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The Numerical Solution of Some Optimal Control Systems with Constant and Pantograph Delays via Bernstein Polynomials

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Abstract. In this paper, we present a numerical method based on Bernstein polynomials to solve optimal control systems with constant and pantograph delays. Constant or pantograph delays may appear in state-control or both. We derive delay operational matrix and pantograph operational matrix for Bernstein polynomials then, these are utilized to reduce the solution of optimal control with constant and pantograph delay to the solution of nonlinear programming. In truth, the principal problem can be transferred to the quadratic programming problem. Some examples are included to demonstrate the validity and applicability of the technique.

Keywords: Optimal control with pantograph systems, Optimal control with time delay, Pantograph delay differential equation, Bernstein polynomials.

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1. Introduction

Time delay can be obtained from organic, chemical, electronic and transportation systems [17]. It is crucial and complicated to obtain the analytic solution of the optimal control for delay systems, so it is intriguing to work with numerical approaches to solving optimal control with time delay systems. During the past decade, enormous effort has been spent on the development of computational methods for generating solutions of delay systems. Wu. et al introduced a method for solving optimal control of switched systems with time delay [27]. Kharatishvili [18] approached this issue by extending the Pontryagin's maximum principle to time delay systems. With [20] authors presented a technique for solving time delay optimal control problems based on Block plus function and Legendre polynomials. Also, Wang [25] solved delay optimal control with the same procedure. Palanisamy and Rao [24], solved this issue using Walsh functions. The least square method based on Bezier curves is used in solving linear optimal control with time delay systems [14]. Recently, a number of articles have considered operational matrices for optimal control problems such as [6, 8, 9, 10, 11].

The pantograph systems are one of the most sorts of delay differential equations as well as performs a crucial role in describing various different phenomena such as electrodynamics [4]. The pantograph equation studied by using the Adomian decomposition method [5]. The stableness of the Runge Kutta procedure was considered for a class of pantograph equations of the neutral form [29]. Taylor polynomials were employed for the approximate solution of a linear Pantograph equation [23]. In [12] Chebyshev wavelets presented to unravel the linear quadratic optimal control problem with pantograph systems. Furthermore, the solving of multi-pantograph equation systems via spectral tau method is studied in [7], and a new Legendre operational technique for delay fractional optimal control problems is presented in [3].

In the recent years, Bernstein polynomials are employed in many articles, for instance, having [16] authors have used them to unravel fractional optimal control problems. Besides within [1], they are presented for solving optimal control problem of time-varying Singular Systems. Specified methods in these articles are based on Lagrangian method, in the point of fact for solving the problems according to these papers we must solve nonlinear algebraic systems which need more computation and time, but with the proposed method in this paper, it is only needed to solve quadratic programming problems(QPP). Indeed in this paper, Bernstein polynomials are used to solve optimal control with constant and pantograph delay systems. We derive an operational matrix of pantograph, an operational matrix of delay, a constant operational matrix of integration and an operational matrix of differentiation for Bernstein polynomials. These matrices are used to reduce the pre-mentioned optimal control systems to a QPP.

2. Properties of Bernstein polynomials

The Bernstein polynomials of mth degree are defined on the interval $[0, t_f]$ as follows: [2]

$$B_{i,m}(t) = \binom{m}{i} \frac{1}{(t_f - t_0)^m} t^i (t_f - t)^{m-i}, \ i = 0, 1, \ldots, m. \tag{2.1}$$

By utilizing binomial expansion of $(t_f - t)^{m-i}$, we have now

$$\begin{aligned} B_{i,m}(t) &= \binom{m}{i} \left(\frac{1}{(t_f)^m}\right) t^i (t_f - t)^{m-i}, \\ &= \left(\frac{1}{(t_f)^m}\right) t^i \binom{m}{i} \left(\sum_{k=0}^{m-i} (-1)^k \binom{m-i}{k} t^k t_f^{m-i-k}\right), \\ &= \left(\frac{1}{(t_f)^m}\right) \sum_{k=0}^{m-i} (-1)^k \binom{m}{i} \binom{m-i}{k} t^{i+k} t_f^{m-i-k}, \ i = 0, 1, \ldots, m. \end{aligned} \tag{2.2}$$

one may define vector A_{i+1} as:

$$\begin{aligned} A_{i+1} = \frac{1}{(t_f)^m} \Bigg[& \overbrace{0\,0 \cdots\ 0}^{i \text{ times}} \ (-1)^0 \binom{m}{i} t_f^{m-i} \ (-1)^1 \binom{m}{i} \binom{m-i}{1} t_f^{m-i-1} \ldots \\ & (-1)^{m-i-1} \binom{m}{i} \binom{m-i}{m-i-1} t_f \ (-1)^{m-i} \binom{m}{i} \binom{m-i}{m-i} \Bigg]_{1 \times (m+1)} \end{aligned}$$

and

$$\phi(t) = \left[B_{0,m}(t) \ B_{1,m}(t) \ \ldots \ B_{m,m}(t)\right]^T, \tag{2.3}$$

Then we can write

$$\phi(t) = A T_m(t), \tag{2.4}$$

where

$$T_m(t) = \begin{bmatrix} 1 \\ t \\ \vdots \\ t^m \end{bmatrix}, \quad A = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_{m+1} \end{bmatrix}. \tag{2.5}$$

Thus A is an $(m+1) \times (m+1)$ matrix which A_{i+1} is $i+1$ row of A.

