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**On the non-abelian tensor square of all groups of order dividing $p^5$**

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In this paper we consider all groups of order dividing $p^5$. We obtain the explicit structure of the non-abelian tensor square, non-abelian exterior square, tensor center, exterior center, the third homotopy group of suspension of an Eilenberg-MacLane space $k(G, 1)$ and $\nabla(G)$ of such groups.

*Keywords*: non-abelian tensor square; $p$-group.

AMS Subject Classification: 20F14, 20F99

## 1. Introduction

The concept of non-abelian tensor product of groups was born by Brown and Loday [2], [3], which is an applied topic in $K$-theory and homotopy theory. The readers can find some spacious result on this topic in [1], [2], [4] and [11]. The non-abelian tensor square $G \otimes G$ of the group $G$ is the group generated by the symbols $g \otimes h$, subject to the relations

$$gg' \otimes h = ({}^{g}g' \otimes^{g} h)(g \otimes h) \text{ and } g \otimes hh' = (g \otimes h)({}^{h}g \otimes^{h} h')$$

for all $g, g', h, h' \in G$, where $G$ acts on itself by conjugation via ${}^{g}g' = gg'g^{-1}$. The tensor square is a special case of non-abelian tensor product $G \otimes H$ of two arbitrary groups $G$ and $H$. The exterior square $G \wedge G$ is obtained by imposing the additional relation $g \otimes g = 1_{\otimes}$ on $G \otimes G$. Recall that [4] describes the maps $\kappa : G \otimes G \to G'$ and $\kappa' : G \wedge G \to G'$ which are both homomorphisms of groups. The kernel of $\kappa$ which is denoted by $J_2(G)$, is isomorphic to the third homotopy group of suspension

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of an Eilenberg-MacLane space $k(G,1)$ and the kernel of $\kappa'$ which is isomorphic to $\mathcal{M}(G)$, is the Schur multiplier of $G$ (see [3] and [4] for more details).
The following commutative diagram with exact rows and central extensions as columns involves the third integral homology group of $G$ and Whitehead functor (see [4] for details)

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
1 \longrightarrow & H_3(G) & \longrightarrow & \Gamma(G/G') & \longrightarrow J_2(G) & \longrightarrow & H_2(G) & \longrightarrow 0 & \\
 & \| & & \downarrow & \downarrow & & \downarrow & & \\
 & H_3(G) & \longrightarrow & \Gamma(G/G') & \longrightarrow G\otimes G & \longrightarrow & G\wedge G & \longrightarrow 1 & \\
 & & & & \kappa\downarrow & & \kappa'\downarrow & & \\
 & & & & G' & = & G' & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Following the terminology in [5], we consider the notations of tensor center and exterior center, $Z^{\otimes}(G) = \{g \in G | g \otimes h = 1_{G\otimes G}, \text{ for all } h \in G\}$, and $Z^{\wedge}(G) = \{g \in G | g \wedge h = 1_{G\wedge G}, \text{ for all } h \in G\}$, respectively. $Z^{\wedge}(G)$ is a subgroup of $Z(G)$ which has the property that $G$ is capable if and only if $Z^{\wedge}(G) = 1$ that is, whether $G$ is isomorphic to $E/Z(E)$ for some group $E$. Ellis proved $Z^{\wedge}(G)$ is isomorphic to the epicenter of $G$ which is denoted by $Z^*(G)$, that is defined as follows

**Definition 1.1.** $Z^*(G)$ is the intersection of all subgroups of the form $\psi(Z(G))$ where $\psi : E \to G$ is an arbitrary surjective homomorphism with $\ker \psi \subseteq Z(G)$.

The next lemmas show the relations between $Z^*(G)$, $Z^{\otimes}(G)$ and $Z^{\wedge}(G)$, which play important role in this paper

**Lemma 1.1.** [5] *Let $G$ be any group. Then $Z^*(G) \cong Z^{\wedge}(G)$.*

**Lemma 1.2.** [5] *Let $G$ be any group. Then $Z^{\otimes}(G) \leq Z^{\wedge}(G)$.*

The presentation of groups of order dividing $p^5$ have been determined in [7] and [10], for the convenience of the readers we bring here the presentation of all groups of order dividing $p^5$ as listed explicitly from [7].

**Theorem 1.1.** [7] *Let $w$ be a primitive root of finite domain $\mathbb{F}_p$ of order $p$ $(p > 3)$ and let $a, b$ and $k$ be integers where $a \in w_3 = \{x \in \mathbb{F}_p | x^3 = 1\}, b \in w_4 = \{x \in \mathbb{F}_p | x^4 = 1\}$ and $k \in \{1\ldots, \frac{p-1}{2}\}$. Let $G_i = \langle g_1, g_2, g_3, g_4, g_5 | R\rangle$, since this groups are polycyclic, they have polycyclic presentation, so they satisfy:*
$g_i^{r_i} = g_{i+1}^{a_{i+1}} \ldots g_5^{a_5}$

$[g_i, g_j] = g_{j+1}^{a_{i+j,j+1}} \dots g_5^{a_{i+j,5}}$.
*For the summery we omitted the relations:* $[g_i, g_j] = 1$ *and* ${g_i}^p = 1$ *for any* $i, j$ *where* $1 \leq i, j \leq 5$. *Then:*

$G_1 = \langle g_1, g_2, g_3, g_4, g_5 | g_1^p = g_2, g_2^p = g_3, g_3^p = g_4, g_4^p = g_5 \rangle$
$G_2 = \langle g_1, g_2, g_3, g_4, g_5 | [g_2, g_1] = g_3, g_1^p = g_4, g_2^p = g_5 \rangle$
$G_3 = \langle g_1, g_2, g_3, g_4, g_5 | [g_2, g_1] = g_3, [g_3, g_1] = g_4, [g_3, g_2] = g_5 \rangle$
$G_4 = \langle g_1, g_2, g_3, g_4, g_5 | [g_2, g_1] = g_3, [g_3, g_1] = g_4, [g_3, g_2] = g_5, g_2^p = g_5 \rangle$
$G_5 = \langle g_1, g_2, g_3, g_4, g_5 | [g_2, g_1] = g_3, [g_3, g_1] = g_4, [g_3, g_2] = g_5, g_2^p = g_4 \rangle$
$G_6 = \langle g_1, g_2, g_3, g_4, g_5 | [g_2, g_1] = g_3, [g_3, g_1] = g_4, [g_3, g_2] = g_5, g_2^p = g_4^w \rangle$
$G_7 = \langle g_1, g_2, g_3, g_4, g_5 | [g_2, g_1] = g_3, [g_3, g_1] = g_4, [g_3, g_2] = g_5, g_1^p = g_4, g_2^p = g_5 \rangle$
$G_8 = \langle g_1, g_2, g_3, g_4, g_5 | [g_2, g_1] = g_3, [g_3, g_1] = g_4 g_5, [g_3, g_2] = g_5, g_1^p = g_4, g_2^p = g_5 \rangle$
$G_9 = \langle g_1, g_2, g_3, g_4, g_5 | [g_2, g_1] = g_3, [g_3, g_1] = g_4 g_5^w, [g_3, g_2] = g_5, g_1^p = g_4, g_2^p = g_5 \rangle$
$G_{10} = \langle g_1, g_2, g_3, g_4, g_5 | [g_2, g_1] = g_3, [g_3, g_1] = g_5^w, [g_3, g_2] = g_4, g_1^p = g_4, g_2^p = g_5 \rangle$
$G_{11_k} = \langle g_1, g_2, g_3, g_4, g_5 | [g_2, g_1] = g_3, [g_3, g_1] = g_4, [g_3, g_2] = g_5^{w^k}, g_1^p = g_4, g_2^p = g_5 \rangle$
$G_{12_k} = \langle g_1, g_2, g_3, g_4, g_5 | [g_2, g_1] = g_3, [g_3, g_1] = g_4 g_5^{w^k}, [g_3, g_2] = g_4^{w^{k-1}} g_5, g_1^p = g_4, g_2^p = g_5 \rangle$
$G_{13} = \langle g_1, g_2, g_3, g_4, g_5 | g_1^p = g_3, g_2^p = g_4, g_3^p = g_5 \rangle \cong C_{p^3} \times C_{p^2}$
$G_{14} = \langle g_1, g_2, g_3, g_4, g_5 | [g_2, g_1] = g_5, g_1^p = g_3, g_2^p = g_4, g_3^p = g_5 \rangle$
$G_{15} = \langle g_1, g_2, g_3, g_4, g_5 | [g_2, g_1] = g_3, g_1^p = g_4, g_4^p = g_5 \rangle$
$G_{16} = \langle g_1, g_2, g_3, g_4, g_5 | [g_2, g_1] = g_3, g_1^p = g_4, [g_3, g_1] = g_5 \rangle$
$G_{17} = \langle g_1, g_2, g_3, g_4, g_5 | [g_2, g_1] = g_3, g_1^p = g_4, [g_3, g_1] = g_5, g_2^p = g_5 \rangle$
$G_{18} = \langle g_1, g_2, g_3, g_4, g_5 | [g_2, g_1] = g_3, g_1^p = g_4, [g_3, g_1] = g_5^w, g_2^p = g_5 \rangle$
$G_{19} = \langle g_1, g_2, g_3, g_4, g_5 | [g_2, g_1] = g_3, g_1^p = g_4, [g_3, g_1] = g_5, g_4^p = g_5 \rangle$
$G_{20} = \langle g_1, g_2, g_3, g_4, g_5 | [g_2, g_1] = g_3, g_1^p = g_4, [g_3, g_1] = g_5, [g_3, g_2] = g_5 \rangle$
$G_{21} = \langle g_1, g_2, g_3, g_4, g_5 | [g_2, g_1] = g_3, g_1^p = g_4, [g_3, g_1] = g_5, [g_3, g_2] = g_5, g_2^p = g_5 \rangle$
$G_{22} = \langle g_1, g_2, g_3, g_4, g_5 | [g_2, g_1] = g_3, g_1^p = g_4, [g_3, g_1] = g_5, [g_3, g_2] = g_5, g_4^p = g_5 \rangle$
$G_{23} = \langle g_1, g_2, g_3, g_4, g_5 | [g_2, g_1] = g_3, g_1^p = g_4, [g_3, g_1] = g_5^w, [g_3, g_2] = g_5^w, g_4^p = g_5 \rangle$
$G_{24} = \langle g_1, g_2, g_3, g_4, g_5 | [g_2, g_1] = g_3, g_1^p = g_4, g_2^p = g_3, [g_3, g_1] = g_5, [g_4, g_2] = g_5^{p-1}, g_3^p = g_5 \rangle$
$G_{25} = \langle g_1, g_2, g_3, g_4, g_5 | [g_2, g_1] = g_3, g_1^p = g_4, g_2^p = g_3, g_4^p = g_5 \rangle$
$G_{26} = \langle g_1, g_2, g_3, g_4, g_5 | g_1^p = g_3, g_3^p = g_4, g_4^p = g_5 \rangle \cong C_{p^4} \times C_p$
$G_{27} = \langle g_1, g_2, g_3, g_4, g_5 | g_1^p = g_3, g_3^p = g_4, g_4^p = g_5, [g_2, g_1] = g_5 \rangle$
$G_{28} = \langle g_1, g_2, g_3, g_4, g_5 | [g_2, g_1] = g_3, [g_3, g_1] = g_4, [g_4, g_1] = g_5 \rangle$
$G_{29_a} = \langle g_1, g_2, g_3, g_4, g_5 | [g_2, g_1] = g_3, [g_3, g_1] = g_4, [g_4, g_1] = g_5^{w^a}, g_2^p = g_5 \rangle$
$G_{30} = \langle g_1, g_2, g_3, g_4, g_5 | [g_2, g_1] = g_3, [g_3, g_1] = g_4, [g_4, g_1] = g_5, g_1^p = g_5 \rangle$
$G_{31} = \langle g_1, g_2, g_3, g_4, g_5 | [g_2, g_1] = g_3, [g_3, g_1] = g_4, [g_4, g_1] = g_5, [g_3, g_2] = g_5 \rangle$

