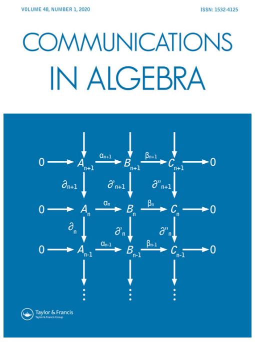

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Nilpotent Lie algebras of class 4 with the derived subalgebra of dimension 3

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ABSTRACT
The paper is devoted to give a full classification of all finite-dimensional nilpotent Lie algebras L of class 4 with $\dim L^2 = 3$. Moreover, we specify the capable ones.

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1. Introduction

It is well known that the classification of nilpotent Lie algebras is a classical problem. Several classifications of nilpotent Lie algebras of dimension at most 7 over various ground fields are available in the literature (see [3, 6, 4]). It is not easy to classify nilpotent Lie algebras with an arbitrary dimension. Hence, we are interested to classify nilpotent Lie algebras by focusing on a different aspect rather than the dimension. For a given Lie algebra L with $\dim L^2 = 1,$ the structure of L is given in [9]. When $\dim L^2 = 2,$ we gave the structure of L when L is of class 3 and with some restrictions for class 2 in [11]. The purpose of this paper is to describe a classification of all nilpotent Lie algebras of class 4 with the derived subalgebra of dimension 3. We also determine which one of these Lie algebras is capable (a Lie algebra L is called capable provided that $L \cong H/Z(H)$ for some Lie algebra H).

2. Preliminaries

This section is devoted to give some elementary and known results that we will need for the next investigations. All Lie algebras in this paper are finite dimensional over any arbitrary field.

We recall the concept of a central product of two Lie algebras A and B.

Definition 2.1. The Lie algebra L is a central product of A and B, if $L = A + B,$ where A and B are ideals of L such that $[A, B] = 0$ and $A \cap B \subseteq Z(L)$. We denote the central product of two Lie algebras A and B by $A \dot{+} B$.

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A Lie algebra L is called Heisenberg provided that $Z(L) = L^2$ and $\dim L^2 = 1$. Such Lie algebras are odd dimensional with basis $v_1, v_2, \ldots, v_m, v$ and the only nonzero multiplication between basis elements is given by $[v_{2i-1}, v_{2i}] = -[v_{2i}, v_{2i-1}] = v$ for all i, $1 \leq i \leq m$. $A(n)$ and $H(m)$ will denote the abelian Lie algebra of dimension n and the Heisenberg Lie algebra of dimension $2m + 1$, respectively. It is shown that Heisenberg Lie algebras are a central product of some of its ideals ([7, Lemma 3.3]).

For a given Lie algebra L, the upper central series

$$0 = Z_0(L) \subseteq Z_1(L) \subseteq Z_2(L) \subseteq \ldots$$

is defined by setting $Z_1(L)$ equal to its center $Z(L)$, and taking $Z_{c+1}(L)$ to be the pre-image in L of $Z(L/Z_c(L))$.

The $(c+1)$-th term of the lower central series of L is denoted by L^{c+1} where $L^1 = L, L^{c+1} = [L^c, L]$, inductively.

Let $cl(L)$ denotes the nilpotency class of a Lie algebra L. Recall that an n-dimensional nilpotent Lie algebra L is said to be nilpotent of maximal class if $cl(L) = n - 1$. For a Lie algebra L of maximal class, we have $\dim(L/L^2) = 2$, $Z_i(L) = L^{n-i}$ and $\dim(L^j/L^{j+1}) = 1$ for all i, $0 \leq i \leq n-1$ and for all j, $2 \leq j \leq n-1$ (see [2] for more information).

The Lie algebras in this article are given with multiplication tables with respect to fixed bases with trivial products of the form $[x, y] = 0$ omitted. From [4], the only Lie algebra of maximal class of dimension 4 is isomorphic to

$$L_{4,3} = \langle x_1, \ldots, x_4 | [x_1, x_2] = x_3, [x_1, x_3] = x_4 \rangle,$$

and there are exactly two non-isomorphic Lie algebras of maximal class of dimension 5 that are isomorphic to

$$L_{5,6} = \langle x_1, \ldots, x_5 | [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = x_5, [x_2, x_3] = x_5 \rangle$$

and

$$L_{5,7} = \langle x_1, \ldots, x_5 | [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = x_5 \rangle,$$

respectively. From [8], a Lie algebra S is called a stem Lie algebra if $Z(S) \subseteq S^2$. We say a Lie algebra L is a semidirect sum of an ideal I by a subalgebra K if $L = I + K$ and $I \cap K = 0$. The semidirect sum of an ideal I by a subalgebra K is denoted by $K \ltimes I$.

Lemma 2.2. [11, Lemma 4.1] *Let L be a 5-dimensional nilpotent stem Lie algebra of class 3 and $\dim L^2 = 2$. Then*

$$L \cong L_{5,5} = \langle x_1, \ldots, x_5 | [x_1, x_2] = x_3, [x_1, x_3] = x_5, [x_2, x_4] = x_5 \rangle.$$

Moreover, $L_{5,5} = I \rtimes \langle x_4 \rangle$ in which

$$I = \langle x_1, x_2, x_3, x_5 | [x_1, x_2] = x_3, [x_1, x_3] = x_5 \rangle \cong L_{4,3}, \text{ and } [I, \langle x_4 \rangle] = \langle x_5 \rangle = Z(L_{5,5}).$$

Remark 2.3. According to the classification of Lie algebras of dimension 6 in [3], 6-dimensional nilpotent stem Lie algebras of class 4 with the derived subalgebra of dimension 3 are listed as follows:

Over a field $\mathbb{F}$ of characteristic different from 2:

$$L_{6,11} = \langle x_1, \ldots, x_6 | [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = [x_2, x_3] = [x_2, x_5] = x_6 \rangle,$$
$$L_{6,12} = \langle x_1, \ldots, x_6 | [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = [x_2, x_5] = x_6 \rangle.$$
$$L_{6,13} = \langle x_1, \ldots, x_6 | [x_1, x_2] = x_3, [x_1, x_3] = [x_2, x_4] = x_5, [x_1, x_5] = [x_3, x_4] = x_6 \rangle.$$

Over a field $\mathbb{F}$ of characteristic 2:

$$L_{6,11} = \langle x_1, \ldots, x_6 | [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = [x_2, x_3] = [x_2, x_5] = x_6 \rangle,$$
$$L_{6,12} = \langle x_1, \ldots, x_6 | [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = [x_2, x_5] = x_6 \rangle.$$
$$L^{(2)}_{6,1} = \langle x_1, \ldots, x_6 | [x_1, x_2] = x_3, [x_1, x_3] = x_5,$$
$$[x_2, x_4] = x_5 + x_6, [x_1, x_5] = [x_3, x_4] = x_6 \rangle.$$

Lemma 2.4. *Let L be a 6-dimensional nilpotent stem Lie algebra of class 4 and $\dim L^2 = 3$. Then L is isomorphic to exactly one of the following Lie algebras.*

(1) $L \cong L_{6,11} = I_1 \rtimes \langle x_5 \rangle$, *in which* $I_1 = \langle x_1, x_2, x_3, x_4, x_6 | [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = [x_2, x_3] = x_6 \rangle \cong L_{5,6}$ and $[I_1, \langle x_5 \rangle] = \langle x_6 \rangle = Z(I_1) = Z(L_{6,11})$.
(2) $L \cong L_{6,12} = I_2 \rtimes \langle x_5 \rangle$, *in which* $I_2 = \langle x_1, x_2, x_3, x_4, x_6 | [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = x_6 \rangle \cong L_{5,7}$ *and* $[I_2, \langle x_5 \rangle] = \langle x_6 \rangle = Z(I_2) = Z(L_{6,12})$.
(3) $L \cong L_{6,13} = I_3 \rtimes \langle x_4 \rangle$, *in which* $I_3 = \langle x_1, x_2, x_3, x_5, x_6 | [x_1, x_2] = x_3, [x_1, x_3] = x_5, [x_1, x_5] = x_6 \rangle \cong L_{5,7}$ *and* $[I_3, \langle x_4 \rangle] = \langle x_5, x_6 \rangle = Z_2(I_3) = Z_2(L_{6,13})$.
(4) $L \cong L^{(2)}_{6,1} = I_3 \rtimes \langle x_4 \rangle$, *in which* $I_4 = \langle x_1, x_2, x_3, x_5, x_6 | [x_1, x_2] = x_3, [x_1, x_3] = x_5, [x_1, x_5] = x_6 \rangle \cong L_{5,7}$ *and* $[I_4, \langle x_4 \rangle] = \langle x_5, x_6 \rangle = Z_2(I_3) = Z_2(L^{(2)}_{6,1})$.

