $\bar{I}_i := [\inf_{x\in X} f_i(x), \sup_{x\in X} f_i(x)]$. Indeed, assumptions (II) and (III) should be considered on a set containing $\bar{I}_i$. Then [28, Theorem 2] is a consequence of Corollary 4.2 of the current paper. The following example shows that the properly efficient solutions of (1) and (27) may not be the same if one assumes (I) –(III) with I_i's instead of $\bar{I}_i$'s.

Example 4.3: Consider the multi-objective problem

$$\begin{array}{ll} \min & \begin{bmatrix} x \\ 1-x \end{bmatrix} \\ \text{s.t.} & 0 \le x \le 1, \end{array} \tag{28}$$

As the above problem is linear, all efficient solutions are properly efficient [1]. In addition, $\inf_{0\le x\le 1} f_1(x) = \inf_{0\le x\le 1} f_2(x) = 0$. Let $\phi : \mathbb{R}^2_+ \to \mathbb{R}^2$ be given by $\phi(y) = [\, y_1{}^2, y_2{}^4 \,]^T$. It is easily seen that the example fulfils all assumptions (I) –(III) listed above. Here, $\bar{x} = 1$ is a properly efficient solution of (28), but not for the transformed problem. This follows from the fact that the transformed problem is convex and $\bar{x} \notin \operatorname{argmin}\{\lambda_1 x^2 + \lambda_2(1-x)^4 : 0 \le x \le 1\}$ for each $\lambda \in \operatorname{int}(\mathbb{R}^2_+)$.

5. Conclusion

In this paper, we provided some theorems for analysing a unified scalarization approach as well as a general objective transformation, regarding proper efficiency. In addition to establishing fundamental important results, we showed that several well-known results existing in the literature can be obtained as a by-product of these new theorems. These results not only address a unified framework for examination of the scalarization techniques, they pave the road for introducing and analysing new scalarization methods.

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