$G_{32_a} = \langle g_1, g_2, g_3, g_4, g_5 | [g_2, g_1] = g_3, [g_3, g_1] = g_4, [g_4, g_1] = g_5^{w^a}, [g_3, g_2] = g_5^{w^a}, g_2^p = g_5 \rangle$

$G_{33_b} = \langle g_1, g_2, g_3, g_4, g_5 | [g_2, g_1] = g_3, [g_3, g_1] = g_4, [g_4, g_1] = g_5^{w^b}, [g_3, g_2] = g_5^{w^b}, g_1^p = g_5 \rangle$

$G_{34} = \langle g_1, g_2, g_3, g_4, g_5 | [g_2, g_1] = g_4, [g_3, g_1] = g_5 \rangle$

$G_{35} = \langle g_1, g_2, g_3, g_4, g_5 | [g_2, g_1] = g_4, g_3^p = g_5 \rangle$

$G_{36} = \langle g_1, g_2, g_3, g_4, g_5 | [g_2, g_1] = g_4, [g_3, g_2] = g_5, g_3^p = g_5 \rangle$

$G_{37} = \langle g_1, g_2, g_3, g_4, g_5 | [g_3, g_2] = g_4, g_3^p = g_5 \rangle$

$G_{38} = \langle g_1, g_2, g_3, g_4, g_5 | [g_3, g_1] = g_4, [g_3, g_2] = g_5, g_3^p = g_5 \rangle$

$G_{39} = \langle g_1, g_2, g_3, g_4, g_5 | [g_2, g_1] = g_4, [g_3, g_2] = g_5, g_3^p = g_4 \rangle$

$G_{40} = \langle g_1, g_2, g_3, g_4, g_5 | [g_3, g_2] = g_5, g_2^p = g_4, g_3^p = g_5 \rangle$

$G_{41} = \langle g_1, g_2, g_3, g_4, g_5 | [g_3, g_1] = g_4, [g_3, g_2] = g_5, g_2^p = g_4, g_3^p = g_5 \rangle$

$G_{42} = \langle g_1, g_2, g_3, g_4, g_5 | [g_2, g_1] = g_4, [g_3, g_2] = g_5, g_2^p = g_4, g_3^p = g_5 \rangle$

$G_{43} = \langle g_1, g_2, g_3, g_4, g_5 | g_2^p = g_4, g_3^p = g_5 \rangle$

$G_{44} = \langle g_1, g_2, g_3, g_4, g_5 | [g_2, g_1] = g_4, [g_3, g_1] = g_5, g_2^p = g_4, g_3^p = g_5 \rangle$

$G_{45} = \langle g_1, g_2, g_3, g_4, g_5 | [g_2, g_1] = g_5, g_2^p = g_4, g_3^p = g_5 \rangle$

$G_{46} = \langle g_1, g_2, g_3, g_4, g_5 | [g_2, g_1] = g_4 g_5, [g_3, g_1] = g_5, g_2^p = g_4, g_3^p = g_5 \rangle$

$G_{47} = \langle g_1, g_2, g_3, g_4, g_5 | [g_3, g_1] = g_5, g_2^p = g_4, g_3^p = g_5 \rangle$

$G_{48_k} = \langle g_1, g_2, g_3, g_4, g_5 | [g_2, g_1] = g_4, [g_3, g_1] = g_5^{w^k}, g_2^p = g_4, g_3^p = g_5 \rangle$

$G_{49} = \langle g_1, g_2, g_3, g_4, g_5 | [g_2, g_1] = g_5^{w^k}, [g_3, g_1] = g_4, g_2^p = g_4, g_3^p = g_5 \rangle$

$G_{50_k} = \langle g_1, g_2, g_3, g_4, g_5 | [g_2, g_1] = g_4 g_5^{w^k}, [g_3, g_1] = g_4^{w^{k-1}} g_5, g_2^p = g_4, g_3^p = g_5 \rangle$

$G_{51} = \langle g_1, g_2, g_3, g_4, g_5 | g_1^p = g_4, g_4^p = g_5 \rangle$

$G_{52} = \langle g_1, g_2, g_3, g_4, g_5 | g_1^p = g_4, g_4^p = g_5, [g_3, g_2] = g_5 \rangle$

$G_{53} = \langle g_1, g_2, g_3, g_4, g_5 | g_1^p = g_4, g_4^p = g_5, [g_3, g_1] = g_5 \rangle$

$G_{54} = \langle g_1, g_2, g_3, g_4, g_5 | [g_2, g_1] = g_4, [g_4, g_2] = g_5 \rangle$

$G_{55} = \langle g_1, g_2, g_3, g_4, g_5 | [g_2, g_1] = g_4, [g_4, g_2] = g_5, g_3^p = g_5 \rangle$

$G_{56} = \langle g_1, g_2, g_3, g_4, g_5 | [g_2, g_1] = g_4, [g_4, g_2] = g_5, g_2^p = g_5 \rangle$

$G_{57} = \langle g_1, g_2, g_3, g_4, g_5 | [g_2, g_1] = g_4, [g_4, g_2] = g_5, g_1^p = g_5 \rangle$

$G_{58} = \langle g_1, g_2, g_3, g_4, g_5 | [g_2, g_1] = g_4, [g_4, g_2] = g_5^w, g_1^p = g_5 \rangle$

$G_{59} = \langle g_1, g_2, g_3, g_4, g_5 | [g_2, g_1] = g_4, [g_4, g_2] = g_5, [g_3, g_1] = g_5 \rangle$

$G_{60} = \langle g_1, g_2, g_3, g_4, g_5 | [g_2, g_1] = g_4, [g_4, g_2] = g_5, [g_3, g_1] = g_5, g_3^p = g_5 \rangle$

$G_{61} = \langle g_1, g_2, g_3, g_4, g_5 | [g_2, g_1] = g_4, [g_4, g_2] = g_5, [g_3, g_1] = g_5, g_2^p = g_5 \rangle$

$G_{62} = \langle g_1, g_2, g_3, g_4, g_5 | [g_2, g_1] = g_4, [g_4, g_2] = g_5, [g_3, g_1] = g_5, g_1^p = g_5 \rangle$

$G_{63} = \langle g_1, g_2, g_3, g_4, g_5 | [g_2, g_1] = g_4, [g_4, g_2] = g_5^w, [g_3, g_1] = g_5^w, g_1^p = g_5 \rangle$

$G_{64} = \langle g_1, g_2, g_3, g_4, g_5 | [g_2, g_1] = g_5 \rangle$

$G_{65} = \langle g_1, g_2, g_3, g_4, g_5 | [g_2, g_1] = g_5, [g_4, g_3] = g_5 \rangle$

$G_{66} = \langle g_1, g_2, g_3, g_4, g_5 | g_1^p = g_5 \rangle$
$G_{67} = \langle g_1, g_2, g_3, g_4, g_5 | [g_2, g_1] = g_5, g_1^p = g_5 \rangle$
$G_{68} = \langle g_1, g_2, g_3, g_4, g_5 | [g_2, g_1] = g_5, g_3^p = g_5 \rangle$
$G_{69} = \langle g_1, g_2, g_3, g_4, g_5 | [g_2, g_1] = g_5, [g_4, g_3] = g_5, g_4^p = g_5 \rangle$
$G_{70} \cong C_p^5$

This paper is devoted to obtain the structure of $G \wedge G, Z^{\wedge}(G), G \otimes G, Z^{\otimes}(G)$, $\Pi_3(SK(G,1))$ the third homotopy group of suspension of an Eilenberg-MacLane space $k(G,1)$ and $\nabla(G)$ for all groups of order dividing $p^5$.
The authors in [6] obtained the structure of $G \wedge G, Z^{\wedge}(G), G \otimes G, Z^{\otimes}(G)$ and $\Pi_3(SK(G,1))$ for all groups of order $p^4$. The same motivation allows us to write the present article. It is instructive to note that at the same time and in process of preparation and finalizing this article we noticed [8]. But it is worth noting however Hatui et al. have the same results originated from the classification of James in [10], they claimed the presentation of their article is a bit unusual while we obtained our results straightforward.
Our work is more shorter and the readers can find it more understandable while we get the results with different argument. Just to obtain the non-abelian exterior square of groups $G_2, G_{10}, G_{11,2}, G_{17}, G_{18}, G_{28}, G_{31}, G_{40}, G_{41}, G_{45}, G_{48,2}$ and $G_{49}$ due to same argument, we prefer to refer to [8].
The Schur multiplier and epicenter of all groups of order dividing $p^5$ have been determined in [8] using ad-hoc methods and in unpublished work [9] using an algorithmic approach. The following theorems give us the structure of the Schur multiplier and the epicenter of all groups of order dividing $p^5$.