Proof. By using Remark 2.3, we get $L \cong L_{6,11}$, $L \cong L_{6,12}$, $L \cong L_{6,13}$, or $L \cong L^{(2)}_{6,1}$. Let $L \cong L_{6,11}$. It is easy to check that $L_{6,11} = I_1 \rtimes \langle x_5 \rangle$, in which $I_1 = \langle x_1, x_2, x_3, x_4, x_6 | [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = [x_2, x_3] = x_6 \rangle \cong L_{5,6}$ and $[I_1, \langle x_5 \rangle] = \langle x_6 \rangle = Z(I_1) = Z(L_{6,11})$. Similarly, we can see that $L_{6,12} = I_2 \rtimes \langle x_5 \rangle$, in which $I_2 = \langle x_1, x_2, x_3, x_4, x_6 | [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = x_6 \rangle \cong L_{5,7}$ and $[I_2, \langle x_5 \rangle] = \langle x_6 \rangle = Z(I_2) = Z(L_{6,12})$. Now, let $L \cong L_{6,13}$ or $L \cong L^{(2)}_{6,1}$. Again, $Z(L_{6,13}) = \langle x_6 \rangle$, $Z_2(L_{6,13}) = \langle x_5, x_6 \rangle$ and $L_{6,13} = I_3 + \langle x_4 \rangle$, where $I_3 = \langle x_1, x_2, x_3, x_5, x_6 | [x_1, x_2] = x_3, [x_1, x_3] = x_5, [x_1, x_5] = x_6 \rangle \cong L_{5,7}$ and $[I_3, \langle x_4 \rangle] = \langle x_5, x_6 \rangle = Z_2(I_3) = Z_2(L_{6,13})$. Clearly, $Z(L^{(2)}_{6,1}) = \langle x_6 \rangle$, $Z_2(L^{(2)}_{6,1}) = \langle x_5, x_6 \rangle$ and $L^{(2)}_{6,1} = I_4 + \langle x_4 \rangle$, where $I_4 = \langle x_1, x_2, x_3, x_5, x_6 | [x_1, x_2] = x_3, [x_1, x_3] = x_5, [x_1, x_5] = x_6 \rangle \cong L_{5,7}$ and $[I_4, \langle x_4 \rangle] = \langle x_5, x_6 \rangle = Z_2(I_3) = Z_2(L^{(2)}_{6,1})$. The result follows. □

For a given Lie algebra L and a subalgebra X of it, $[L, \underbrace{X, \ldots, X}_{i-times}]$ is denoted by $[L,_i X]$.

Lemma 2.5. *Let L be an n-dimensional nilpotent Lie algebra of class c. Then $L^{c-i} \not\subseteq Z_i(L)$ for all i, $0 \leq i \leq c-1$.*

Proof. By contrary, let $L^{c-i} \subseteq Z_i(L)$ for some i, $0 \leq i \leq c-1$. Then, $L^c = [L^{c-1}, L] = \underbrace{[L, \ldots, L]}_{c-times} = [L^{c-i},_i L]$. Since $L^{c-i} \subseteq Z_i(L)$ and $[Z_i(L), L] \subseteq Z_{i-1}(L)$, we have $L^c = [L^{c-i},_i L] \subseteq [Z_i(L),_i L] = [[Z_i(L), L],_{i-1} L] \subseteq [Z_{i-1}(L),_{i-1} L]$. By a similar way, we can see that

$$L^c \subseteq [Z_{i-1}(L),_{i-1} L] \subseteq \ldots \subseteq Z_0(L) = 0$$

and so $L^c = 0$. It is a contradiction, since $cl(L) = c$. □

The next two results are obtained in [1, Lemma 2.3 and Theorem 2.4] for p-groups. We need to state and prove the same results for the class of Lie algebras.

Lemma 2.6. *Let L be an n-dimensional nilpotent Lie algebra of class c such that $\dim L^2 = c-1$ and I be an ideal of dimension i, $(0 \leq i \leq c-1)$ contained in L^2. Then $I = L^{c-i+1}$.*

Proof. Clearly, $\dim L^j = c - j + 1$, where $1 \leq j \leq c$. We proceed by induction on $c - i + 1$. If $i = c - 1$, the result follows easily. Let $c - i > 1$ and M/I be an ideal of dimension 1 such that $M/I \subseteq (L/I)^2 \cap Z(L/I)$, so M is an $(i+1)$-dimensional ideal of L such that $I \subsetneqq M \subseteq L^2$. By using the induction hypothesis, $M = L^{c-i}$. Since $M/I \subseteq Z(L/I)$, we have $L^{c-i+1} = [L, L^{c-i}] = [L, M] \subseteq I$. Now, both I and L^{c-i+1} are of dimension i, and hence $I = L^{c-i+1}$. The result follows. □

Proposition 2.7. *Let L be an n-dimensional nilpotent Lie algebra of class c such that $\dim L^2 = c - 1$. Then $L^2 \cap Z_i(L) = L^{c-i+1}$ for all i, $0 \leq i \leq c - 1$.*

Proof. By contrary, let $L^{c-i+1} \subsetneqq L^2 \cap Z_i(L)$ for some i, $0 \leq i \leq c-1$. Since $\dim L^{c-i+1} = i$, we get $k = \dim(L^2 \cap Z_i(L)) \geq i + 1$ and so Lemma 2.6 implies $L^2 \cap Z_i(L) = L^{c-k+1} \subseteq L^{c-i}$. Since

$$i + 1 \leq \dim L^2 \cap Z_i(L) = \dim L^{c-k+1} = k \leq \dim L^{c-i} = i + 1,$$

we have $L^2 \cap Z_i(L) = L^{c-i}$. It is a contradiction since $L^{c-i} \not\subseteq Z_i(L)$, by Lemma 2.5. This completes the proof. □

Now, we are able to prove the following result which is useful in the rest.

Proposition 2.8. *Let L be an n-dimensional Lie algebra of class c such that $\dim L^2 = c - 1$. Then L is a stem Lie algebra if and only if $Z(L) = L^c \cong A(1)$.*

Proof. Let L be a stem Lie algebra. Then by Proposition 2.7, we have

$$L^c \subseteq Z(L) \subseteq L^2 \cap Z(L) = L^c.$$

This implies $\dim Z(L) = \dim L^c = 1$, as required. □

3. Main results

We are going to give the structure of all nilpotent Lie algebras of class 4 with the derived subalgebra of dimension 3. Moreover, we determine which one of these Lie algebras is capable.

The following proposition is a useful instrument.