3. Function Approximation

Let $H = L^2[0, t_f]$ is Hilbert space with the inner product which is described by $\langle f, g\rangle = \int_0^{t_f} fg\ dt$. Also $Y = span\{B_{0,m}, B_{1,m}, \cdots, B_{m,m}\}$ is a finite dimensional and closed subspace. As a result Y a is complete subspace of H. Suppose f is an arbitrary element in H, now f has a unique best approximation out of Y, such as $y_0 \in Y$, that is:

$$\exists\ y_0 \in Y, \quad s.t.\ \forall\ y \in Y, \quad \|f - y_0\|_2 \leqslant \|f - y\|_2, \tag{3.1}$$

where $\|f\|_2^2 = \langle f, f\rangle$, (For details see [26])

Since $y_0 \in Y$, so there exist the unique coefficients $c_0, c_1, \ldots, c_m$ such that

$$f(t) \simeq y_0 = \sum_{i=0}^{m} c_i B_{i,m} = c^T\phi(t), \tag{3.2}$$

where $\phi(t)$ is defined in (2.3) and $c^T = [c_0, c_1, \ldots, c_m]$. Thus vector c^T can be obtained by:

$$c^T\langle\phi, \phi\rangle = \langle f, \phi\rangle, \tag{3.3}$$

where

$$\langle f, \phi\rangle = \int_0^{t_f} f\phi\ dt = [\langle f, B_{0,m}\rangle\ \langle f, B_{1,m}\rangle\ \ldots\ \langle f, B_{m,m}\rangle], \tag{3.4}$$

and $\langle\phi, \phi\rangle$ is an $(m+1)\times(m+1)$ matrix and is called as dual matrix of ϕ, where is denoted by Q, so

$$c^T = (\int_0^{t_f} f(t)\phi(t)^T\ dt)Q^{-1}. \tag{3.5}$$

By exerting (2.4), we have:

$$Q = \langle\phi, \phi\rangle = \int_0^{t_f} \phi(t)\phi(t)^T\ dt = \int_0^{t_f} (AT_m(t))(AT_m(t))^T\ dt$$
$$= A[\int_0^{t_f} T_m(t)T_m(t)^T\ dt]A^T = AHA^T, \tag{3.6}$$

where

$$H = \begin{bmatrix} t_f & \frac{t_f^2}{2} & \frac{t_f^3}{3} & \cdots & \frac{t_f^{m+1}}{m+1} \\ \frac{t_f^2}{2} & \frac{t_f^3}{3} & \frac{t_f^4}{4} & \cdots & \frac{t_f^{m+2}}{m+2} \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{t_f^{m+1}}{m+1} & \frac{t_f^{m+2}}{m+2} & \frac{t_f^{m+3}}{m+3} & \cdots & \frac{t_f^{2m+1}}{2m+1} \end{bmatrix}. \tag{3.7}$$

Lemma 3.1. *Consider the real-valued function g, where $g \in C^{m+1}[0, t_f]$, as well as $Y = span\{B_{0,m}, B_{1,m}, \cdots, B_{m,m}\}$. If $c^T\phi(t)$ be the best approximation g out of Y then the mean error bound is presented as follows:*

$$\|g - c^T\phi(t)\|_2 \leqslant \frac{Mt_f^{\frac{2m+3}{2}}}{(m+1)!\sqrt{2m+3}}, \tag{3.8}$$

where

$$M = max \mid g^{m+1}(\eta) \mid, \ \eta \in [0, t_f],$$

Proof. We consider the Taylor polynomial of order m for function g at $t_0 = 0$, and denote it by $y_1(t)$, so

$$y_1(t) = g(t_0) + g'(t_0)(t - t_0) + \cdots + g^m(t_0)\frac{(t - t_0)^m}{m!}.$$

We know that

$$|g(t) - y_1(t)| \leqslant \mid g^{m+1}(\eta) \mid \frac{(t - t_0)^{m+1}}{(m + 1)!}, \tag{3.9}$$
$$\eta \in [t_0, t_f].$$

Since $c^T \phi(t)$ is the best approximation to g from Y, and $y_1 \in Y$, by using (3.9) we obtain:

$$\begin{aligned}
&\|g - c^T \phi(t)\|_2^2 \leqslant \|g - y_1\|_2^2 = \int_0^{t_f} \mid g(t) - y_1(t) \mid^2 \mathrm{d}t, \\
&\leqslant \int_0^{t_f} \left[g^{m+1}(\eta)\frac{t^{m+1}}{(m+1)!}\right]^2 \mathrm{d}t \leqslant \frac{M^2}{(m+1)!^2} \int_0^{t_f} t^{2m+2}\mathrm{d}t, \\
&\leqslant \frac{M^2 t_f^{2m+3}}{(m+1)!^2(2m+3)}.
\end{aligned}$$

Therefore by taking the square root we obtain the bound indicated in (3.8). □

This lemma shows that the error reduces to zero as m increases. In other words this, lemma expresses that the approximated solution is obtained by the presented method convergences to the exact solution when, m tends to infinity.

4. Operational Matrices

In this section, we describe obtaining some operational matrices on Bernstein polynomials that can reduce the basic dynamical systems to QPP.