**Theorem 1.2.** *Let $G$ be a group of order dividing $p^5$ where $p > 3$ is an odd prime. Then*

$$\mathcal{M}(G) \cong \begin{cases} 1 & \text{if } G \cong G_1, G_7, \ldots, G_9, G_{11,1}, G_{12_k}, G_{24}, G_{27}. \\ \mathbb{Z}_p & \text{if } G \cong G_4, \ldots, G_6, G_{10}, G_{11,2}, G_{12_k}, G_{19}, G_{21}, G_{22}, G_{23}, \\ & \quad G_{25}, G_{29_a}, G_{30}, G_{32_a}, G_{33_b}, G_{42}, G_{44}, G_{46}, G_{48,1}, G_{50_k}. \\ \mathbb{Z}_p^{(2)} & \text{if } G \cong G_{15}, \ldots, G_{18}, G_{20}, G_{41}, G_{47}, G_{52}, G_{53}. \\ \mathbb{Z}_p^{(3)} & \text{if } G \cong G_2, G_3, G_{28}, G_{31}, G_{36}, G_{38}, G_{39}, G_{40}, G_{51}, G_{55}, \ldots, G_{58}, \\ & \quad G_{60}, \ldots, G_{63}. \\ \mathbb{Z}_p^{(4)} & \text{if } G \cong G_{35}, G_{37}, G_{54}, G_{59}. \\ \mathbb{Z}_p^{(5)} & \text{if } G \cong G_{65}, G_{67}, \ldots, G_{69}. \\ \mathbb{Z}_p^{(6)} & \text{if } G \cong G_{34}, G_{66}. \\ \mathbb{Z}_p^{(7)} & \text{if } G \cong G_{64}. \\ \mathbb{Z}_p^{(10)} & \text{if } G \cong G_{70}. \\ \mathbb{Z}_{p^2} & \text{if } G \cong G_{13}, G_{48,2}, G_{49}. \\ \mathbb{Z}_{p^2} \oplus \mathbb{Z}_p & \text{if } G \cong G_{45}. \\ \mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(2)} & \text{if } G \cong G_{43}. \end{cases}$$

**Theorem 1.3.** *Let $G$ be a group of order dividing $p^5$ where $p > 3$ is an odd prime. Then*

$$Z^*(G) \cong \begin{cases} 1 & \text{if } G \cong G_2,\ G_3\ G_{10},\ G_{11,2},\ G_{17},\ G_{18},\ G_{28},\ G_{31},\ G_{34},\ G_{40},\ G_{41},\ G_{43},\ G_{45}, \\ & \quad G_{48,2},\ G_{49},\ G_{54},\ G_{59},\ G_{17},\ G_{64},\ G_{70}. \\ \mathbb{Z}_p & \text{if } G \cong G_4,\ldots,G_6,\ G_{10},\ G_{14},\ G_{16},\ G_{25},\ G_{29_a},\ G_{30},\ G_{32_a},\ G_{33_b},\ G_{35}, \\ & \quad G_{36},\ldots,G_{39},\ G_{55},\ldots,G_{58},\ G_{61},\ldots,G_{63},\ G_{65},\ldots,G_{69}. \\ \mathbb{Z}_p^{(2)} & \text{if } G \cong G_7,\ldots,G_9,\ G_{11,1},\ G_{12_k},\ G_{21},\ G_{42},\ G_{44},\ G_{46},\ G_{47},\ G_{48,1},\ G_{50_k}. \\ \mathbb{Z}_{p^2} & \text{if } G \cong G_{15},\ G_{20},\ G_{22},\ G_{23},\ G_{51},\ldots,G_{53}. \\ \mathbb{Z}_{p^3} & \text{if } G \cong G_{26},\ G_{27}. \\ \mathbb{Z}_{p^5} & \text{if } G \cong G_{70}. \end{cases}$$

Here, we are going to obtain the structure of $\nabla(G)$ of groups of order dividing $p^5$ by the following theorems

**Theorem 1.4.** [1] *If $G^{ab}$ has no elements of order 2, then $\nabla(G) \cong \Gamma(G^{ab})$.*

Given an abelian group $A$, from [13], $\Gamma(A)$ is used to denote the abelian group with generators $\gamma(a)$, for $a \in A$, by defining relations

(*i*). $\gamma(a^{-1}) = \gamma(a)$.
(*ii*). $\gamma(abc)\gamma(a)\gamma(b)\gamma(c) = \gamma(ab)\gamma(bc)\gamma(ca)$,

for all $a, b, c \in A$. $\Gamma$ is called Whitehead's universal quadratic functor.
From [4], we have

**Theorem 1.5.** *Let $G$ and $H$ be abelian groups. Then*

*(i)* $\Gamma(G \times H) \cong \Gamma(G) \times \Gamma(H) \times (G \otimes H)$,
*(ii)*

$$\Gamma(\mathbb{Z}_n) = \begin{cases} \mathbb{Z}_n & n \text{ is odd} \\ \mathbb{Z}_{2n} & n \text{ is even} \end{cases}$$

*where* $\mathbb{Z}_n = \langle x | x^n = e \rangle$ *for* $n \geq 0$.

Now we give the structure of $\nabla(G)$ when $G$ is a group of order dividing $p^5$.

**Lemma 1.3.** *Let $G$ be a group of order $p^5$, where $p > 3$ is an odd prime. Then*

$$\nabla(G) \cong \begin{cases} \mathbb{Z}_p^{(3)} & \text{if } G \cong \ G_3, \ldots, G_{12_k},\, G_{28}, \ldots, G_{33_b}. \\ \mathbb{Z}_{p^5} & \text{if } G \cong \ G_1. \\ \mathbb{Z}_p^{(6)} & \text{if } G \cong \ G_{34},\, G_{36},\, G_{38},\, G_{39},\, G_{41},\, G_{42},\, G_{44},\, G_{46}, \\ & \quad G_{48_k}, \ldots, G_{50_k},\, G_{54}, \ldots, G_{63}. \\ \mathbb{Z}_p^{(10)} & \text{if } G \cong \ G_{64},\, G_{65},\, G_{67}, \ldots, G_{69}. \\ \mathbb{Z}_p^{(15)} & \text{if } G \cong \ G_{70}. \\ \mathbb{Z}_{p^2}^{(3)} & \text{if } G \cong \ G_2,\, G_{14}. \\ \mathbb{Z}_{p^2}^{(4)} \oplus \mathbb{Z}_p^{(5)} & \text{if } G \cong \ G_{43}. \\ \mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(5)} & \text{if } G \cong \ G_{35},\, G_{37},\, G_{40},\, G_{45},\, G_{47},\, G_{52},\, G_{53}. \\ \mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(9)} & \text{if } G \cong \ G_{66}. \\ \mathbb{Z}_{p^3} \oplus \mathbb{Z}_p^{(2)} & \text{if } G \cong \ G_{13},\, G_{15},\, G_{25},\, G_{27}. \\ \mathbb{Z}_{p^3} \oplus \mathbb{Z}_p^{(5)} & \text{if } G \cong \ G_{51}. \\ \mathbb{Z}_{p^4} \oplus \mathbb{Z}_p^{(2)} & \text{if } G \cong \ G_{26}. \end{cases}$$

**Proof.** For abelian groups Theorem 1.5, shows the result. Let $G$ be isomorphic to one of the groups $G_3, \ldots, G_{12_k}, G_{28}, \ldots, G_{33_b}$. Since $G^{ab} \cong \mathbb{Z}_p^{(2)}$, clearly using Theorems 1.4 and 1.5 we have $\nabla(G) \cong \Gamma(\mathbb{Z}_p^{(2)}) \cong \Gamma(\mathbb{Z}_p) \oplus \Gamma(\mathbb{Z}_p) \oplus (\mathbb{Z}_p \otimes \mathbb{Z}_p) \cong \mathbb{Z}_p^{(3)}$. The proof of the rest of groups is similar. □

Brown and Loday in [4] described the role of $J_2(G)$ in algebraic topology, they showed the third homotopy group of suspension of an Eilenberg-MacLane space $k(G,1)$ satisfied the condition $\Pi_3(SK(G,1)) \cong J_2(G)$. Here we are going to obtain the structure of third homotopy groups for groups of order dividing $p^5$. Blyth et al. [1] proved the following theorem that helps us to obtain $J_2(G)$ of $p$-groups of order $p^5$.