Theorem 3.1. *Let L be an n-dimensional nilpotent stem Lie algebra of class 4 $(n \geq 6)$ and $\dim L^2 = 3$ such that*

(a) *$L = I + K$ for subalgebras K and I of L such that I is a maximal class subalgebra of dimension 5,*
(b) *$I \cap K \subseteq Z(I) \cap K^2$, $[I, K] \subseteq Z(I)$ and $\dim(K^2) \leq 1$.*

Then we have:

(i) *If K is a non-trivial abelian Lie algebra, then L is isomorphic to $L_{6,11}$ or $L_{6,12}$.*
(ii) *If K is a non-abelian Lie algebra, then L is isomorphic to exactly one of the following Lie algebras.*
   (1) *$L \cong \langle x_1, \ldots, x_5, a_j, b_j \mid [x_1, x_i] = x_{i+1}, [x_2, x_3] = [a_j, b_j] = x_5, 1 \leq i \leq 4, 1 \leq j \leq m\rangle$ and $n = 2m + 5$ for all $m \geq 1$.*
   (2) *$L \cong \langle x_1, \ldots, x_5, a_j, b_j \mid [x_1, x_i] = x_{i+1}, [a_j, b_j] = x_5, 1 \leq i \leq 4, 1 \leq j \leq m\rangle$ and $n = 2m + 5$ for all $m \geq 1$.*
   (3) *$L \cong \langle x_1, \ldots, x_6, a_j, b_j \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = [x_2, x_3] = [x_2, x_6] = [a_j, b_j] = x_5\rangle$ and $n = 2m + 6$ for all $m \geq 1$.*

(4) $L \cong \langle x_1, ..., x_6, a_j, b_j | [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = [a_j, b_j] = [x_2, x_6] = x_5 \rangle$ *and* $n = 2m + 6$ *for all* $m \geq 1$.

Proof.

(i) We have $[I, K] \subseteq Z(I)$, so I is an ideal of L and hence $I \cong L_{5,6}$ or $I \cong L_{5,7}$. Since I is a Lie algebra of maximal class of dimension 5 so $Z(I) = I^4 \cong A(1)$, we also have $\dim L^2 = \dim I^2 = 3$. Therefore, $L^2 = I^2$. Proposition 2.8 implies $Z(L) = L^3 \cong A(1)$ and hence

$$L^3 = Z(L) = Z(I) = I^3 \cong A(1).$$

Since $K \cap I = 0$, we obtain $\dim K = \dim L - \dim I = n - 5$ and so $L = I \rtimes K$, in which $K \cong A(n-5)$. We claim that $[I, K] = Z(I)$. By contrary, assume that $[I, K] = 0$. Since K is abelian, we have $K \subseteq Z(L) = Z(I) \subseteq I$, which shows $I \cap K \neq 0$, so we have a contradiction. Thus, $[I, K] = Z(L) = Z(I)$.

We claim that $K \cong A(1)$.

First assume that $n = 6$. Lemma 2.4 implies $L \cong I_1 \rtimes K \cong L_{6,11}$ or $L \cong I_2 \rtimes K \cong L_{6,12}$, in which $I_1 \cong L_{5,6}$, $I_2 \cong L_{5,7}$, $[I_1, K] = Z(L)$ and $[I_2, K] = Z(L)$.

Now, let $n \geq 7$ and $K = \oplus_{i=1}^{n-5} \langle z_i \rangle$. In this case, we show that $K \cap I \neq 0$, which is impossible, so this case cannot occur. First let $I \cong L_{5,6}$. By using the Jacobi identity, for all i, $1 \leq i \leq n-5$, we have

$$[z_i, x_3] = [z_i, [x_1, x_2]] = [z_i, x_1, x_2] + [x_2, z_i, x_1] = 0$$

and

$$[z_i, x_4] = [z_i, [x_1, x_3]] = [z_i, x_1, x_3] + [x_3, z_i, x_1] = 0$$

since $[z_i, x_1], [z_i, x_3]$, and $[x_2, z_i]$ are central. Thus, $[z_i, x_3] = [z_i, x_4] = 0$ for all i, $1 \leq i \leq n-5$. Now, let $[z_i, x_1] = \alpha_i x_5$ and $\alpha_i \neq 0$ for all i, $1 \leq i \leq n-5$. Putting $z_i' = z_i + \alpha_i x_4$, we have $[z_i', x_1] = [z_i + \alpha_i x_4, x_1] = [z_i, x_1] + \alpha_i [x_4, x_1] = \alpha_i x_5 - \alpha_i x_5 = 0$. Also, we obtain $[z_i', x_4] = 0$. Thus, $[z_i', x_1] = [z_i', x_3] = [z_i', x_4] = 0$ for all i, $1 \leq i \leq n-5$. Now let $[z_i', x_2] = \alpha x_5$ and $[z_j', x_2] = \beta x_5$, in which $\alpha \neq 0$ and $\beta \neq 0$ for $i \neq j$ for some fixed i, j. Put $d_{ij} = \beta z_i' - \alpha z_j'$. We have $d_{ij} \neq 0$ and $[d_{ij}, x_2] = [\beta z_i' - \alpha z_j', x_2] = \beta \alpha x_5 - \beta \alpha x_5 = 0$ and so $[d_{ij}, x_2] = 0$. On the other hand, $[d_{ij}, x_1] = [d_{ij}, x_2] = [d_{ij}, x_3] = [d_{ij}, x_4] = 0$. Therefore, $[d_{ij}, I] = 0$ and hence $0 \neq d_{ij} \in Z(L) = Z(I) = \langle x_5 \rangle$. Since

$$\begin{aligned} d_{ij} &= \beta z_i' - \alpha z_j' = \beta(z_i - \alpha_i x_4) - \alpha(z_j - \alpha_j x_4) \\ &= \beta z_i - \alpha z_j + (\alpha \alpha_j - \beta \alpha_i) x_4 \in Z(I), \end{aligned}$$

$0 \neq \beta z_i - \alpha z_j \in K \cap I = 0$, which is a contradiction. Thus, $n = 6$, $K \cong A(1)$, $L = I \rtimes \langle z_1 \rangle$, and $[x_2, z_1] = x_5$, as required. Now, considering the classification of nilpotent Lie algebras L of dimension 6 with $\dim L^2 = 3$ which is given in [4], using Lemma 2.4 and our assumption, we should have $L \cong L_{6,11}$. If $I \cong L_{5,7}$, then by a similar way, one can see that $L \cong L_{6,12}$.

(ii) By a similar method used in the part (i), we have $L^2 = I^2$ and $Z(L) = Z(I) = L^4 \cong A(1)$. If $K^2 \cap Z(I) = 0$, then $K^2 \subseteq L^2 \cap K \subseteq I \cap K = 0$ and so K is abelian, which is a contradiction. So, $K^2 = Z(I)$. Hence, $I \cap K = K^2 = Z(I) = Z(L) \cong A(1)$, and so $\dim(K) = \dim(L) - \dim(I) + \dim(I \cap K) = n - 5 + 1 = n - 4$. We know that $\dim K^2 = 1$, so [12, Theorem 3.6] implies $K \cong K_1 \oplus A$, in which $K_1 \cong H(m)$ and $A \cong A(n - 2m - 5)$. Now, let $A = 0$. Since $2m + 1 = \dim(K_1) = n - 4$, we have $n = 2m + 5$. We are going to show that $[I, K_1] = 0$. In fact, we show that there exists $I_1 \cong L_{5,6}$ or $I_1 \cong L_{5,7}$ and $K_2 \cong H(m)$ with $[I_1, K_2] = 0$ and $L = I_1 \dotplus K_2$. First let $m = 1$. We have $\dim L = 6$ and $K_1 = \langle x, y, x_5 | [x, y] = x_5 \rangle$, since $K_1 \cong H(1)$. By using the Jacobi identity, we have

$$[x,x_3]=[x,[x_1,x_2]]=[x,x_1,x_2]+[x_2,x,x_1]=0,$$
$$[x,x_4]=[x,[x_1,x_3]]=[x,x_1,x_3]+[x_3,x,x_1]=0,$$
$$[y,x_3]=[y,[x_1,x_2]]=[y,x_1,x_2]+[x_2,y,x_1]=0$$