4.1. **Operational matrix of differentiation.** Suppose D is an $(m+1) \times (m+1)$ differentiation operational matrix of mth-degree Bernstein polynomials over $[0, t_f]$, then

$$\frac{d}{dt}\phi(t) = A\Lambda B \ \phi(t) = D\phi(t), \quad 0 \leqslant t \leqslant t_f. \tag{4.1}$$

where $D = A\Lambda B$, while A is the matrix defined in (2.5) and Λ is the following $(m + 1) \times (m)$ matrix

$$\Lambda = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & m \end{bmatrix},$$

and B is an $(m) \times (m+1)$ matrix can be expressed by $B_{k+1} = A^{-1}_{k+1}$ for $k = 0, 1, \dots, m$, where A^{-1}_{k+1} is $k+1$ row of A^{-1}. For more information see [28]

4.2. **Operational matrix of integral.** Assume

$$\int_0^t \phi(x)\, dx = P\phi(t), \qquad 0 \leqslant t \leqslant 1, \tag{4.2}$$

then P is the operational matrix of integral and $P = A\Lambda^{'}B^{'}$, where A is the matrix defined in (2.4) when $t_f = 1$, also $B^{'}$ and $\Lambda^{'}$ defined as follows:

$$\Lambda^{'} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{2} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & \frac{1}{m+1} \end{bmatrix}, \quad B^{'} = \begin{bmatrix} A^{-1}_2 \\ A^{-1}_3 \\ \vdots \\ A^{-1}_{m+1} \\ c^T_{m+1} \end{bmatrix}.$$

where

$$c_{m+1} = \frac{Q^{-1}}{2m+2} \begin{bmatrix} \frac{\binom{m}{0}}{\binom{2m+1}{m+1}} \\ \frac{\binom{m}{1}}{\binom{2m+1}{m+2}} \\ \vdots \\ \frac{\binom{m}{m}}{\binom{2m+1}{2m+1}} \end{bmatrix}.$$

While Q^{-1} is the inverse of Q defined in (3.6), when $t_f = 1$. For more information see [28]

4.3. **Operational matrix of pantograph delay.** We derive the operational matrix of pantograph $D_p(\tau)$, that is given by

$$\phi(\frac{t}{\tau}) = D_p(\tau)\phi(t), \qquad 0 \leqslant t \leqslant t_f, \tag{4.3}$$

The (i,j)-th entry of $D_p(\tau)$ is

$$[D_p(\tau)](i,j) = \begin{cases} \binom{j}{j-i} \dfrac{(\tau-1)^{j-i}}{\tau^j} & j \geqslant i \\ 0 & j < i. \end{cases}$$

For $i = 0, 1, \ldots, m$ and $j = 0, 1, \ldots, m$, the matrix of $D_p(\tau)$ is an upper triangular matrix and defined as:

$$D_p(\tau) = \begin{bmatrix} 1 & \frac{(\tau-1)}{\tau} & \frac{(\tau-1)^2}{\tau^2} & \cdots & \frac{(\tau-1)^m}{\tau^m} \\ 0 & \frac{1}{\tau} & \frac{2(\tau-1)}{\tau^2} & \cdots & \frac{m(\tau-1)^{(m-1)}}{\tau^m} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{\tau^m} \end{bmatrix}. \tag{4.4}$$

4.4. **Operational matrix of constant delay.** The operational matrix of delay $D(\tau)$ is given by

$$\phi(t-\tau) = D(\tau)\ \phi(t), \qquad 0 \leqslant t \leqslant 1, \tag{4.5}$$

Let the $(m+1) \times (m+1)$ matrix of $D(\tau)$ is

$$D(\tau) = \begin{bmatrix} D_{0,0}(\tau) & D_{0,1}(\tau) & D_{0,2}(\tau) & \ldots & D_{0,m}(\tau) \\ D_{1,0}(\tau) & D_{1,1}(\tau) & D_{2,2}(\tau) & \ldots & D_{1,m}(\tau) \\ \vdots & \vdots & \vdots & & \vdots \\ D_{m,0}(\tau) & D_{m,1}(\tau) & D_{m,2}(\tau) & \ldots & D_{m,m}(\tau) \end{bmatrix}. \tag{4.6}$$

The (i, j)-th element of delay operational matrix, where is:

$$D_{i,j}(\tau) = (-1)^i \left[\sum_{k=0}^{i} (-1)^{i+k} \binom{j}{j-i+k} \frac{(m-i+k)!}{k!(m-i)!} \tau^{2k} \right] (\tau+1)^{m-i-j} \tau^{j-i},$$

For

$$i = 0, 1, \ldots, m \textit{ and } j = 0, 1, \ldots, m.$$

4.5. **Operational matrix of constant integral.** Define

$$\int_0^\tau \phi(t)\ dt = Z\phi(t), \qquad 0 \leqslant t \leqslant 1,$$

the matrix Z is called the constant operational matrix of integral, $i+1$ th row of Z is

$$Z_i = K_i\ [\overbrace{1\ 1\ \cdots\ 1}^{m+1\ times}], \qquad i = 1, 2 \ldots, m+1.$$

where

$$K_i = \frac{1}{m+1} \tau^i \left[\sum_{j=0}^{m-i+1} (-1)^j \binom{m+1}{i+j} \binom{i+j-1}{j} \tau^j \right],\ i = 1, 2 \ldots, m+1$$

Thus the $(m+1)\times(m+1)$ matrix Z is

$$Z = \begin{bmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_{m+1} \end{bmatrix}. \tag{4.7}$$