**Theorem 1.6.** [1] *Let $G$ be a group such that $G^{ab}$ is a finitely generated abelian group with no elements of order 2. Then $J_2(G) \cong \Gamma(G^{ab}) \times \mathcal{M}(G)$.*

**Lemma 1.4.** *Let $G$ be a group of order dividing $p^5$, where $p$ ($p > 3$) is an odd*

*prime. Then*

$$J_2(G) \cong \begin{cases} \mathbb{Z}_p^{(3)} & \text{if } G \cong G_7, \ldots, G_9,\ G_{11,1},\ G_{12_k}. \\ \mathbb{Z}_p^{(4)} & \text{if } G \cong G_4, \ldots, G_6\ G_{10},\ G_{11,2},\ G_{29_a},\ G_{11,1},\ G_{30},\ 32_a,\ or\ G_{33_b}. \\ \mathbb{Z}_p^{(7)} & \text{if } G \cong G_{42},\ G_{44},\ G_{46},\ G_{48,1},\ G_{50_k}. \\ \mathbb{Z}_p^{(9)} & \text{if } G \cong G_{36},\ G_{38},\ G_{39},\ G_{55}, \ldots, G_{58},\ G_{60}, \ldots, G_{63}. \\ \mathbb{Z}_p^{(10)} & \text{if } G \cong G_{54},\ G_{59}. \\ \mathbb{Z}_p^{(12)} & \text{if } G \cong G_{34}. \\ \mathbb{Z}_p^{(15)} & \text{if } G \cong G_{65},\ G_{67}, \ldots, G_{69}. \\ \mathbb{Z}_p^{(17)} & \text{if } G \cong G_{64}. \\ \mathbb{Z}_p^{(25)} & \text{if } G \cong G_{70}. \\ \mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(2)} & \text{if } G \cong G_{24}. \\ \mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(3)} & \text{if } G \cong G_{19}. \\ \mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(4)} & \text{if } G \cong G_{16}, \ldots, G_{18},\ G_{20}, \ldots, G_{22}. \\ \mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(6)} & \text{if } G \cong G_{45},\ G_{48,2},\ G_{49}. \\ \mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(8)} & \text{if } G \cong G_{40}. \\ \mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(9)} & \text{if } G \cong G_{35},\ G_{37}. \\ \mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(15)} & \text{if } G \cong G_{66}. \\ \mathbb{Z}_{p^2}^{(3)} \oplus \mathbb{Z}_p & \text{if } G \cong G_{14}. \\ \mathbb{Z}_{p^2}^{(3)} \oplus \mathbb{Z}_p^{(3)} & \text{if } G \cong G_2. \\ \mathbb{Z}_{p^2}^{(4)} \oplus \mathbb{Z}_p^{(5)} & \text{if } G \cong G_{43}. \\ \mathbb{Z}_{p^3} \oplus \mathbb{Z}_p^{(3)} & \text{if } G \cong G_{25},\ G_{27}. \\ \mathbb{Z}_{p^3} \oplus \mathbb{Z}_p^{(8)} & \text{if } G \cong G_{51}. \\ \mathbb{Z}_{p^3} \oplus \mathbb{Z}_{p^2}^{(3)} & \text{if } G \cong G_{13}. \\ \mathbb{Z}_{p^4} \oplus \mathbb{Z}_p^{(3)} & \text{if } G \cong G_{26}. \\ \mathbb{Z}_{p^5} & \text{if } G \cong G_1. \end{cases}$$

**Proof.** Using Theorems 1.2, 1.3 and 1.6, we have the result. □

## 2. non-abelian tensor square and non-abelian exterior square of all groups of order dividing $p^5$

This section illustrates our main results, our goal is to obtain non-abelian exterior square and non-abelian tensor square of all groups of order dividing $p^5$. The following theorem is a key tool to obtain the structure of $G \wedge G$ .

**Theorem 2.1.** [5] *Let $G$ be a group and $N \trianglelefteq G$. Then $G/N \wedge G/N \cong G \wedge G$ if and only if $N \leq Z^\wedge(G)$.*

Our main theorem in this context is the following.

**Theorem 2.2.** *Let $G$ be a group of order dividing $p^5$, where $p(p > 3)$ is an odd*

*prime. Then*

$$G\wedge G \cong \begin{cases} 1 & \text{if } G\cong\ G_1. \\ \mathbb{Z}_p & \text{if } G\cong\ G_{26},\ G_{27}. \\ \mathbb{Z}_p^{(3)} & \text{if } G\cong\ G_7,\dots,G_9,\ G_{11,1},\ G_{12_k},\ G_{15},\ G_{19},\ G_{21},\dots,G_{23},\ G_{42},\ G_{44}, \\ & \quad G_{46},\ G_{47},\ G_{481},\ G_{50_k},\ G_{51},\ G_{52},\ G_{53}. \\ \mathbb{Z}_p^{(4)} & \text{if } G\cong\ G_4,\dots,G_6,\ G_{16},\ G_{20},\ G_{29_a},\ G_{30},\ G_{32_a},\ G_{33_b}. \\ \mathbb{Z}_p^{(5)} & \text{if } G\cong\ G_{35},\dots,G_{39},\ G_{55},\dots,G_{58},\ G_{60},\dots,G_{63}. \\ \mathbb{Z}_p^{(6)} & \text{if } G\cong\ G_3,\ G_{28},\ G_{31},\ G_{54},\ G_{59},\ G_{65},\dots,G_{69}. \\ E_1\times\mathbb{Z}_p^{(3)} & \text{if } G\cong\ G_{28},\ G_{31}. \\ \mathbb{Z}_p^{(8)} & \text{if } G\cong\ G_{34},\ G_{64}. \\ \mathbb{Z}_p^{(10)} & \text{if } G\cong\ G_{70}. \\ \mathbb{Z}_{p^2} & \text{if } G\cong\ G_{13},\ G_{14},\ G_{24},\ G_{25}. \\ \mathbb{Z}_{p^2}\oplus\mathbb{Z}_p^{(2)} & \text{if } G\cong\ G_2,\ G_{10},\ G_{11,2},\ G_{17},\ G_{18},\ G_{40},\ G_{41},\ G_{43},\ G_{45},\ G_{48,2},\ G_{49}. \end{cases}$$

**Proof.**

(*i.*) Let $G \cong G_1, G_{13}, G_{26}, G_{43}, G_{51}, G_{66}$ or $G_{70}$.
Since the derived subgroup of an abelian group $G$ is trivial, we have $G\wedge G \cong \mathcal{M}(G)$. Now the result follows by Theorem 1.2.

(*ii.*) Let $G \cong G_7, \dots, G_9, G_{11,1}, G_{12_k}, G_{24}$ or $G_{27}$.
Since Theorem 1.2 implies $\mathcal{M}(G) = 1$, we have $G \wedge G \cong G'$ and the result follows.

(*iii.*) Let $G \cong G_4, G_5, G_6, G_{16}, G_{20}, G_{29_a}, G_{30}, G_{32_a}$ or $G_{33_b}$.
Using Theorem 2.1 and putting $N = Z^\wedge(G) = \mathbb{Z}_p$, we have $G \wedge G \cong G/N \wedge G/N$, now using Table 3. of [6], we have $G \wedge G \cong \mathbb{Z}_p^{(4)}$.

(*iv*). Let $G \cong G_3$.
This group is a capable group and we know $|G \wedge G| = |\mathcal{M}(G)||(G')|$, so by using Theorem 1.2, we have $|G \wedge G| = p^6$. Using [12] , the exponent of $G \otimes G$ is $p$, since the exponent of $G$ is $p$. Therefore the exponent of $G \wedge G$ is $p$.
On the other hand $G \wedge G$ is an abelian group.
We have $(G \wedge G)' = \langle [x, y] \wedge [x', y'] | x, y, x', y' \in G\rangle = \langle [(x \wedge y),(x' \wedge y')] | x, y, x', y' \in G\rangle = 1$,
since $G' = \langle [g_i, g_j] | i, j \in \{3, 4, 5\}\rangle$ it is enough to prove that
$(g_3 \wedge g_4) = 1, (g_3 \wedge g_5) = 1$ and $(g_4 \wedge g_5) = 1$.
Hence $G \wedge G$ is an elementary abelian group and then $G \wedge G \cong \mathbb{Z}_p^{(6)}$.

(*v*). Let $G \cong G_{34}, G_{54}, G_{59}$ or $G_{64}$.
The proof for this groups is completely similar to (*iii*).

(*vi*). Let $G \cong G_2, G_{10}, G_{11,2}, G_{17}, G_{18}, G_{28}, G_{31}, G_{40}, G_{41}, G_{45}, G_{48,2}$ or $G_{49}$.
This groups are also capable and we use the result of [8]. Following notation and terminology of [8], $G_2 = \phi_2(221)d, G_{10} = \phi_6(221)d_0$,
$G_{11,2} = \phi_6(221)b_{1/2}(p-1), G_{17} = \phi_3(221)b_r, G_{18} = \phi_3(221)b_r$,

$G_{28} = \phi_9(1^5), G_{31} = \phi_{10}(1^5), G_{40} = \phi_2(221)a, G_{41} = \phi_4(221)b,$
$G_{45} = \phi_2(221)c, G_{48,2} = \phi_4(221)d_{1/2}(p-1)$ and $G_{49} = \phi_4(221)f_0$.
Here $E_1$ denotes the non-abelian group of order $p^3$ and exponent $p$.

$(vii)$. The proof for the rest of groups is completely similar to $(iii)$. □

The next theorem states the structure of the tensor square of groups with respect to the direct products of two groups.

**Theorem 2.3.** [4] *Let $G$ and $H$ be groups. Then* $(G \times H) \otimes (G \times H) \cong (G \otimes G) \times (G^{ab} \otimes H^{ab}) \times (H^{ab} \otimes G^{ab}) \times (H \otimes H)$.

Blyth et al. in [1] give us the structure of the non-abelian tensor square of a group $G$ with $G^{ab}$ finitely generated as follows.

**Theorem 2.4.** [1] *Let $G$ be a group such that $G^{ab}$ is finitely generated. If $G^{ab}$ has no elements of order 2 or if $G'$ has a complement in $G$, then* $G \otimes G \cong \Gamma(G^{ab}) \times G \wedge G$.