and

$$[y,x_4]=[y,[x_1,x_3]]=[y,x_1,x_3]+[x_3,y,x_1]=0.$$

Thus, $[x,x_3]=[x,x_4]=[y,x_3]=[y,x_4]=0$. Now, let $[y,x_2]=\eta' x_5$ for some scalar η'. By taking $x_2'=x_2-\eta' x_4$, $y'=y+x_1$, $x_3'=x_3-\eta' x_5$, we have $[x_1,x_2]=x_3'$ and $[y,x_2]=0$. Similarly, $[x,x_2]=0$. Now, let $[x,x_1]=\alpha x_5$ and $[y,x_1]=\beta x_5$ for some scalars α and β. Then by taking $x'=x+\alpha x_4$ and $y'=y+\beta x_4$, we have $[x',x_1]=[y',x_1]=0$. Thus, $L=I\dotplus K$, where $K\cong H(1)$ and I is a maximal class Lie algebra of dimension 5. Thus, $L\cong\langle x_1,\ldots,x_5,a,b|[x_1,x_i]=x_{i+1},[x_2,x_3]=[a,b]=x_5,1\le i\le 4\rangle$ or $L\cong\langle x_1,\ldots,x_5,a,b|[x_1,x_i]=x_{i+1},[a,b]=x_5,1\le i\le 4\rangle$. Now, let $m\geq 2$ and $H(m)=\langle a_1,b_1,\ldots,a_m,b_m,z|[a_l,b_l]=z,1\le l\le m\rangle$. [7, Lemma 3.3] implies that $H(m)=T_1\dotplus\ldots\dotplus T_m$, in which $T_i\cong H(1)$ for all $i,1\le i\le m$. With the same procedure used in case (i) and changing the variables we can see that $[T_i,I]=0$ for all i, $1\le i\le m$. So $[I,K_1]=0$ and hence $L=I\dotplus K$. Since $m\geq 2$, we have $L=(I\dotplus T_1)\dotplus(T_2\dotplus\ldots\dotplus T_m)$ such that $I\dotplus T_1\cong\langle x_1,\ldots,x_5,a,b|[x_1,x_i]=x_{i+1},[x_2,x_3]=[a,b]=x_5,1\le i\le 4\rangle$, $K_1\cong H(m-1)$ or $I\dotplus T_1\cong\langle x_1,\ldots,x_5,a,b|[x_1,x_i]=x_{i+1},[a,b]=x_5,1\le i\le 4\rangle$ and $T_2\dotplus\ldots\dotplus T_m\cong H(m-1)$. Thus, $L\cong\langle x_1,\ldots,x_5,a_j,b_j|[x_1,x_i]=x_{i+1},[x_2,x_3]=[a_j,b_j]=x_5,1\le i\le 4,1\le j\le m\rangle$ or $L\cong\langle x_1,\ldots,x_5,a_j,b_j|[x_1,x_i]=x_{i+1},[a_j,b_j]=x_5,1\le i\le 4,1\le j\le m\rangle$ and $n=2m+5$ for all $m\geq 1$. It completes cases (1) and (2) of (ii).

Let $A\neq 0$. Then, $n\neq 2m-5$. Thus, $L=I+(K_1\oplus A)$ such that $[I,K]\subseteq Z(L)=Z(I)$. We are going to show that $A\cong A(1)$, $[I,K_1]=0$ and $[I,A]=Z(I)=Z(L)$. Similar to the cases (1) and (2) of (ii), we can see that $[I,K_1]=0$. We claim that $[I,A]\neq 0$. By contrary, let $[A,K_1]=[I,A]=0$, so $A\subseteq Z(L)=Z(I)$. Since $A\cap I=0$, we have $A=0$, which is a contradiction. So we have $[I,A]=Z(L)$. We claim that $\dim A=1$. Let $\dim A\geq 2$. Similar to the proof of part (i), we have $[z_1,x_1]=[z_2,x_1]=[z_1,x_3]=[z_2,x_3]=[z_1,x_4]=[z_2,x_4]=0$ where $z_1,z_2\in A$ and $z_1\neq z_2$. Now, let $[z_1,x_2]=\alpha x_5$ and $[z_2,x_2]=\beta x_5$ for some nonzero scalars α and β. Putting $z_1'=\beta z_1-\alpha z_2$, we have $[z_1',x_2]=[\beta z_1-\alpha z_2,x_2]=\beta\alpha x_5-\beta\alpha x_5=0$ and so $[z_1',x_2]=0$. Hence, $[z_1',x_1]=[z_1',x_2]=[z_1',x_3]=[z_1',x_4]=0$. Therefore, $[z_1',I]=0$ and hence, $z_1'\in Z(L)=Z(I)=\langle x_5\rangle=K_1^2$. So $z_1'\in K_1$ which is a contradiction since $K_1\cap A=0$. Hence, $A\cong A(1)$ and consequently $n=2m+6$. Thus, $L=(I\rtimes A)\dotplus K_1$ such that $[I,A]=Z(L)=Z(I)$. By using part (i), we have $I\rtimes A\cong L_{6,11}$ or $I\rtimes A\cong L_{6,12}$. The proof of cases $(ii)(3)$ and $(ii)(4)$ is completed. □

Proposition 3.2. *Let L be an n-dimensional nilpotent stem Lie algebra of class 4 $(n\geq 7)$ over a field $\mathbb{F}$ of characteristic different from 2 and $\dim L^2=3$ such that*

(a) *$L=I+K$, for a subalgebra K of L and a subalgebra $I\cong L_{6,13}$ of L,*
(b) *$I\cap K\subseteq Z(I)\cap K^2$, $[I,K]\subseteq Z(I)$ and $\dim(K^2)\leq 1$.*

Then the following results hold.

(i) *If K is a non-trivial abelian Lie algebra, then L is isomorphic to exactly one of the following Lie algebras.*
   (1) $L\cong\langle x_1,\ldots,x_7|[x_1,x_2]=x_3,[x_1,x_3]=[x_2,x_4]=x_5,[x_1,x_5]=[x_3,x_4]=[x_2,x_7]=[x_4,x_7]=x_6\rangle$.
   (2) $L\cong\langle x_1,\ldots,x_7|[x_1,x_2]=x_3,[x_1,x_3]=[x_2,x_4]=x_5,[x_1,x_5]=[x_3,x_4]=[x_4,x_7]=x_6\rangle$.
   (3) $L\cong\langle x_1,\ldots,x_7|[x_1,x_2]=x_3,[x_1,x_3]=[x_2,x_4]=x_5,[x_1,x_5]=[x_3,x_4]=[x_2,x_7]=x_6\rangle$.

(4) $L \cong \langle x_1, \ldots, x_8 | [x_1, x_2] = x_3, [x_1, x_3] = [x_2, x_4] = x_5, [x_1, x_5] = [x_3, x_4] = [x_2, x_7] = [x_4, x_8] = x_6 \rangle$.

(ii) *If K is a non-abelian Lie algebra, then L is isomorphic to exactly one of the following Lie algebras.*

(1) $L \cong \langle x_1, \ldots, x_6, a_i, b_i | [x_1, x_2] = x_3, [x_1, x_3] = [x_2, x_4] = x_5, [x_1, x_5] = [x_3, x_4] = [a_i, b_i] = x_6,\ 1 \leq i \leq m \rangle$ *and* $n = 2m + 6$ *for all* $m \geq 1$.

(2) $L \cong \langle x_1, \ldots, x_6, x_7, a_i, b_i | [x_1, x_2] = x_3, [x_1, x_3] = [x_2, x_4] = x_5, [x_1, x_5] = [x_3, x_4] = [x_2, x_7] = [x_4, x_7] = [a_i, b_i] = x_6, 1 \leq i \leq m \rangle$ *and* $n = 2m + 7$ *for all* $m \geq 1$.