5. The Approximated Solution of Delays Optimal Control Systems

5.1. **Constant delay.** Consider the following time-invariant delay approximated system,

$$\dot{x}(t) = Ax(t) + Bu(t) + Cx(t-\tau), \qquad 0 \leqslant t \leqslant 1, \tag{5.1}$$

$$x(0) = x_0, \tag{5.2}$$

$$x(t) = \theta(t), \qquad -\tau \leqslant t \leqslant 0, \tag{5.3}$$

where $x(t)$ is a l vector state function, $u(t)$ is a q vector control function, A, B and C are matrices of appropriate dimensions, x_0 is a constant specified vector, and θ is arbitrary known function. The problem is to find the pair of optimal solution $(x^*(.), u^*(.))$ which satisfies (5.1), (5.2) and (5.3) and minimizes the following quadratic objective functional

$$J = \frac{1}{2}\int_0^{t_f} \left[x^T(t)Px(t) + u^T(t)Ru(t)\right]\, \mathbf{d}t, \tag{5.4}$$

where P and R are matrices of appropriate dimensions, and they are respectively the symmetric semi positive and positive definite matrices.
We approximate $x(t)$ and $u(t)$ by $X(t)$ and $U(t)$, respectively. Let

$$X(t) = \left[X_1(t)\ X_2(t)\ \cdots\ X_l(t)\right]^T, \tag{5.5}$$

$$U(t) = \left[U_1(t)\ U_2(t)\ \cdots\ U_q(t)\right]^T. \tag{5.6}$$

Each $X_i(t)$ and $U_j(t)$ where $i = 1, \cdots, l$ and $j = 1, \cdots, q$ can be written in term of Bernstein polynomials as follows:

$$X_i(t) = X_i^T\phi(t),$$
$$U_j(t) = U_j^T\phi(t),$$

where $\phi(t) = [B_{0,m}(t)\ B_{1,m}(t)\ \ldots\ B_{m,m}(t)]^T$ and $X_i^T = [\alpha_{0i}\ \alpha_{1i}\ \ldots\ \alpha_{mi}]$. Similarly, we have

$$x(0) \simeq X(0) = c^T\phi(t),$$
$$\theta(t-\tau) \simeq \theta(t) = \theta^T\phi(t),$$

where c and θ are $(m+1)\times l$ matrices, thus

$$X_i(0) = c_i^T \phi(t),$$
$$\theta_i(t-\tau) = \theta_i^T \phi(t), \quad i = 1, 2, \ldots, l$$

where

$$c_i = [c_{0i}\ c_{1i}\ \cdots\ c_{mi}]^T,$$
$$\theta_i = [\theta_{0i}\ \theta_{1i}\ \cdots\ \theta_{mi}]^T,$$

We can also write $X_i(t-\tau)$ in term of Bernstein polynomials as follows:

$$X_i(t-\tau) = \begin{cases} \theta_i(t-\tau) = \theta_i^T \phi(t), & 0 \leqslant t \leqslant \tau, \\ X_i^T \phi(t-\tau) = X_i^T D(\tau)\phi(t), & \tau \leqslant t \leqslant 1, \quad i = 1, 2, \ldots, l \end{cases}$$

where $D(\tau)$ is delay operational matrix given in (4.6), also we have:

$$\int_0^t X_i(t^{'})dt^{'} = \int_0^t X_i^T \phi(t^{'})dt^{'} = X_i^T P\phi(t), \tag{5.7}$$

$$\int_0^t U_j(t^{'})dt^{'} = \int_0^t U_j^T \phi(t^{'})dt^{'} = U_j^T P\phi(t), \tag{5.8}$$

$$\int_0^\tau \phi(t)dt = Z\phi(t), \tag{5.9}$$

$$\int_0^t X_i(t^{'}-\tau)dt^{'} = \int_0^\tau X_i(t^{'}-\tau)dt^{'} + \int_\tau^t X_i(t^{'}-\tau)dt^{'}, \quad i = 1, 2, \ldots, l \tag{5.10}$$

one can show that

$$\int_0^\tau X_i(t^{'}-\tau)dt^{'} = \int_0^\tau \theta_i^T \phi(t^{'})dt^{'} = \theta_i^T \int_0^\tau \phi(t^{'})dt^{'} = \theta_i^T Z\phi(t), \tag{5.11}$$

$$\begin{aligned}\int_\tau^t X_i(t^{'}-\tau)dt^{'} &= \int_\tau^t X_i^T D(\tau)\phi(t^{'})dt^{'} = \int_0^t X_i^T D(\tau)\phi(t^{'})dt^{'} - \int_0^\tau X_i^T D(\tau)\phi(t^{'})dt^{'} \\ &= X_i^T D(\tau)P\phi(t) - X_i^T D(\tau)Z\phi(t), \quad i = 1, 2, \ldots, l\end{aligned} \tag{5.12}$$

Thus

$$\int_0^t X_i(t^{'}-\tau)dt^{'} = \theta_i^T Z\phi(t) + X_i^T D(\tau)P\phi(t) - X_i^T D(\tau)Z\phi(t). \quad i = 1, 2, \ldots, l. \tag{5.13}$$

Now back to the optimal control problem (5.1)-(5.4). By integrating (5.1) from 0 to t, and using (5.7)-(5.13) we obtain:

$$X_i(t) - X_i(0) = \int_0^t \dot{X}_i(t^{'})dt^{'} = A\int_0^t X_i(t^{'})dt^{'} + B\int_0^t U_j(t^{'})dt^{'} + C\int_0^t X_i(t^{'}-\tau)dt^{'}$$

Then,

$$X_i^T \phi(t) - c_i^T \phi(t) = AX_i^T P\phi(t) + BU_j^T P\phi(t) + C\theta_i^T Z\phi(t) + CX_i^T D(\tau)P\phi(t) - CX_i^T D(\tau)Z\phi(t).$$