**Theorem 2.5.** *Let $G$ be a group of order dividing $p^5$ where $p(p > 3)$ is an odd*

*prime. Then*

$$G \otimes G \cong \begin{cases} \mathbb{Z}_p^{(6)} & \textit{if } G \cong\ G_7, \ldots, G_9,\ G_{11,1},\ \textit{or } G_{12_k}. \\ \mathbb{Z}_p^{(7)} & \textit{if } G \cong\ G_4, \ldots, G_6,\ G_{11,1},\ G_{29_a},\ G_{30},\ G_{32_a},\ G_{33_b}. \\ E_1 \times \mathbb{Z}_p^{(6)} & \textit{if } G \cong\ G_{28}\ \textit{or } G_{31}. \\ \mathbb{Z}_p^{(9)} & \textit{if } G \cong\ G_3,\ G_{42},\ G_{44},\ G_{46},\ G_{48,1},\ G_{50_k}. \\ \mathbb{Z}_p^{(11)} & \textit{if } G \cong\ G_{36},\ G_{38},\ G_{39},\ G_{55}, \ldots, G_{58},\ G_{60}, \ldots, G_{63}. \\ \mathbb{Z}_p^{(12)} & \textit{if } G \cong\ G_{54},\ G_{59}. \\ \mathbb{Z}_p^{(14)} & \textit{if } G \cong\ G_{34}. \\ \mathbb{Z}_p^{(16)} & \textit{if } G \cong\ G_{65}, \ldots, G_{69}. \\ \mathbb{Z}_p^{(18)} & \textit{if } G \cong\ G_{64}. \\ \mathbb{Z}_p^{(25)} & \textit{if } G \cong\ G_{70}. \\ \mathbb{Z}_{p^2}^{(4)} & \textit{if } G \cong\ G_{14}. \\ \mathbb{Z}_{p^2}^{(2)} \oplus \mathbb{Z}_p^{(2)} & \textit{if } G \cong\ G_{24}. \\ \mathbb{Z}_{p^2}^{(4)} \oplus \mathbb{Z}_p^{(2)} & \textit{if } G \cong\ G_2. \\ \mathbb{Z}_{p^2}^{(2)} \oplus \mathbb{Z}_p^{(4)} & \textit{if } G \cong\ G_{17},\ G_{18}. \\ \mathbb{Z}_{p^2}^{(2)} \oplus \mathbb{Z}_p^{(5)} & \textit{if } G \cong\ G_{10},\ G_{11,2},\ G_{19},\ G_{21}, \ldots, G_{23}. \\ \mathbb{Z}_{p^2}^{(4)} \oplus \mathbb{Z}_p^{(5)} & \textit{if } G \cong\ G_{43}. \\ \mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(6)} & \textit{if } G \cong\ G_{16},\ G_{20}. \\ \mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(8)} & \textit{if } G \cong\ G_{41},\ G_{47},\ G_{48,2},\ G_{49},\ G_{52},\ G_{53}. \\ \mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(10)} & \textit{if } G \cong\ G_{35},\ G_{37}. \\ \mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(15)} & \textit{if } G \cong\ G_{66}. \\ \mathbb{Z}_{p^2}^{(2)} \oplus \mathbb{Z}_p^{(7)} & \textit{if } G \cong\ G_{40},\ G_{45}. \\ \mathbb{Z}_{p^3} \oplus \mathbb{Z}_p^{(3)} & \textit{if } G \cong\ G_{27}. \\ \mathbb{Z}_{p^3} \oplus \mathbb{Z}_p^{(5)} & \textit{if } G \cong\ G_{15}. \\ \mathbb{Z}_{p^3} \oplus \mathbb{Z}_p^{(8)} & \textit{if } G \cong\ G_{51}. \\ \mathbb{Z}_{p^3} \oplus \mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(2)} & \textit{if } G \cong\ G_{25}. \\ \mathbb{Z}_{p^3} \oplus \mathbb{Z}_{p^2}^{(3)} & \textit{if } G \cong\ G_{13}. \\ \mathbb{Z}_{p^4} \oplus \mathbb{Z}_p^{(3)} & \textit{if } G \cong\ G_{26}. \\ \mathbb{Z}_{p^5} & \textit{if } G \cong\ G_1. \end{cases}$$

**Proof.** Using Theorems 1.3, 2.2 and 2.4, we have the result. □

## 3. tensor center and exterior center of all groups of order dividing $p^5$

The purpose of this section is to obtain the structure of $Z^{\otimes}(G)$ and $Z^{\wedge}(G)$ when $G$ is a group of order dividing $p^5$. In the following theorem the tensor center of an arbitrary finite abelian group is determined.

**Lemma 3.1.** [5] *Let $G$ be a finite abelian $p$-group of order $p^5$. Then $Z^{\otimes}(G) = 1$.*

Following theorem is a key tool for the next investigations.

**Theorem 3.1.** [5] *Let $G$ be a group and $N \trianglelefteq G$. Then $G/N \otimes G/N \cong G \otimes G$ if and only if $N \leq Z^{\otimes}(G)$.*

The following corollary is a straightforward consequence of Lemma 1.3 and Theorem1.4.

**Corollary 3.1.** *Let $G$ be a group of order dividing $p^5$ where $p$ $(p > 3)$ is an odd prime. Then*

$$Z^{\wedge}(G) \cong \begin{cases} 1 & \text{if } G \cong\ G_2,\, G_3\ G_{10},\, G_{11,2},\, G_{17},\, G_{18},\, G_{28},\, G_{31},\, G_{34},\, G_{40},\, G_{41}, \\ & \quad G_{43},\, G_{45},\, G_{48,2},\, G_{49},\, G_{54},\, G_{59},\, G_{17},\, G_{64},\, G_{70}. \\ \mathbb{Z}_p & \text{if } G \cong\ G_4, \ldots, G_6,\, G_{10},\, G_{14},\, G_{16},\, G_{25},\, G_{29_a},\, G_{30},\, G_{32_a},\, G_{33_b},\, G_{35}, \\ & \quad G_{36}, \ldots, G_{39},\, G_{55}, \ldots, G_{58},\, G_{61}, \ldots, G_{63},\, G_{65}, \ldots, G_{69}. \\ \mathbb{Z}_p^{(2)} & \text{if } G \cong\ G_7, \ldots, G_9,\, G_{11,1},\, G_{12_k},\, G_{21},\, G_{42},\, G_{44},\, G_{46},\, G_{47},\, G_{48,1}, \\ & \quad G_{50_k}. \\ \mathbb{Z}_{p^2} & \text{if } G \cong\ G_{15},\, G_{20},\, G_{22},\, G_{23},\, G_{51}, \ldots, G_{53}. \\ \mathbb{Z}_{p^3} & \text{if } G \cong\ G_{26},\, G_{27}. \\ \mathbb{Z}_{p^5} & \text{if } G \cong\ G_{70}. \end{cases}$$

In next theorem we obtain tensor center of all groups of order dividing $p^5$.

**Theorem 3.2.** *Let $G$ be a group of order dividing $p^5$ where $p$ $(p > 3)$ is prime. Then*

$$Z^{\otimes}(G) \cong \begin{cases} 1 & \text{if } G \cong\ G_1, \ldots, G_3,\, G_{10},\, G_{11,2}, G_{13},\, G_{15}, \ldots, G_{18},\, G_{20},\, G_{25},\, G_{26}, \\ & \quad G_{28},\, G_{31},\, G_{34},\, G_{35},\, G_{37},\, G_{40},\, G_{41},\, G_{45},\, G_{48,2},\, G_{49},\, G_{51},\, G_{54}, \\ & \quad G_{59},\, G_{64},\, G_{66},\, G_{70}. \\ \mathbb{Z}_p & \text{if } G \cong\ G_4, \ldots, G_6,\, G_{14},\, G_{19},\, G_{21}, \ldots, G_{24},\, G_{27},\, G_{29_a},\, G_{30},\, G_{32_a}, \\ & \quad G_{33_b},\, G_{36},\, G_{38},\, G_{39},\, G_{52},\, G_{53},\, G_{55}, \ldots, G_{58}\ G_{60}, \ldots, G_{63}, \\ & \quad G_{67}, \ldots, G_{69}. \\ \mathbb{Z}_p^{(2)} & \text{if } G \cong\ G_7, \ldots, G_9,\, G_{11,1},\, G_{12_k},\, G_{42},\, G_{44},\, G_{46},\, G_{48,1},\, G_{50_k}. \end{cases}$$

**Proof.**

$(i)$. Let $G$ be isomorphic to one of groups $G_2, G_3, G_{10}, G_{11,2}, G_{17}, G_{18}, G_{28}, G_{31} G_{34}, G_{40}, G_{41}, G_{43}, G_{45}, G_{48,2}, G_{49}, G_{59}, G_{64}, G_{18}$ or $G_{70}$. Clearly by using lemmas 1.2 and 3.1, $G$ is capable and $Z^{\otimes}(G) \leq Z^{\wedge}(G) = Z^{*}(G) \cong 1$, so $Z^{\otimes}(G) \cong 1$.

$(ii)$. Let $G \cong G_1, G_{13}, G_{26}, G_{43}, G_{51}, G_{66}$ or $G_{70}$. By Theorem 3.1, $Z^{\otimes}(G) \cong 1$.

$(iii)$. Let $G \cong G_4$. Using Theorem 3.1 and put $N \cong \mathbb{Z}_p$, using Table 3. of [6], we have $G/N \otimes G/N \cong \mathbb{Z}_p^{(7)}$. On the other hand by Theorem 2.5, since $G \otimes G \cong \mathbb{Z}_p^{(7)}$, we have $\mathbb{Z}_p \leq Z^{\otimes}(G)$. Again using lemmas 1.2 and 3.1, we have $Z^{\otimes}(G) \cong \mathbb{Z}_p$.

$(iv)$. Let $G$ be isomorphic to one of groups $G_5, G_6, G_{14}, G_{24}, G_{29_a}, G_{30}, G_{33_b}, G_{34}, G_{36}, G_{38}, G_{39}, G_{55}, G_{56}, G_{57}, G_{58}, G_{60}, G_{61}, G_{62}, G_{63}, G_{65}, G_{67}, G_{68}$ or $G_{69}$. The proof is completely similar to $(iii)$.