(3) $L \cong \langle x_1, \ldots, x_6, x_7, a_i, b_i | [x_1, x_2] = x_3, [x_1, x_3] = [x_2, x_4] = x_5, [x_1, x_5] = [x_3, x_4] = [x_2, x_7] = [a_i, b_i] = x_6, 1 \leq i \leq m \rangle$ *and* $n = 2m + 7$ *for all* $m \geq 1$.

(4) $L \cong \langle x_1, \ldots, x_6, x_7, a_i, b_i | [x_1, x_2] = x_3, [x_1, x_3] = [x_2, x_4] = x_5, [x_1, x_5] = [x_3, x_4] = [x_4, x_7] = [a_i, b_i] = x_6, 1 \leq i \leq m \rangle$ *and* $n = 2m + 7$ *for all* $m \geq 1$.

(5) $L \cong \langle x_1, \ldots, x_6, x_7, x_8, a_i, b_i | [x_1, x_2] = x_3, [x_1, x_3] = [x_2, x_4] = x_5, [x_1, x_5] = [x_3, x_4] = [x_2, x_7] = [x_4, x_8] = [a_i, b_i] = x_6, 1 \leq i \leq m \rangle$ *and* $n = 2m + 8$ *for all* $m \geq 1$.

Proof. It is similar to the proof of Theorem 3.1. □

Proposition 3.3. *Let L be an n-dimensional class 4 nilpotent stem Lie algebra over a field* $\mathbb{F}$ *of characteristic 2* $(n \geq 7)$ *and* $\dim L^2 = 3$ *such that*

(a) $L = I + K$, *for a subalgebra K of L and a subalgebra* $I \cong L_{6,1}^{(2)}$,
(b) $I \cap K \subseteq Z(I) \cap K^2$, $[I, K] \subseteq Z(I)$ *and* $\dim(K^2) \leq 1$,.

Then the following results hold.

(i) *If K is a non-trivial abelian Lie algebra, then L is isomorphic to exactly one of the following Lie algebras.*

(1) $L \cong \langle x_1, \ldots, x_7 | [x_1, x_2] = x_3, [x_1, x_3] = x_5, [x_2, x_4] = x_5 + x_6, [x_1, x_5] = [x_3, x_4] = [x_2, x_7] = [x_4, x_7] = x_6 \rangle$.

(2) $L \cong \langle x_1, \ldots, x_7 | [x_1, x_2] = x_3, [x_1, x_3] = x_5, [x_2, x_4] = x_5 + x_6, [x_1, x_5] = [x_3, x_4] = [x_2, x_7] = x_6 \rangle$.

(3) $L \cong \langle x_1, \ldots, x_7 | [x_1, x_2] = x_3, [x_1, x_3] = x_5, [x_2, x_4] = x_5 + x_6, [x_1, x_5] = [x_3, x_4] = [x_4, x_7] = x_6. \rangle$

(4) $L \cong \langle x_1, \ldots, x_8 | [x_1, x_2] = x_3, [x_1, x_3] = x_5, [x_2, x_4] = x_5 + x_6, [x_1, x_5] = [x_3, x_4] = [x_2, x_7] = [x_4, x_8] = x_6 \rangle$.

(ii) *If K is a non-abelian Lie algebra, then L is isomorphic to exactly one of the following Lie algebras.*

(1) $L \cong \langle x_1, \ldots, x_6, a_i, b_i | [x_1, x_2] = x_3, [x_1, x_3] = x_5, [x_2, x_4] = x_5 + x_6, [x_1, x_5] = [x_3, x_4] = [a_i, b_i] = x_6,\ 1 \leq i \leq m \rangle$ *and* $n = 2m + 6$ *for all* $m \geq 1$.

(2) $L \cong \langle x_1, \ldots, x_6, x_7, a_i, b_i | [x_1, x_2] = x_3, [x_1, x_3] = x_5, [x_2, x_4] = x_5 + x_6, [x_1, x_5] = [x_3, x_4] = [x_2, x_7] = [x_4, x_7] = [a_i, b_i] = x_6, 1 \leq i \leq m \rangle$ *and* $n = 2m + 7$ *for all* $m \geq 1$.

(3) $L \cong \langle x_1, \ldots, x_7, a_i, b_i | [x_1, x_2] = x_3, [x_1, x_3] = x_5, [x_2, x_4] = x_5 + x_6, [x_1, x_5] = [x_3, x_4] = [x_2, x_7] = [a_i, b_i] = x_6, 1 \leq i \leq m \rangle$ *and* $n = 2m + 7$ *for all* $m \geq 1$.

(4) $L \cong \langle x_1, \ldots, x_7, a_i, b_i | [x_1, x_2] = x_3, [x_1, x_3] = x_5, [x_2, x_4] = x_5 + x_6, [x_1, x_5] = [x_3, x_4] = [x_4, x_7] = [a_i, b_i] = x_6, 1 \leq i \leq m \rangle$ *and* $n = 2m + 7$ *for all* $m \geq 1$.

(5) $L \cong \langle x_1, \ldots, x_8, a_i, b_i | [x_1, x_2] = x_3, [x_1, x_3] = x_5, [x_2, x_4] = x_5 + x_6, [x_1, x_5] = [x_3, x_4] = [x_2, x_7] = [x_4, x_8] = [a_i, b_i] = x_6, 1 \leq i \leq m \rangle$ *and* $n = 2m + 8$ *for all* $m \geq 1$.

Proof. The result is obtained by a same way used in the proof of Theorem 3.1. □

We are ready to determine the central factors of all stem Lie algebras T such that $cl(T) = 4$ and $\dim T^2 = 3$.

Lemma 3.4. *Let T be an n-dimensional stem Lie algebra such that* $cl(T) = 4$ *and* $\dim T^2 = 3$. *Then* $T/Z(T) \cong L_{4,3} \oplus A(n-4)$ *or* $T/Z(T) \cong L_{5,5} \oplus A(n-5)$.

Proof. By using Proposition 2.8, we have $T^2/Z(T) \cong A(2)$. Since $T/Z(T)$ is capable, [11, Theorem 5.3] implies that $T/Z(T) \cong L_{4,3} \oplus A(n-4)$ or $T/Z(T) \cong L_{5,5} \oplus A(n-5)$. It completes the proof. □

In the following theorem, we determine the structure of all stem Lie algebras of class 4 with the derived subalgebra of dimension 3.

Theorem 3.5. *Let T be an n-dimensional stem Lie algebra such that $cl(T) = 4$ and $\dim T^2 = 3$. Then T is isomorphic to exactly one of the following Lie algebras.*

Over a field $\mathbb{F}$ of characteristic different from 2:

(1) $T_1 \cong L_{5,6}$.
(2) $T_2 \cong L_{5,7}$.
(3) $T_3 \cong L_{6,11}$.
(4) $T_4 \cong L_{6,12}$.
(5) $T_5 \cong \langle x_1, \ldots, x_5, a_j, b_j \mid [x_1, x_i] = x_{i+1}, [x_2, x_3] = [a_j, b_j] = x_5, 1 \leq i \leq 4, 1 \leq j \leq m\rangle$ *and* $n = 2m+5$ *for all* $m \geq 1$.
(6) $T_6 \cong \langle x_1, \ldots, x_5, a_j, b_j \mid [x_1, x_i] = x_{i+1}, [a_j, b_j] = x_5, 1 \leq i \leq 4, 1 \leq j \leq m\rangle$ *and* $n = 2m+5$ *for all* $m \geq 1$.
(7) $T_7 \cong \langle x_1, \ldots, x_6, a_j, b_j \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = [x_2, x_3] = [x_2, x_6] = [a_j, b_j] = x_5\rangle$ *and* $n = 2m+6$ *for all* $m \geq 1$.
(8) $T_8 \cong \langle x_1, \ldots, x_6, a_j, b_j \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = [a_j, b_j] = [x_2, x_6] = x_5\rangle$ *and* $n = 2m+6$ *for all* $m \geq 1$.
(9) $T_9 \cong L_{6,13}$.
(10) $T_{10} \cong \langle x_1, \ldots, x_7 \mid [x_1, x_2] = x_3, [x_1, x_3] = [x_2, x_4] = x_5, [x_1, x_5] = [x_3, x_4] = [x_2, x_7] = [x_4, x_7] = x_6\rangle$.
(11) $T_{11} \cong \langle x_1, \ldots, x_7 \mid [x_1, x_2] = x_3, [x_1, x_3] = [x_2, x_4] = x_5, [x_1, x_5] = [x_3, x_4] = [x_4, x_7] = x_6\rangle$.
(12) $T_{12} \cong \langle x_1, \ldots, x_7 \mid [x_1, x_2] = x_3, [x_1, x_3] = [x_2, x_4] = x_5, [x_1, x_5] = [x_3, x_4] = [x_2, x_7] = x_6\rangle$.
(13) $T_{13} \cong \langle x_1, \ldots, x_8 \mid [x_1, x_2] = x_3, [x_1, x_3] = [x_2, x_4] = x_5, [x_1, x_5] = [x_3, x_4] = [x_2, x_7] = [x_4, x_8] = x_6\rangle$.
(14) $T_{14} \cong \langle x_1, \ldots, x_6, a_i, b_i \mid [x_1, x_2] = x_3, [x_1, x_3] = [x_2, x_4] = x_5, [x_1, x_5] = [x_3, x_4] = [a_i, b_i] = x_6, \ 1 \leq i \leq m\rangle$ *and* $n = 2m+6$ *for all* $m \geq 1$.
(15) $T_{15} \cong \langle x_1, \ldots, x_7, a_i, b_i \mid [x_1, x_2] = x_3, [x_1, x_3] = [x_2, x_4] = x_5, [x_1, x_5] = [x_3, x_4] = [x_2, x_7] = [x_4, x_7] = [a_i, b_i] = x_6, 1 \leq i \leq m\rangle$ *and* $n = 2m+7$ *for all* $m \geq 1$.
(16) $T_{16} \cong \langle x_1, \ldots, x_7, a_i, b_i \mid [x_1, x_2] = x_3, [x_1, x_3] = [x_2, x_4] = x_5, [x_1, x_5] = [x_3, x_4] = [x_2, x_7] = [a_i, b_i] = x_6, 1 \leq i \leq m\rangle$ *and* $n = 2m+7$ *for all* $m \geq 1$.
(17) $T_{17} \cong \langle x_1, \ldots, x_7, a_i, b_i \mid [x_1, x_2] = x_3, [x_1, x_3] = [x_2, x_4] = x_5, [x_1, x_5] = [x_3, x_4] = [x_4, x_7] = [a_i, b_i] = x_6, 1 \leq i \leq m\rangle$ *and* $n = 2m+7$ *for all* $m \geq 1$.
(18) $T_{18} \cong \langle x_1, \ldots, x_8, a_i, b_i \mid [x_1, x_2] = x_3, [x_1, x_3] = [x_2, x_4] = x_5, [x_1, x_5] = [x_3, x_4] = [x_2, x_7] = [x_4, x_8] = [a_i, b_i] = x_6, 1 \leq i \leq m\rangle$ *and* $n = 2m+8$ *for all* $m \geq 1$.

Over a field $\mathbb{F}$ of characteristic 2:

(1) $T_1 \cong L_{5,6}$.
(2) $T_2 \cong L_{5,7}$.
(3) $T_3 \cong L_{6,11}$.
(4) $T_4 \cong L_{6,12}$.
(5) $T_5 \cong \langle x_1, \ldots, x_5, a_j, b_j \mid [x_1, x_i] = x_{i+1}, [x_2, x_3] = [a_j, b_j] = x_5, 1 \leq i \leq 4, 1 \leq j \leq m\rangle$ *and* $n = 2m+5$ *for all* $m \geq 1$.
(6) $T_6 \cong \langle x_1, \ldots, x_5, a_j, b_j \mid [x_1, x_i] = x_{i+1}, [a_j, b_j] = x_5, 1 \leq i \leq 4, 1 \leq j \leq m\rangle$ *and* $n = 2m+5$ *for all* $m \geq 1$.
(7) $T_7 \cong \langle x_1, \ldots, x_6, a_j, b_j \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = [x_2, x_3] = [x_2, x_6] = [a_j, b_j] = x_5\rangle$ *and* $n = 2m+6$ *for all* $m \geq 1$.

(8) $T_8 \cong \langle x_1, ..., x_6, a_j, b_j | [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = [a_j, b_j] = [x_2, x_6] = x_5 \rangle$ *and* $n = 2m + 6$ *for all* $m \geq 1$.

(9) $T_{19} \cong L_{6,1}^{(2)}$.

(10) $T_{20} \cong \langle x_1, ..., x_7 | [x_1, x_2] = x_3, [x_1, x_3] = x_5, [x_2, x_4] = x_5 + x_6, [x_1, x_5] = [x_3, x_4] = [x_2, x_7] = [x_4, x_7] = x_6 \rangle$.

(11) $T_{21} \cong \langle x_1, ..., x_7 | [x_1, x_2] = x_3, [x_1, x_3] = x_5, [x_2, x_4] = x_5 + x_6, [x_1, x_5] = [x_3, x_4] = [x_2, x_7] = x_6 \rangle$.

(12) $T_{22} \cong \langle x_1, ..., x_7 | [x_1, x_2] = x_3, [x_1, x_3] = x_5, [x_2, x_4] = x_5 + x_6, [x_1, x_5] = [x_3, x_4] = [x_4, x_7] = x_6. \rangle$

(13) $T_{23} \cong \langle x_1, ..., x_8 | [x_1, x_2] = x_3, [x_1, x_3] = x_5, [x_2, x_4] = x_5 + x_6, [x_1, x_5] = [x_3, x_4] = [x_2, x_7] = [x_4, x_8] = x_6 \rangle$.

(14) $T_{24} \cong \langle x_1, ..., x_6, a_i, b_i | [x_1, x_2] = x_3, [x_1, x_3] = x_5, [x_2, x_4] = x_5 + x_6, [x_1, x_5] = [x_3, x_4] = [a_i, b_i] = x_6, 1 \leq i \leq m \rangle$ *and* $n = 2m + 6$ *for all* $m \geq 1$.

(15) $T_{25} \cong \langle x_1, ..., x_7, a_i, b_i | [x_1, x_2] = x_3, [x_1, x_3] = x_5, [x_2, x_4] = x_5 + x_6, [x_1, x_5] = [x_3, x_4] = [x_2, x_7] = [x_4, x_7] = [a_i, b_i] = x_6, 1 \leq i \leq m \rangle$ *and* $n = 2m + 7$ *for all* $m \geq 1$.

(16) $T_{26} \cong \langle x_1, ..., x_7, a_i, b_i | [x_1, x_2] = x_3, [x_1, x_3] = x_5, [x_2, x_4] = x_5 + x_6, [x_1, x_5] = [x_3, x_4] = [x_2, x_7] = [a_i, b_i] = x_6, 1 \leq i \leq m \rangle$ *and* $n = 2m + 7$ *for all* $m \geq 1$.

(17) $T_{27} \cong \langle x_1, ..., x_7, a_i, b_i | [x_1, x_2] = x_3, [x_1, x_3] = x_5, [x_2, x_4] = x_5 + x_6, [x_1, x_5] = [x_3, x_4] = [x_4, x_7] = [a_i, b_i] = x_6, 1 \leq i \leq m \rangle$ *and* $n = 2m + 7$ *for all* $m \geq 1$.

(18) $T_{28} \cong \langle x_1, ..., x_8, a_i, b_i | [x_1, x_2] = x_3, [x_1, x_3] = x_5, [x_2, x_4] = x_5 + x_6, [x_1, x_5] = [x_3, x_4] = [x_2, x_7] = [x_4, x_8] = [a_i, b_i] = x_6, 1 \leq i \leq m \rangle$ *and* $n = 2m + 8$ *for all* $m \geq 1$.