With eliminating ϕ and putting whole of terms on the left-hand side of equalization, we obtain:

$$\begin{aligned} R_1^i =& X_i^T - c_i^T - AX_i^T P - BU_j^T P - C\theta_i^T Z - \\ & CX_i^T D(\tau)P + CX_i^T D(\tau)Z = 0, \quad i = 1, 2, \dots, l. \end{aligned} \tag{5.14}$$

It is clear R_1^i is $m+1$ vector for $i = 1, 2, \dots, l$. Finally, we approximate of the objective function (5.4) as follows:

$$\begin{aligned} x^T(t)Px(t) &= \sum_{i=1}^{l} p_{ii} x_i^2 + 2\sum_{i=1}^{l} \sum_{j=i+1}^{l-i} p_{ij} x_i x_j, \\ &= \sum_{i=1}^{l} p_{ii} X_i^T \phi(t)\phi(t)^T X_i + 2\sum_{i=1}^{l} \sum_{j=i+1}^{l-i} p_{ij} X_i^T \phi(t)\phi(t)^T X_j, \\ u^T(t)Ru(t) &= \sum_{i=1}^{q} r_{ii} u_i^2 + 2\sum_{i=1}^{l} \sum_{j=i+1}^{q-i} r_{ij} u_i u_j, \\ &= \sum_{i=1}^{q} r_{ii} U_i^T \phi(t)\phi(t)^T U_i + 2\sum_{i=1}^{l} \sum_{j=i+1}^{q-i} r_{ij} U_i^T \phi(t)\phi(t)^T U_j, \end{aligned} \tag{5.15}$$

By substituting (5.15) into (5.4), we obtain:

$$\begin{aligned} R_2 = & \frac{1}{2}\sum_{i=1}^{l} p_{ii} X_i^T \left[\int_0^1 \phi(t)\phi^T(t)dt\right] X_i + \sum_{i=1}^{l} \sum_{j=i+1}^{l-i} p_{ij} X_i^T \left[\int_0^1 \phi(t)\phi^T(t)dt\right] X_j \\ & + \frac{1}{2}\sum_{i=1}^{q} r_{ii} U_i^T \left[\int_0^1 \phi(t)\phi^T(t)dt\right] U_i + \sum_{i=1}^{q} \sum_{j=i+1}^{q-i} r_{ij} U_i^T \left[\int_0^1 \phi(t)\phi^T(t)dt\right] U_j \end{aligned} \tag{5.16}$$

Then (5.4) can be rewritten as follows:

$$R_2 = \frac{1}{2}\sum_{i=1}^{l} p_{ii} X_i^T Q X_i + \sum_{i=1}^{l} \sum_{j=i+1}^{l-i} p_{ij} X_i^T Q X_j + \frac{1}{2}\sum_{i=1}^{q} r_{ii} U_i^T Q U_i + \sum_{i=1}^{q} \sum_{j=i+1}^{q-i} r_{ij} U_i^T Q U_j \tag{5.17}$$

where

$$\int_0^1 \phi(t)\phi^T(t)\mathrm{d}t = Q,$$

we may recall that Q is the same at (3.6) when $t_f = 1$. Also, we may recall that Q and P are positive definite matrices, so R_2 is a non-negative quadratic form. Now the delay optimal control problem (5.1)- (5.4) can be reduced to the following quadratic programming problem (QPP).

$$\begin{aligned} \min \quad & M\sum_{i=1}^{l} \|R_1^i\|^2 + R_2, \\ s.t. \quad & X_i^T \phi(0) = x_{0i}. \quad i = 1, 2, \dots, l \end{aligned} \tag{5.18}$$

Since the equation on R_1^i is linear equality, the original problem can be reformulated as the quadratic programming problem. The new problem consists only the entries of the vectors X and U. The norm we used in this optimization problem is the Euclidean norm, and M is a penalty parameter. The QPP (5.18) can be solved by many software, in this paper we used the package of Mathematica 10 to solve this problem. The more articles about solving optimal control problem with approximated method utilize Lagrangian approach, however, we approximate the main problem with QPP that can be solved by many subroutine algorithms and software. In other words by exerting of this method the main problem transfer to a QPP which applying it is easy and interesting with more accurate and less computation.

5.2. **Pantograph delay.** Now consider the following linear time-varying pantograph system

$$\dot{x}(t) = E(t)x(t) + G(t)u(t) + F(t)x(\frac{t}{\tau}) + H(t), \qquad 0 \leqslant t \leqslant t_f, \tag{5.19}$$

$$x(0) = x_0, \tag{5.20}$$

where $x(t), u(t) \in R$, and x_0 is a known constant, and $F(t)$, $G(t)$, $E(t)$ and $H(t)$ are specified functions and are given. The propose is to find the pair of optimal solution $(x^*(.), u^*(.))$ where satisfies (5.19) and (5.20) while minimizes the following quadratic objective functional,

$$J = \frac{1}{2} \int_0^{t_f} x^T(t)Px(t) + u^T(t)Ru(t)\ \mathbf{d}t, \tag{5.21}$$

where P and R are appropriate given constants.
We approximate $x(t)$ and $u(t)$ by $X(t)$ and $U(t)$, respectively. We assume

$$\begin{aligned} X(t) &= X^T\phi(t),\\ U(t) &= U^T\phi(t),\\ E(t) &\simeq E^T\phi(t),\\ F(t) &\simeq F^T\phi(t),\\ H(t) &\simeq H^T\phi(t), \end{aligned} \tag{5.22}$$