$(v)$. Let $G \cong G_7$. Using Theorem 3.1 and put $N \cong \mathbb{Z}_p^{(2)}$, the Table 3. of [6] shows $G/N \otimes G/N \cong \mathbb{Z}_p^{(6)}$. On the other hand, by Theorem 2.5, since $G \otimes G \cong \mathbb{Z}_p^{(6)}$, $\mathbb{Z}_p^{(2)} \leq Z^{\otimes}(G)$. Again using Lemmas 1.2 and 3.1, we have $Z^{\otimes}(G) \cong \mathbb{Z}_p^{(2)}$.

$(vi)$. Let $G$ be isomorphic to one of groups $G_8, G_9, G_{11,1}, G_{42}, G_{44}, G_{46}, G_{48,1}$ or $G_{50_k}$. The proof is completely similar to $(v)$.

$(vii)$. Let $G$ be isomorphic to one of the groups $G_{15}, G_{16}, G_{20}, G_{35}$ or $G_{37}$. There is no normal subgroup $N$ such that $G \otimes G \cong G/N \otimes G/N$, so $Z^{\otimes}(G) \cong 1$. □

Here we summarize the results of the paper in the next tables. In the first table $cl(G)$( the nilpotency class of $G$), $\mathcal{M}(G), Z(G), G', G^{ab}(G/G'), \nabla(G)$, $J_2(G)$ and in the second table $cl(G), \mathcal{M}(G), G \wedge G, Z^{\wedge}(G), G \otimes G, Z^{\otimes}(G)$ of all groups of order dividing $p^5$ are given.

Table 1.

| **Type of** $G$ | $cl(G)$ | $\mathcal{M}(G)$ | $Z(G)$ | $G'$ | $G^{ab}$ | $\nabla(G)$ | $J_2(G)$ |
|---|---|---|---|---|---|---|---|
| $G_1$ | 1 | 1 | $\mathbb{Z}_{p^5}$ | 1 | $\mathbb{Z}_{p^5}$ | $\mathbb{Z}_{p^5}$ | $\mathbb{Z}_{p^5}$ |
| $G_2$ | 2 | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p$ | $\mathbb{Z}_{p^2}^{(2)}$ | $\mathbb{Z}_{p^2}^{(3)}$ | $\mathbb{Z}_p^{(3)} \oplus \mathbb{Z}_{p^2}^{(3)}$ |
| $G_3$ | 3 | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(6)}$ |
| $G_4$ | 3 | $\mathbb{Z}_p$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(4)}$ |
| $G_5$ | 3 | $\mathbb{Z}_p$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(4)}$ |
| $G_6$ | 3 | $\mathbb{Z}_p$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(4)}$ |
| $G_7$ | 3 | 1 | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(3)}$ |
| $G_8$ | 3 | 1 | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(3)}$ |
| $G_9$ | 3 | 1 | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(3)}$ |
| $G_{10}$ | 3 | $\mathbb{Z}_p$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(4)}$ |
| $G_{11,1} = G_{11_k}$ if $k \neq \frac{p-1}{2}$ | 3 | 1 | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(3)}$ |
| $G_{11,2} = G_{11_k}$ if $k = \frac{p-1}{2}$ | 3 | $\mathbb{Z}_p$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(4)}$ |
| $G_{12_k}$ | 3 | 1 | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(3)}$ |
| $G_{13}$ | 1 | $\mathbb{Z}_{p^2}$ | $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_{p^3}$ | 1 | $\mathbb{Z}_{p^3} \oplus \mathbb{Z}_{p^2}$ | $\mathbb{Z}_{p^3} \oplus \mathbb{Z}_{p^2}^{(2)}$ | $\mathbb{Z}_{p^2}^{(3)} \oplus \mathbb{Z}_{p^3}$ |
| $G_{14}$ | 2 | $\mathbb{Z}_p$ | $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p$ | $\mathbb{Z}_p$ | $\mathbb{Z}_{p^2}^{(2)}$ | $\mathbb{Z}_{p^2}^{(3)}$ | $\mathbb{Z}_{p^2}^{(3)} \oplus \mathbb{Z}_p$ |
| $G_{15}$ | 2 | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p$ | $\mathbb{Z}_p$ | $\mathbb{Z}_{p^3} \oplus \mathbb{Z}_p$ | $\mathbb{Z}_p^{(2)} \oplus \mathbb{Z}_{p^3}$ | $\mathbb{Z}_p^{(4)} \oplus \mathbb{Z}_{p^3}$ |
| $G_{16}$ | 3 | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p$ | $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(4)} \oplus \mathbb{Z}_{p^2}$ |
| $G_{17}$ | 3 | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p$ | $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(4)} \oplus \mathbb{Z}_{p^2}$ |
| $G_{18}$ | 3 | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p$ | $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(4)} \oplus \mathbb{Z}_{p^2}$ |
| $G_{19}$ | 3 | $\mathbb{Z}_p$ | $\mathbb{Z}_{p^2}$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p$ | $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(3)} \oplus \mathbb{Z}_{p^2}$ |
| $G_{20}$ | 3 | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p$ | $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(4)} \oplus \mathbb{Z}_{p^2}$ |

| **Type of $G$** | $cl(G)$ | $\mathcal{M}(G)$ | $Z(G)$ | $G'$ | $G^{ab}$ | $\nabla(G)$ | $J_2(G)$ |
|---|---|---|---|---|---|---|---|
| $G_{21}$ | 3 | $\mathbb{Z}_p$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p$ | $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(3)} \oplus \mathbb{Z}_{p^2}$ |
| $G_{22}$ | 3 | $\mathbb{Z}_p$ | $\mathbb{Z}_{p^2}$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p$ | $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(3)} \oplus \mathbb{Z}_{p^2}$ |
| $G_{23}$ | 3 | $\mathbb{Z}_p$ | $\mathbb{Z}_{p^2}$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p$ | $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(3)} \oplus \mathbb{Z}_{p^2}$ |
| $G_{24}$ | 3 | 1 | $\mathbb{Z}_p$ | $\mathbb{Z}_{p^2}$ | $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p$ | $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(2)} \oplus \mathbb{Z}_{p^2}$ |
| $G_{25}$ | 2 | $\mathbb{Z}_p$ | $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p$ | $\mathbb{Z}_p$ | $\mathbb{Z}_{p^3} \oplus \mathbb{Z}_p$ | $\mathbb{Z}_{p^3} \oplus \mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(3)} \oplus \mathbb{Z}_{p^3}$ |
| $G_{26}$ | 1 | $\mathbb{Z}_p$ | $\mathbb{Z}_{p^4} \oplus \mathbb{Z}_p$ | 1 | $\mathbb{Z}_{p^4} \oplus \mathbb{Z}_p$ | $\mathbb{Z}_{p^4} \oplus \mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(3)} \oplus \mathbb{Z}_{p^4}$ |
| $G_{27}$ | 2 | 1 | $\mathbb{Z}_{p^3}$ | $\mathbb{Z}_p$ | $\mathbb{Z}_{p^3} \oplus \mathbb{Z}_p$ | $\mathbb{Z}_{p^3} \oplus \mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(2)} \oplus \mathbb{Z}_{p^3}$ |
| $G_{28}$ | 4 | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p$ | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(6)}$ |
| $G_{29_a}$ | 4 | $\mathbb{Z}_p$ | $\mathbb{Z}_p$ | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(4)}$ |
| $G_{30}$ | 4 | $\mathbb{Z}_p$ | $\mathbb{Z}_p$ | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(4)}$ |
| $G_{31}$ | 4 | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p$ | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(6)}$ |
| $G_{32_a}$ | 4 | $\mathbb{Z}_p$ | $\mathbb{Z}_p$ | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(4)}$ |
| $G_{33_b}$ | 4 | $\mathbb{Z}_p$ | $\mathbb{Z}_p$ | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(4)}$ |
| $G_{34}$ | 2 | $\mathbb{Z}_p^{(6)}$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(6)}$ | $\mathbb{Z}_p^{(12)}$ |
| $G_{35}$ | 2 | $\mathbb{Z}_p^{(4)}$ | $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p$ | $\mathbb{Z}_p$ | $\mathbb{Z}_p^{(2)} \oplus \mathbb{Z}_{p^2}$ | $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(5)}$ | $\mathbb{Z}_p^{(9)} \oplus \mathbb{Z}_{p^2}$ |
| $G_{36}$ | 2 | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(6)}$ | $\mathbb{Z}_p^{(9)}$ |
| $G_{37}$ | 2 | $\mathbb{Z}_p^{(4)}$ | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p$ | $\mathbb{Z}_p^{(2)} \oplus \mathbb{Z}_{p^2}$ | $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(5)}$ | $\mathbb{Z}_p^{(9)} \oplus \mathbb{Z}_{p^2}$ |
| $G_{38}$ | 2 | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(6)}$ | $\mathbb{Z}_p^{(9)}$ |
| $G_{39}$ | 2 | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(6)}$ | $\mathbb{Z}_p^{(9)}$ |
| $G_{40}$ | 2 | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p$ | $\mathbb{Z}_p^{(2)} \oplus \mathbb{Z}_{p^2}$ | $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(5)}$ | $\mathbb{Z}_p^{(8)} \oplus \mathbb{Z}_{p^2}$ |
| $G_{41}$ | 2 | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(6)}$ | $\mathbb{Z}_p^{(8)}$ |
| $G_{42}$ | 2 | $\mathbb{Z}_p$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(6)}$ | $\mathbb{Z}_p^{(7)}$ |
| $G_{43}$ | 1 | $\mathbb{Z}_p^{(2)} \oplus \mathbb{Z}_{p^2}$ | $\mathbb{Z}_{p^2}^{(2)} \oplus \mathbb{Z}_p$ | 1 | $\mathbb{Z}_{p^2}^{(2)} \oplus \mathbb{Z}_p$ | $\mathbb{Z}_{p^2}^{(3)} \oplus \mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_{p^2}^{(4)} \oplus \mathbb{Z}_p^{(5)}$ |
| $G_{44}$ | 2 | $\mathbb{Z}_p$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(6)}$ | $\mathbb{Z}_p^{(7)}$ |
| $G_{45}$ | 2 | $\mathbb{Z}_p \oplus \mathbb{Z}_{p^2}$ | $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p$ | $\mathbb{Z}_p$ | $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(5)}$ | $\mathbb{Z}_p^{(6)} \oplus \mathbb{Z}_{p^2}^{(2)}$ |
| $G_{46}$ | 2 | $\mathbb{Z}_p$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(6)}$ | $\mathbb{Z}_p^{(7)}$ |
| $G_{47}$ | 2 | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p$ | $\mathbb{Z}_p$ | $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(5)}$ | $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(7)}$ |
| $G_{48,1} = G_{48_k}$ if $k \neq \frac{p-1}{2}$ | 2 | $\mathbb{Z}_p$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(6)}$ | $\mathbb{Z}_p^{(7)}$ |
| $G_{48,2} = G_{48_k}$ if $k = \frac{p-1}{2}$ | 2 | $\mathbb{Z}_{p^2}$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(6)}$ | $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(6)}$ |
| $G_{49}$ | 2 | $\mathbb{Z}_{p^2}$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(6)}$ | $\mathbb{Z}_p^{(6)} \oplus \mathbb{Z}_{p^2}$ |
| $G_{50_k}$ | 2 | $\mathbb{Z}_p$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(6)}$ | $\mathbb{Z}_p^{(7)}$ |
| $G_{51}$ | 1 | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_{p^3} \oplus \mathbb{Z}_p^{(2)}$ | 1 | $\mathbb{Z}_{p^3} \oplus \mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_{p^3} \oplus \mathbb{Z}_p^{(5)}$ | $\mathbb{Z}_p^{(8)} \oplus \mathbb{Z}_{p^3}$ |
| $G_{52}$ | 2 | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_{p^3}$ | $\mathbb{Z}_p$ | $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(5)}$ | $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(7)}$ |
| $G_{53}$ | 2 | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p$ | $\mathbb{Z}_p$ | $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(5)}$ | $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(7)}$ |