Proof. Since $cl(T) = 4$, we have $\dim T \geq 5$. If $\dim T = 5$, then T is a 5-dimensional Lie algebra of maximal class and hence by looking at the classification of nilpotent Lie algebras of dimension 5 given in [4], we should have $T \cong L_{5,6}$ or $T \cong L_{5,7}$. Cases (1) and (2) are obtained.

Assume that $\dim T \geq 6$. Proposition 2.8 and Lemma 3.4 imply $T/Z(T) \cong L_{4,3} \oplus A(n-4)$ or $T/Z(T) \cong L_{5,5} \oplus A(n-5)$, where $Z(T) = T^4 \cong A(1)$. First let $T/Z(T) \cong L_{4,3} \oplus A(n-4)$. There exist two ideals $I_1/Z(T)$ and $I_2/Z(T)$ of $T/Z(T)$ such that $I_1/Z(T) \cong L_{4,3}$ and $I_2/Z(T) \cong A(n-5)$. Since $T^2/Z(T) = (I_1^2 + Z(T))/Z(T)$, we have $T^2 = I_1^2 + Z(T)$ and $Z(T) = T^4$. Using [14, Lemma 1], we have $T^2 = I_1^2$ and so $cl(T) = cl(I_1) = 4$. Hence, I_1 is a 5-dimensional Lie algebra of maximal class and hence $I_1 \cong L_{5,6}$ or $I_1 \cong L_{5,7}$. Now, $Z(T) = Z(I_1) \cong A(1)$ because $Z(T) \cap I_1 \subseteq Z(I_1) \cong A(1)$ and $\dim Z(T) = 1$. Since $Z(T) \subseteq I_1 \cap I_2 \subseteq Z(T)$, we have $I_1 \cap I_2 = Z(T) = Z(I_1)$. Therefore

$$Z_2(T)/Z(T) = Z(T/Z(T)) = Z(I_1/Z(T)) \oplus I_2/Z(T) = Z_2(I_1)/Z(T) + I_2/Z(T).$$

Since I_1 is maximal class of dimension 5, we have $Z_2(T) = Z_2(I_1) + I_2 = I_1^3 + I_2$. Now, we are going to determine the structure of I_2. Since $I_2/Z(T) \cong A(n-5)$, we have $I_2^2 \subseteq Z(T) \cong A(1)$ and hence $cl(I_2) \leq 2$. Since $\dim T/Z(T) \geq 5$, we have $\dim I_2 \geq 2$. Let $cl(I_2) = 1$. Then, $[I_1, I_2] = I_1 \cap I_2 = Z(T) = Z(I_1) \cong A(1)$, because if $[I_1, I_2] = 0$ since I_2 is abelian, $I_2 \subseteq Z(T) \cong A(1)$, which is a contradiction, since $\dim I_2 \geq 2$. Hence, $I_2 = Z(T) \oplus A$ in which $A \cong A(n-5)$ and $[I_1, I_2] = Z(T)$. Now, $Z(T) \subseteq I_1, A \cap I_1 = 0$ and $I_1 \cap I_2 = Z(T)$ so $T = I_1 + I_2 = I_1 + Z(T) + A = I_1 \rtimes A$. Using the proof of Theorem 3.1 (*i*), we have $T \cong I_1 \rtimes K \cong L_{6,11}$, in which $K \cong A(1)$ and $[K, I_1] = Z(T)$ or $T \cong I_1 \rtimes K \cong L_{6,12}$ in which $K \cong A(1)$ and $[K, I_1] = Z(T)$. These are cases (3) and (4).

Now, let $cl(I_2) = 2$. Since $I_2^2 = I_1 \cap I_2 = Z(T) = Z(I_1) \cong A(1)$, [12, Theorem 3.6] implies $I_2 \cong H(m) \oplus A(n-2m-5)$. First, assume that $A(n-2m-5) = 0$. Then $n = 2m+5$ and $I_2 \cong H(m)$. Using Theorem 3.1 (*ii*) (1) and (2), we can similarly obtain that $[I_1, I_2] = 0$ and $T = I_1 \dotplus I_2$, where $I_2 \cong H(m)$. Cases (5) and (6) follow.

Now, let $A(n-2m-5) \neq 0$. Then, $n \neq 2m+5$ and hence, $T = I_1 + (K \oplus A)$ where $K \cong H(m)$, $A \cong A(n-2m-5)$ and $[I_1, K \oplus A] \subseteq Z(T) = Z(I_1)$. By using the proof of Theorem 3.1 (*ii*), we have $[I_1, K] = 0$. Now, we claim that $[I_1, A] = Z(T) \cong A(1)$. Let $[I_1, A] = 0$. Since $[K, A] = 0$, we have $A \subseteq Z(T) = Z(I_1) = Z(K) = K^2 \cong A(1)$. It is a contradiction, since $A \cap K = 0$.

Therefore, $[I_1, A] = Z(T)$ and hence $T \cong (I_1 \rtimes A) \dotplus K$ where $A \cong A(n-2m-5)$ and $[I_1, A] = Z(T) = Z(I_1)$. Similar to cases (3) and (4), one can obtain that $A \cong A(1)$, so $n = 2m+6$ and $[I_1, A] = Z(T)$. Therefore, $T = (I_1 \rtimes A) \dotplus K$ in which $A \cong A(1)$ and $[I_1, A] = Z(T)$. These are cases (7) and (8), by Theorem 3.1 (*ii*) (3) and (4).

Now, let $T/Z(T) \cong L_{5,5} \oplus A(n-5)$. By a similar way to the above, we can obtain all cases 9–28, by using Propositions 3.2 and 2.2. The result follows. □

The following result shows that the capability of the direct product of a non-abelian Lie algebra T of class 4 with $\dim T^2 = 3$ and an abelian Lie algebra depends only on the capability of its non-abelian factor.

Theorem 3.6. *Let L be a finite dimensional nilpotent Lie algebra of class 4 and $\dim L^2 = 3$. Then $L = T \oplus A$ such that $Z(T) = L^2 \cap Z(L) = L^4 = T^4$ and $Z^*(L) = Z^*(T)$, where A is an abelian Lie algebra.*

Proof. By using [7, Proposition 3.1], $L = T \oplus A$ such that $Z(T) = L^2 \cap Z(L)$ and $Z^*(L) = Z^*(T)$, where A is an abelian Lie algebra. Since T is stem, Proposition 2.8 implies $Z(T) = T^4$, as required. □

The following corollary is an immediate consequence of Theorems 3.5 and 2.5.

Corollary 3.7. *Let L be a finite dimensional nilpotent Lie algebra of class 4 and $\dim L^2 = 3$. Then $L = T \oplus A$ in which $T \cong T_i$ for some i, $1 \leq i \leq 28$ and A is an abelian Lie algebra.*

As mentioned a Lie algebra L is called capable provided that $L \cong H/Z(H)$ for some Lie algebra H. The notion of the epicenter $Z^*(L)$ for a Lie algebra L was defined in [13]. It is shown that L is capable if and only if $Z^*(L) = 0$. Another notion having a close relation to the capability is the concept of the exterior square of Lie algebras, which is denoted by $L \wedge L$, for a Lie algebra L, was introduced in [5]. Our approach is on the concept of the exterior center $Z^{\wedge}(L)$, the set of all elements l of L for which $l \wedge l' = 0_{L \wedge L}$ for all $l' \in L$. Niroomand et al. in [12] showed $Z^{\wedge}(L) = Z^*(L)$ for any finite-dimensional Lie algebra L. In the following results, we classify all non-capable stem Lie algebras T of class 4 with $\dim T^2 = 3$.