where X and U are unknown vectors but E, F and H are specified known vectors. With the operational matrix of pantograph (4.4) we have

$$X(\frac{t}{\tau}) = X^T\phi(\frac{t}{\tau}) = X^T D_p(\tau)\phi(t),$$

also with the operational matrix of differentiation (4.1) over $[0, t_f]$ we have

$$\dot{X}(t) = X^T D\phi(t), \qquad 0 \leqslant t \leqslant t_f. \tag{5.23}$$

So, we obtain

$$E(t)X(t) \simeq E^T\phi(t)\phi^T(t)X = E^T\widehat{X}\phi(t), \tag{5.24}$$

$$G(t)U(t) \simeq G^T\phi(t)\phi^T(t)U = G^T\widehat{U}\phi(t), \tag{5.25}$$

$$F(t)X(\frac{t}{\tau}) \simeq F^T\phi(t)\phi^T(t)(XD_p(\tau))^T = F^T\widehat{X_1}, \tag{5.26}$$

where in the last expression $X_1 = (XD_p(\tau))^T$. We need to mention that $\widehat{X}$, $\widehat{U}$ and $\widehat{X_1}$ are product operational matrix given in [28], then by substituting (5.23)-(5.26) into (5.19) we can obtain

$$X^T D\phi(t) = E^T\widehat{X}\phi(t) + G^T\widehat{U}\phi(t) + F^T\widehat{X_1}\phi(t) + H^T\phi(t),$$

Similarly, with eliminating ϕ and putting whole of terms on the left-hand side of equalization, we have

$$R_1 = X^T D - E^T\widehat{X} - G^T\widehat{U} - F^T\widehat{X_1} - H^T = 0, \tag{5.27}$$

since P and R are constant, similarly for the objective function, we have:

$$R_2 = P\ X^T QX + R\ U^T QU, \tag{5.28}$$

where the matrix Q is given by (3.6). Finally, the optimal control problem is reduced to the following QPP.

$$\begin{aligned} \min \quad & M\|R_1\|^2 + R_2^2 \\ s.t. \quad & X^T\phi(0) = x_0, \end{aligned} \tag{5.29}$$

while $\|.\|$ is the Euclidean norm and M is a penalty parameter.

6. Illustrative Examples

6.1. Constant delay.

Example 6.1. Consider the following delay differential equation [5].

$$\begin{gathered} \tfrac{d^3x(t)}{dt} = -x(t) - x(t-0.3) + e^{-t+0.3}, \quad 0 \leqslant t \leqslant 1, \\ x(0) = 1,\ \tfrac{dx(0)}{dt} = -1,\ \tfrac{d^2x(0)}{dt} = 1,\ x(t) = e^{-t}. \end{gathered} \tag{6.1}$$

With the exact solution $x(t) = e^{-t}$.
By choosing $m = 8$, we obtain the following solution

$$\begin{aligned} x(t) = {} & (1-t)^8 + 7(1-t)^7 t + 21.5(1-t)^6 t^2 + 37.8325(1-t)^5 t^3 + 41.7063(1-t)^4 t^4 + \\ & 29.4889(1-t)^3 t^5 + 13.0572(1-t)^2 t^6 + 3.30991(1-t) t^7 + 0.36787 t^8, \end{aligned}$$

In Table 1, exact and approximated solution and also absolute error between them are shown

TABLE 1. Exact, estimated values and error of $x(t)$ for Example 6.1

t	approximated	exact	error
0	1	1	0
0.2	0.818727	0.818731	0.000004
0.4	0.670299	0.670320	0.000021
0.6	0.548759	0.548812	0.000053
0.8	0.44925	0.449329	0.000079
1	0.36787	0.367879	0.000009

EXAMPLE 6.2. Consider the following optimal control systems with constant delay [25].

$$\begin{aligned}
\min\ J &= \frac{1}{2}\int_0^1 [x_1(t)+x_2(t)]^2 + u^2(t)\mathrm{d}t, \qquad (6.2)\\
\dot{x}_1(t) &= x_1(t) + x_2(t-\frac{1}{4}),\\
\dot{x}_2(t) &= x_2(t) - 5x_1(t-\frac{1}{4}) - x_2(t-\frac{1}{4}) + u(t),\\
x_1(t) &= 1, \quad x_2(t) = 1, \qquad -\frac{1}{4} \leqslant t \leqslant 0.
\end{aligned}$$

By choosing $m = 8$, the approximate solutions of $x_1(t)$ and $x_2(t)$ with presented method are

$$x_1(t) = (1-t)^8 + 9.29662(1-t)^7t + 43.8422(1-t)^6t^2 + 111.199(1-t)^5t^3 + 145.896(1-t)^4t^4 + 139.172(1-t)^3t^5 + 74.011(1-t)^2t^6 + 21.5528(1-t)t^7 + 2.48454t^8,$$

and

$$x_2(t) = (1-t)^8 + 4.49747(1-t)^7t - 7.43068(1-t)^6t^2 - 34.3441(1-t)^5t^3 - 45.1487(1-t)^4t^4 - 177.469(1-t)^3t^5 - 99.1169(1-t)^2t^6 - 46.3591(1-t)t^7 - 7.72589t^8,$$

The approximated objective function is $J = 2.74573$ and the value of objective function in [25] is $J = 2.79302$. The graph of approximated solution $x_1(t)$ and $x_2(t)$ are plotted in Figure 1 and 2.