| **Type of** $G$ | $cl(G)$ | $\mathcal{M}(G)$ | $Z(G)$ | $G'$ | $G^{ab}$ | $\nabla(G)$ | $J_2(G)$ |
|---|---|---|---|---|---|---|---|
| $G_{54}$ | 3 | $\mathbb{Z}_p^{(4)}$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(6)}$ | $\mathbb{Z}_p^{(10)}$ |
| $G_{55}$ | 3 | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_{p^2}$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(6)}$ | $\mathbb{Z}_p^{(9)}$ |
| $G_{56}$ | 3 | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(6)}$ | $\mathbb{Z}_p^{(9)}$ |
| $G_{57}$ | 3 | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(6)}$ | $\mathbb{Z}_p^{(9)}$ |
| $G_{58}$ | 3 | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(6)}$ | $\mathbb{Z}_p^{(9)}$ |
| $G_{59}$ | 3 | $\mathbb{Z}_p^{(4)}$ | $\mathbb{Z}_p$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(6)}$ | $\mathbb{Z}_p^{(10)}$ |
| $G_{60}$ | 3 | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(6)}$ | $\mathbb{Z}_p^{(9)}$ |
| $G_{61}$ | 3 | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(6)}$ | $\mathbb{Z}_p^{(9)}$ |
| $G_{62}$ | 3 | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(6)}$ | $\mathbb{Z}_p^{(9)}$ |
| $G_{63}$ | 3 | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(6)}$ | $\mathbb{Z}_p^{(9)}$ |
| $G_{64}$ | 2 | $\mathbb{Z}_p^{(7)}$ | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p$ | $\mathbb{Z}_p^{(4)}$ | $\mathbb{Z}_p^{(10)}$ | $\mathbb{Z}_p^{(17)}$ |
| $G_{65}$ | 3 | $\mathbb{Z}_p^{(5)}$ | $\mathbb{Z}_p$ | $\mathbb{Z}_p$ | $\mathbb{Z}_p^{(4)}$ | $\mathbb{Z}_p^{(10)}$ | $\mathbb{Z}_p^{(15)}$ |
| $G_{66}$ | 1 | $\mathbb{Z}_p^{(6)}$ | $\mathbb{Z}_p^{(3)} \oplus \mathbb{Z}_{p^2}$ | 1 | $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(9)}$ | $\mathbb{Z}_p^{(15)} \oplus \mathbb{Z}_{p^2}$ |
| $G_{67}$ | 2 | $\mathbb{Z}_p^{(5)}$ | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p$ | $\mathbb{Z}_p^{(4)}$ | $\mathbb{Z}_p^{(10)}$ | $\mathbb{Z}_p^{(15)}$ |
| $G_{68}$ | 2 | $\mathbb{Z}_p^{(5)}$ | $\mathbb{Z}_p^{(2)} \oplus \mathbb{Z}_p$ | $\mathbb{Z}_p$ | $\mathbb{Z}_p^{(4)}$ | $\mathbb{Z}_p^{(10)}$ | $\mathbb{Z}_p^{(15)}$ |
| $G_{69}$ | 3 | $\mathbb{Z}_p^{(5)}$ | $\mathbb{Z}_p$ | $\mathbb{Z}_p$ | $\mathbb{Z}_p^{(4)}$ | $\mathbb{Z}_p^{(10)}$ | $\mathbb{Z}_p^{(15)}$ |
| $G_{70}$ | 1 | $\mathbb{Z}_p^{(10)}$ | $\mathbb{Z}_p^{(5)}$ | 1 | $\mathbb{Z}_p^{(5)}$ | $\mathbb{Z}_p^{(15)}$ | $\mathbb{Z}_p^{(25)}$ |

Table 2.

| **Type of** $G$ | $cl(G)$ | $\mathcal{M}(G)$ | $G \wedge G$ | $G \otimes G$ | $Z^{\wedge}(G)$ | $Z^{\otimes}(G)$ |
|---|---|---|---|---|---|---|
| $G_1$ | 1 | 1 | 1 | $\mathbb{Z}_{p^5}$ | $\mathbb{Z}_{p^5}$ | 1 |
| $G_2$ | 2 | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_{p^2}^{(4)} \oplus \mathbb{Z}_p^{(2)}$ | 1 | 1 |
| $G_3$ | 3 | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(6)}$ | $\mathbb{Z}_p^{(9)}$ | 1 | 1 |
| $G_4$ | 3 | $\mathbb{Z}_p$ | $\mathbb{Z}_p^{(4)}$ | $\mathbb{Z}_p^{(7)}$ | $\mathbb{Z}_p$ | $\mathbb{Z}_p$ |
| $G_5$ | 3 | $\mathbb{Z}_p$ | $\mathbb{Z}_p^{(4)}$ | $\mathbb{Z}_p^{(7)}$ | $\mathbb{Z}_p$ | $\mathbb{Z}_p$ |
| $G_6$ | 3 | $\mathbb{Z}_p$ | $\mathbb{Z}_p^{(4)}$ | $\mathbb{Z}_p^{(7)}$ | $\mathbb{Z}_p$ | $\mathbb{Z}_p$ |
| $G_7$ | 3 | 1 | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(6)}$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(2)}$ |
| $G_8$ | 3 | 1 | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(6)}$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(2)}$ |
| $G_9$ | 3 | 1 | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(6)}$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(2)}$ |
| $G_{10}$ | 3 | $\mathbb{Z}_p$ | $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(5)}$ | 1 | 1 |
| $G_{11,1} = G_{11_k}$ if $k \neq \frac{p-1}{2}$ | 3 | 1 | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(6)}$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(2)}$ |
| $G_{11,2} = G_{11_k}$ if $k = \frac{p-1}{2}$ | 3 | $\mathbb{Z}_p$ | $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(5)}$ | 1 | 1 |