Theorem 3.8. *Let L be isomorphic to one of Lie algebras $T_5, \ldots, T_8, T_{14}, \ldots, T_{18}, T_{24}, \ldots, T_{27}$ or T_{28}. Then L is non-capable.*

Proof. First assume that $L \cong T_5$. By using the presentation of Theorem 3.5, we have $L = T \dotplus H$ such that $T \cong L_{5,6}$ and $H \cong H(m)$, so [10, Proposition 2.2] implies L is non-capable. The other cases can be obtained by a similar way. □

The following lemma is a useful instrument in the rest.

Lemma 3.9. *Let L be isomorphic to one of Lie algebras $T_{10}, T_{11}, T_{13}, T_{20}, T_{22}$, or T_{23}. Then L is non-capable*

Proof. Using Theorem 3.5 (10), let $L \cong T_{10} = \langle x_1, x_2, \ldots, x_7 | [x_1, x_2] = x_3, [x_1, x_3] = [x_2, x_4] = x_5, [x_1, x_5] = [x_3, x_4] = [x_2, x_7] = [x_4, x_7] = x_6 \rangle$. Clearly, $Z(L) = \langle x_6 \rangle$. We claim that $x_6 \in Z^{\wedge}(L)$. It is sufficient to show that $x_i \wedge x_6 = 0_{L \wedge L}$ for all i, $1 \leq i \leq 7$. Since

$$
\begin{aligned}
x_1 \wedge x_6 &= x_1 \wedge [x_4, x_7] = [x_7, x_1] \wedge x_6 - [x_4, x_1] \wedge x_7 = 0_{L \wedge L}, \\
x_3 \wedge x_6 &= [x_1, x_2] \wedge x_6 = x_1 \wedge [x_2, x_6] - x_2 \wedge [x_1, x_6] = 0_{L \wedge L}, \\
x_5 \wedge x_6 &= [x_1, x_3] \wedge x_6 = x_1 \wedge [x_3, x_6] - x_3 \wedge [x_1, x_6] = 0_{L \wedge L}, \\
x_4 \wedge x_6 &= x_4 \wedge [x_1, x_5] = [x_5, x_4] \wedge x_1 - [x_1, x_4] \wedge x_5 = 0_{L \wedge L}, \\
x_7 \wedge x_6 &= x_7 \wedge [x_1, x_5] = [x_5, x_7] \wedge x_1 - [x_1, x_7] \wedge x_5 = 0_{L \wedge L}
\end{aligned}
$$

and

$$x_2 \wedge x_6 = x_2 \wedge [x_4, x_7] = [x_7, x_2] \wedge x_4 - [x_4, x_2] \wedge x_7 = -x_6 \wedge x_4 -$$
$$[x_4, x_2] \wedge x_7 = -x_5 \wedge x_7 = -[x_1, x_3] \wedge x_7 = -x_1 \wedge [x_3, x_7] + x_3 \wedge [x_1, x_7] = 0_{L \wedge L},$$

$x_6 \in Z^{\wedge}(L)$, as required. Similarly, $\langle x_1, ..., x_6, x_7 | [x_1, x_2] = x_3, [x_1, x_3] = x_5, [x_2, x_4] = x_5 + x_6, [x_1, x_5] = [x_3, x_4] = [x_2, x_7] = [x_4, x_7] = x_6 \rangle$ is non-capable. The other cases can be obtained by a similar way. □

The capable stem Lie algebras of class 4 with the derived subalgebra of dimension 3 are characterized in the following.

Lemma 3.10. *Let L be an n-dimensional stem Lie algebra such that $cl(L) = 4$ and $\dim L^2 = 3$. If L is isomorphic to one of Lie algebras $L \cong L_{5,6}$, $L \cong L_{5,7}$, $L \cong L_{6,11}$, $L \cong L_{6,12}$, $L \cong L_{6,13}$, $L \cong L_{6,1}^{(2)}$, $L \cong T_{12}$, or $L \cong T_{21}$, then L is capable.*

Proof. By using a terminology of [3, 4], we have $L_{6,15} = \langle x_1, ..., x_6 | [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = x_5 = [x_2, x_3], [x_1, x_5] = x_6 = [x_2, x_4] \rangle$ and $L_{6,18} = \langle x_1, ..., x_6 | [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = x_5, [x_1, x_5] = x_6 \rangle$. Clearly, $Z(L_{6,15}) = \langle x_6 \rangle$ and $Z(L_{6,18}) = \langle x_6 \rangle$, so $L_{6,15}/\langle x_6 \rangle \cong L_{5,6}$ and $L_{6,18}/\langle x_6 \rangle \cong L_{5,7}$. Hence, $L_{5,6}$ and $L_{5,7}$ are capable. Consider $H = \langle x_1, ..., x_7 | [x_1, x_2] = x_3, [x_1, x_3] = x_4 = [x_2, x_7], [x_1, x_4] = x_5 = [x_3, x_7], [x_1, x_5] = x_6 = [x_4, x_7] \rangle$. Clearly, $Z(H) = \langle x_6 \rangle$. Hence, $L_{6,13} \cong H/\langle x_6 \rangle$. Therefore, $L_{6,13}$ is capable. The capability of $L_{6,11}$, $L_{6,1}^{(2)}$, $L_{6,12}$, T_{12}, and T_{21} are obtained by a similar way. The proof is completed. □

In the following theorem, all capable stem Lie algebras of class 3 with the derived subalgebra of dimension 3 are given.

Theorem 3.11. *Let T be an n-dimensional stem Lie algebra such that $cl(T) = 4$ and $\dim T^2 = 3$. Then T is capable if and only if T is isomorphic to one of the Lie algebras $L_{5,6}$, $L_{5,7}$, $L_{6,11}$, $L_{6,12}, L_{6,1}^{(2)}$, T_{12}, T_{21}, or $L_{6,13}$.*

Proof. Let T be capable. By Theorem 3.5, Lemmas 3.9, and 2.9, T is isomorphic to one of the Lie algebras $L_{5,6}$, $L_{5,7}$, $L_{6,11}$, $L_{6,12}$, $L_{6,1}^{(2)}$, T_{12}, T_{21}, or $L_{6,13}$. The converse holds by Lemma 3.10. □

The next theorem gives a necessary and sufficient condition for the capability of stem Lie algebras of class 4 with the derived subalgebra of dimension 3.

Theorem 3.12. *Let T be an n-dimensional stem Lie algebra such that $cl(T) = 4$ and $\dim T^2 = 3$. Then T is capable if and only if $4 \leq \dim (T/Z(T)) \leq 6$.*

Proof. The result follows from Lemma 3.4 and Theorem 3.11. □

Now, we are in the position to determine all capable Lie algebras L of class 4 with $\dim L^2 = 3$.

Theorem 3.13. *Let L be an n-dimensional Lie algebra such that $cl(L) = 4$ and $\dim L^2 = 3$. Then L is capable if and only if L is isomorphic to exactly one of the Lie algebras $L_{5,6} \oplus A(n-5)$, $L_{5,7} \oplus A(n-5)$, $L_{6,11} \oplus A(n-6)$, $L_{6,12} \oplus A(n-6)$, $L_{6,1}^{(2)} \oplus A(n-6)$, $L_{6,13} \oplus A(n-6)$, $T_{12} \oplus A(n-7)$, or $T_{21} \oplus A(n-7)$.*

Proof. By using Theorem 3.6, $L = T \oplus A$ such that $Z(T) = L^2 \cap Z(L) = L^4 = T^4 \cong A(1)$ and $Z^*(L) = Z^*(T)$, where A is an abelian Lie algebra. Now, the result follows from Theorem 3.11. □

The following result is obtained from Theorems 3.12 and 3.13.

Corollary 3.14. *Let L be a finite dimensional Lie algebra of class 4 and $\dim L^2 = 3$. Then L is capable if and only if $4 \leq \dim (L/Z(L)) \leq 6$.*

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