6.2. **Pantograph delay.**

EXAMPLE 6.3. [5, 21] Consider the following pantograph differential equation.

$$\begin{aligned}
\frac{dx(t)}{dt} &= \tfrac{1}{2}e^{\frac{t}{2}}x(\tfrac{t}{2}) + \tfrac{1}{2}x(t), \quad 0 \leqslant t \leqslant 1, \qquad (6.3)\\
x(0) &= 1.
\end{aligned}$$

With the exact solution $x(t) = e^t$.

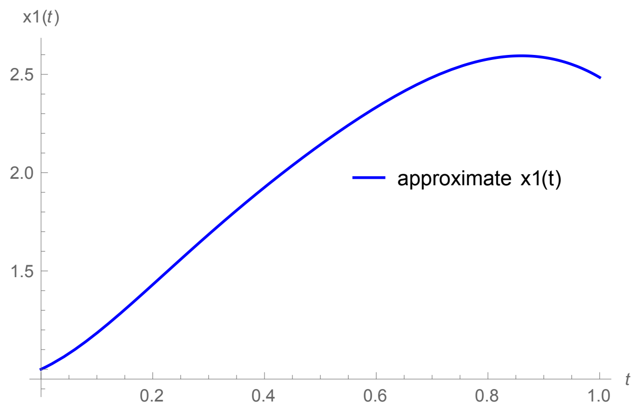

FIGURE 1. The graph of approximated solution $x_1(t)$ for Example 6.2

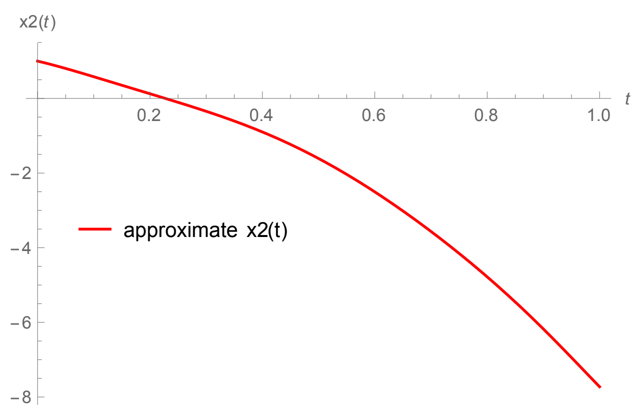

FIGURE 2. The graph of approximated solution $x_2(t)$ for Example 6.2

By choosing $m = 8$, we obtain the following solution

$$x(t) = (1-t)^8 + 9(1-t)^7 t + 35.5078(1-t)^6 t^2 + 80.1918(1-t)^5 t^3 + 113.368(1-t)^4 t^4 + 102.716(1-t)^3 t^5 + 52.2374(1-t)^2 t^6 + 18.8894(1-t) t^7 + 2.68327 t^8,$$

Exact and approximated solution and also absolute error between them are shown in Table 2.

TABLE 2. Exact, estimated values and error of $x(t)$ for Example 6.3

t	exact	approximated	error
0	1	1	0
0.1	1.10517	1.10523	−0.00006
0.2	1.2214	1.22152	−0.00012
0.3	1.34986	1.34984	0.00002
0.4	1.49182	1.49125	0.00057
0.5	1.64872	1.64685	0.00187
0.6	1.82212	1.81786	0.00426
0.7	2.01375	2.00557	0.000053
0.8	2.22554	2.21137	0.01417
0.9	2.4596	2.43674	0.02286
1	2.71828	2.68327	0.03501

EXAMPLE 6.4. [23, 5, 21] Consider the following pantograph differential equation.

$$\begin{aligned} x''(t) &= \frac{3}{4}x(t) + x(\frac{t}{2}) - t^2 + 2, \quad 0 \leqslant t \leqslant 1, \\ x(0) &= 0, \\ x'(0) &= 0. \end{aligned} \tag{6.4}$$

To solve this problem, we choose $m = 2$, suppose

$$\begin{aligned} X(t) &= X^T\phi(t), \\ X(\frac{t}{2}) &= X^T D_p(\tau = 2)\phi(t), \\ X'(t) &= X^T D\phi(t), \\ X''(t) &= X^T D.D\phi(t) = X^T D^2\phi(t), \\ H(t) &= -t^2 + 2 = H^T\phi(t), \end{aligned} \tag{6.5}$$

where

$$D_p(\tau = 2) = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{4} \end{bmatrix}, D = \begin{bmatrix} -2 & -1 & 0 \\ 2 & 0 & -2 \\ 0 & 1 & 2 \end{bmatrix}, H^T = \begin{bmatrix} 2 & 2 & 1 \end{bmatrix}, X^T = \begin{bmatrix} X_0 & X_1 & X_2 \end{bmatrix}. \tag{6.6}$$

Substituting (6.5) into (6.4), we have

$$X^T.D.D = \frac{3}{4}X^T + X^T D_p(\tau) + H^T,$$

then

$$R_1 = X^T.D.D - \frac{3}{4}X^T - X^T D_p(\tau) - H^T.$$

Since

$$\|R_1\|^2 = R_1.R_1^T,$$

thus

$$\|R_1\|^2 = (X^T.D.D-\frac{3}{4}X^T-X^TD_p(\tau)-H).(X^T.D.D-\frac{3}{4}X^T-X^TD_p(\tau)-H)^T. \tag{6.7}$$