| **Type of** $G$ | $cl(G)$ | $\mathcal{M}(G)$ | $G \wedge G$ | $G \otimes G$ | $Z^{\wedge}(G)$ | $Z^{\otimes}(G)$ |
|---|---|---|---|---|---|---|
| $G_{12_k}$ | 3 | 1 | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(6)}$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(2)}$ |
| $G_{13}$ | 1 | $\mathbb{Z}_{p^2}$ | $\mathbb{Z}_{p^2}$ | $\mathbb{Z}_{p^3} \oplus \mathbb{Z}_{p^2}^{(3)}$ | $\mathbb{Z}_p$ | 1 |
| $G_{14}$ | 2 | $\mathbb{Z}_p$ | $\mathbb{Z}_{p^2}$ | $\mathbb{Z}_{p^2}^{(4)}$ | $\mathbb{Z}_p$ | $\mathbb{Z}_p$ |
| $G_{15}$ | 2 | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_{p^3} \oplus \mathbb{Z}_p^{(5)}$ | $\mathbb{Z}_{p^2}$ | 1 |
| $G_{16}$ | 3 | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(4)}$ | $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(6)}$ | $\mathbb{Z}_p$ | 1 |
| $G_{17}$ | 3 | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_{p^2}^{(2)} \oplus \mathbb{Z}_p^{(4)}$ | 1 | 1 |
| $G_{18}$ | 3 | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_{p^2}^{(2)} \oplus \mathbb{Z}_p^{(4)}$ | 1 | 1 |
| $G_{19}$ | 3 | $\mathbb{Z}_p$ | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(5)}$ | $\mathbb{Z}_{p^2}$ | $\mathbb{Z}_p$ |
| $G_{20}$ | 3 | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(4)}$ | $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(6)}$ | $\mathbb{Z}_p$ | 1 |
| $G_{21}$ | 3 | $\mathbb{Z}_p$ | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(5)}$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p$ |
| $G_{22}$ | 3 | $\mathbb{Z}_p$ | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(5)}$ | $\mathbb{Z}_{p^2}$ | $\mathbb{Z}_p$ |
| $G_{23}$ | 3 | $\mathbb{Z}_p$ | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(5)}$ | $\mathbb{Z}_{p^2}$ | $\mathbb{Z}_p$ |
| $G_{24}$ | 3 | 1 | $\mathbb{Z}_{p^2}$ | $\mathbb{Z}_{p^2}^{(2)} \oplus \mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p$ | $\mathbb{Z}_p$ |
| $G_{25}$ | 2 | $\mathbb{Z}_p$ | $\mathbb{Z}_{p^2}$ | $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_{p^3} \oplus \mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p$ | 1 |
| $G_{26}$ | 1 | $\mathbb{Z}_p$ | $\mathbb{Z}_p$ | $\mathbb{Z}_{p^4} \oplus \mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_{p^3}$ | 1 |
| $G_{27}$ | 2 | 1 | $\mathbb{Z}_p$ | $\mathbb{Z}_{p^3} \oplus \mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_{p^3}$ | $\mathbb{Z}_p$ |
| $G_{28}$ | 4 | $\mathbb{Z}_p^{(3)}$ | $E_1 \times \mathbb{Z}_p^{(3)}$ | $E_1 \times \mathbb{Z}_p^{(6)}$ | 1 | 1 |
| $G_{29_a}$ | 4 | $\mathbb{Z}_p$ | $\mathbb{Z}_p^{(4)}$ | $\mathbb{Z}_p^{(7)}$ | $\mathbb{Z}_p$ | $\mathbb{Z}_p$ |
| $G_{30}$ | 4 | $\mathbb{Z}_p$ | $\mathbb{Z}_p^{(4)}$ | $\mathbb{Z}_p^{(7)}$ | $\mathbb{Z}_p$ | $\mathbb{Z}_p$ |
| $G_{31}$ | 4 | $\mathbb{Z}_p^{(3)}$ | $E_1 \times \mathbb{Z}_p^{(3)}$ | $E_1 \times \mathbb{Z}_p^{(6)}$ | 1 | 1 |
| $G_{32_a}$ | 4 | $\mathbb{Z}_p$ | $\mathbb{Z}_p^{(4)}$ | $\mathbb{Z}_p^{(7)}$ | $\mathbb{Z}_p$ | $\mathbb{Z}_p$ |
| $G_{33_b}$ | 4 | $\mathbb{Z}_p$ | $\mathbb{Z}_p^{(4)}$ | $\mathbb{Z}_p^{(7)}$ | $\mathbb{Z}_p$ | $\mathbb{Z}_p$ |
| $G_{34}$ | 2 | $\mathbb{Z}_p^{(6)}$ | $\mathbb{Z}_p^{(8)}$ | $\mathbb{Z}_p^{(14)}$ | 1 | 1 |
| $G_{35}$ | 2 | $\mathbb{Z}_p^{(4)}$ | $\mathbb{Z}_p^{(5)}$ | $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(10)}$ | $\mathbb{Z}_p$ | 1 |
| $G_{36}$ | 2 | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(5)}$ | $\mathbb{Z}_p^{(11)}$ | $\mathbb{Z}_p$ | $\mathbb{Z}_p$ |
| $G_{37}$ | 2 | $\mathbb{Z}_p^{(4)}$ | $\mathbb{Z}_p^{(5)}$ | $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(10)}$ | $\mathbb{Z}_p$ | 1 |
| $G_{38}$ | 2 | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(5)}$ | $\mathbb{Z}_p^{(11)}$ | $\mathbb{Z}_p$ | $\mathbb{Z}_p$ |
| $G_{39}$ | 2 | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(5)}$ | $\mathbb{Z}_p^{(11)}$ | $\mathbb{Z}_p$ | $\mathbb{Z}_p$ |
| $G_{40}$ | 2 | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_{p^2}^{(2)} \oplus \mathbb{Z}_p^{(7)}$ | 1 | 1 |
| $G_{41}$ | 2 | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(8)}$ | 1 | 1 |
| $G_{42}$ | 2 | $\mathbb{Z}_p$ | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(9)}$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(2)}$ |
| $G_{43}$ | 1 | $\mathbb{Z}_p^{(2)} \oplus \mathbb{Z}_{p^2}$ | $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_{p^2}^{(4)} \oplus \mathbb{Z}_p^{(5)}$ | 1 | 1 |
| $G_{44}$ | 2 | $\mathbb{Z}_p$ | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(9)}$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(2)}$ |
| $G_{45}$ | 2 | $\mathbb{Z}_p \oplus \mathbb{Z}_{p^2}$ | $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_{p^2}^{(2)} \oplus \mathbb{Z}_p^{(7)}$ | 1 | 1 |

| Type of $G$ | $cl(G)$ | $\mathcal{M}(G)$ | $G \wedge G$ | $G \otimes G$ | $Z^{\wedge}(G)$ | $Z^{\otimes}(G)$ |
|---|---|---|---|---|---|---|
| $G_{46}$ | 2 | $\mathbb{Z}_p$ | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(9)}$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(2)}$ |
| $G_{47}$ | 2 | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(8)}$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p$ |
| $G_{48,1} = G_{48_k}$ if $k \neq \frac{p-1}{2}$ | 2 | $\mathbb{Z}_p$ | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(9)}$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(2)}$ |
| $G_{48,2} = G_{48_k}$ if $k = \frac{p-1}{2}$ | 2 | $\mathbb{Z}_{p^2}$ | $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(8)}$ | 1 | 1 |
| $G_{49}$ | 2 | $\mathbb{Z}_{p^2}$ | $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(8)}$ | 1 | 1 |
| $G_{50_k}$ | 2 | $\mathbb{Z}_p$ | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(9)}$ | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(2)}$ |
| $G_{51}$ | 1 | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_{p^3} \oplus \mathbb{Z}_p^{(8)}$ | $\mathbb{Z}_{p^2}$ | 1 |
| $G_{52}$ | 2 | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(8)}$ | $\mathbb{Z}_{p^2}$ | $\mathbb{Z}_p$ |
| $G_{53}$ | 2 | $\mathbb{Z}_p^{(2)}$ | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(8)}$ | $\mathbb{Z}_{p^2}$ | $\mathbb{Z}_p$ |
| $G_{54}$ | 3 | $\mathbb{Z}_p^{(4)}$ | $\mathbb{Z}_p^{(6)}$ | $\mathbb{Z}_p^{(12)}$ | 1 | 1 |
| $G_{55}$ | 3 | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(5)}$ | $\mathbb{Z}_p^{(11)}$ | $\mathbb{Z}_p$ | $\mathbb{Z}_p$ |
| $G_{56}$ | 3 | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(5)}$ | $\mathbb{Z}_p^{(11)}$ | $\mathbb{Z}_p$ | $\mathbb{Z}_p$ |
| $G_{57}$ | 3 | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(5)}$ | $\mathbb{Z}_p^{(11)}$ | $\mathbb{Z}_p$ | $\mathbb{Z}_p$ |
| $G_{58}$ | 3 | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(5)}$ | $\mathbb{Z}_p^{(11)}$ | $\mathbb{Z}_p$ | $\mathbb{Z}_p$ |
| $G_{59}$ | 3 | $\mathbb{Z}_p^{(4)}$ | $\mathbb{Z}_p^{(6)}$ | $\mathbb{Z}_p^{(12)}$ | 1 | 1 |
| $G_{60}$ | 3 | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(5)}$ | $\mathbb{Z}_p^{(11)}$ | $\mathbb{Z}_p$ | $\mathbb{Z}_p$ |
| $G_{61}$ | 3 | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(5)}$ | $\mathbb{Z}_p^{(11)}$ | $\mathbb{Z}_p$ | $\mathbb{Z}_p$ |
| $G_{62}$ | 3 | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(5)}$ | $\mathbb{Z}_p^{11)}$ | $\mathbb{Z}_p$ | $\mathbb{Z}_p$ |
| $G_{63}$ | 3 | $\mathbb{Z}_p^{(3)}$ | $\mathbb{Z}_p^{(5)}$ | $\mathbb{Z}_p^{(11)}$ | $\mathbb{Z}_p$ | $\mathbb{Z}_p$ |
| $G_{64}$ | 2 | $\mathbb{Z}_p^{(7)}$ | $\mathbb{Z}_p^{(8)}$ | $\mathbb{Z}_p^{(18)}$ | 1 | 1 |
| $G_{65}$ | 3 | $\mathbb{Z}_p^{(5)}$ | $\mathbb{Z}_p^{(6)}$ | $\mathbb{Z}_p^{(16)}$ | $\mathbb{Z}_p$ | $\mathbb{Z}_p$ |
| $G_{66}$ | 1 | $\mathbb{Z}_p^{(6)}$ | $\mathbb{Z}_p^{(6)}$ | $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(15)}$ | $\mathbb{Z}_p$ | 1 |
| $G_{67}$ | 2 | $\mathbb{Z}_p^{(5)}$ | $\mathbb{Z}_p^{(6)}$ | $\mathbb{Z}_p^{(16)}$ | $\mathbb{Z}_p$ | $\mathbb{Z}_p$ |
| $G_{68}$ | 2 | $\mathbb{Z}_p^{(5)}$ | $\mathbb{Z}_p^{(6)}$ | $\mathbb{Z}_p^{(16)}$ | $\mathbb{Z}_p$ | $\mathbb{Z}_p$ |
| $G_{69}$ | 3 | $\mathbb{Z}_p^{(5)}$ | $\mathbb{Z}_p^{(6)}$ | $\mathbb{Z}_p^{(16)}$ | $\mathbb{Z}_p$ | $\mathbb{Z}_p$ |
| $G_{70}$ | 1 | $\mathbb{Z}_p^{(10)}$ | $\mathbb{Z}_p^{(10)}$ | $\mathbb{Z}_p^{(25)}$ | 1 | 1 |

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