By replacing (6.6) in (6.7), we obtain

$$\begin{aligned}\|R_1\|^2 &= (-2+\frac{3}{2}X_0-\frac{21}{4}X_1+2X_2)^2+(-2+\frac{1}{4}X_0-4X_1+2X_2)^2\\ &\quad+(-1+\frac{7}{4}X_0-\frac{9}{2}X_1+X_2)^2.\end{aligned}$$

For the initial condition $X(0)=0$ and $\dot{X}(0)=0$, we have

$$\begin{aligned}X(0) &= X^T\phi(0) = X_0 = 0,\\ \dot{X}(0) &= X^TD\phi(0) = -2X_0+2X_1 = 0,\end{aligned}$$

so we have, $X_0=0$ and $X_1=0$. Substituting X_0 and X_1 into $\|R_1\|^2$, yields the result:

$$\|R_1\|^2 = 2(-2+2X_2)^2+(-1+X_2)^2.$$

So the quadratic programming is

$$\min \quad 2(-2+2X_2)^2+(-1+X_2)^2.$$

Solving the optimization problem, we have found $X_2=1$. So $X_0=0$, $X_1=0$, and $X_2=1$, then

$$X(t) = X^T\phi(t) = \begin{bmatrix}0 & 0 & 1\end{bmatrix}\begin{bmatrix}(1-t)^2\\ 2t(1-t)\\ t^2\end{bmatrix} = t^2.$$

Then the approximated solution is $X(t)=t^2$ which it is the exact solution.

Example 6.5. Consider the following optimal control systems with pantograph delay [12].

$$\begin{aligned}\min\ J &= \frac{1}{2}\int_0^4 \left[x^2(t)+u^2(t)\right]\mathrm{d}t, \qquad (6.8)\\ \dot{x}(t) &= x(\frac{t}{2})+u(t),\\ x(0) &= 1.\end{aligned}$$

The approximated cost function by the presented method with $m=10$ is $J=0.173968$, Also the approximated cost function by the Chebyshev wavelet method with $M=10,\ K=4$, the Legendre method with $M=10$ [12] and the Bezier method [13] with $n=8$ are, respectively, $J=0.173952$, $J=0.173958$ and $J=0.173187$.

Comparing with other methods, such as the Chebyshev wavelet method, the Legendre method, the Variational method and the bezier method, the proposed method has less time and less calculus since the operational matrix of pantograph of Bernstein polynomial (4.4) is the upper triangular matrix that reduces the computation. Moreover the Bernstein polynomials form a basic for the vector space of continues polynomials. So any polynomial can be written as a linear combination of these polynomials. In the other word, Bernstein polynomials bases only a small number of bases are needed to obtain a satisfactory and good results which is one of the advantage of proposed method. Furthermore, in the Bezier method [13], one has to solve the problem of optimizing the integral (i.e., the objective function in the optimization problem is an integral over $[0, 1]$). So firstly, the integral must be approximated by numerical methods. Secondly, the approximated optimization problem should be solved. But by the presented method in this article, we directly solve quadratic programming problem (QPP) which this is another advantage of our method. The graph of approximated state and control plotted in Figure 3 and 4.

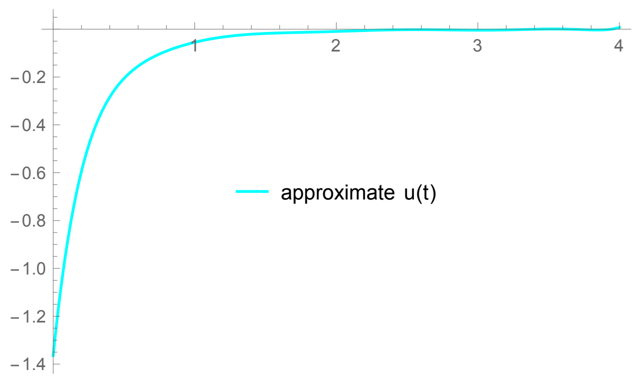

FIGURE 3. The graph of approximated state $x(t)$ for Example 6.5

CONCLUSIONS

In this paper, the operational matrices of pantograph, delay, and constant integration, for Bernstein polynomials are attained. Furthermore, an upper bound for the error of approximation is presented. The presented upper bound of error demonstrates convergent to the exact solution when a degree of the Bernstein polynomials tends to infinity. The Bernstein polynomials are used to solve pre-mentioned optimal control and pantograph and delay differential equations. The problem has been decreased to solve a QPP which can be solved by many software with an alternative accurate. The method is general with less time and calculus, simple to implement, and yields accurate results. The illustrative examples demonstrate that the presented method is valid.

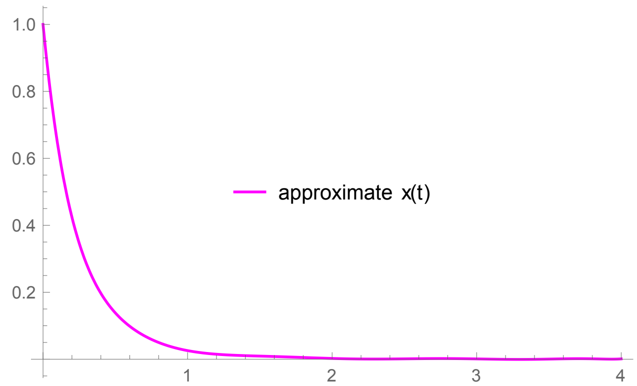

FIGURE 4. The graph of approximated control $u(t)$ for Example 6.5

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