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An efficient approximate method for solving two-dimensional fractional optimal control problems using generalized fractional order of Bernstein functions

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We define a new operational matrix of fractional derivative in the Caputo type and apply a spectral method to solve a two-dimensional fractional optimal control problem (2D-FOCP). To acquire this aim, first we expand the state and control variables based on the fractional order of Bernstein functions. Then we reduce the constraints of 2D-FOCP to a system of algebraic equations through the operational matrix. Now, one can solve straightforward the problem and drive the approximate solution of state and control variables. The convergence of the method in approximating the 2D-FOCP is proved. We demonstrate the efficiency and superiority of the method by comparing the results obtained by the presented method with the results of previous methods in some examples.

Keywords: fractional optimal control problems; fractional power Bernstein functions; operational matrix; Caputo derivative.

1. Introduction

The use of fractional modeling for interpretation of various issues of science and engineering has increased in the recent decade. The applications of fractional calculus can be studied in many scientific studies based on mathematical modeling including physics (classical and quantum mechanics, thermodynamics, etc.) (see, e.g., Cottone *et al.*, 2009; Sapora *et al.*, 2013; Machado, J.T., 2014; Sumelka, 2014; Bohannan & Knauber, 2015; Mondol *et al.*, 2018; Estrada-Rodriguez *et al.*, 2019), economics (Wang *et al.*, 2012), signal and image processing (Nigmatullin *et al.*, 2015; Ullah *et al.*, 2017) and control theory (see Darehmiraki *et al.*, 2016; Bahaa, 2017a; Bahaa, 2017b; Li *et al.*, 2017; Bahaa, 2018; Dadkhah *et al.*, 2018; Rakhshan & Effati, 2018).

Fractional optimal control problems (FOCPs) have been investigated by many researchers due to their applications over the simulation and they can be used for modeling real-world applications in mathematics. A more common case of FOCPs is when the dynamical systems are governed by fractional partial differential equations. We can find broad fields to observe and study such problems, for example, life sciences, populations biology, physiology, thermal systems and so on. Since compute an analytical solution for such problems is very difficult or sometimes impossible, the approximate methods and numerical techniques attract more attention for researchers and scientists.

Many numerical techniques are developed for solving one-dimensional FOCPs; for example, Yousefi *et al.* (2011) used a Legendre multiwavelet collocation method and Ejlali & Hosseini (2017) used the Pseudospectral method. So far, a few articles have been done to solve two-dimensional FOCPs (2D-FOCPs). Özdemir *et al.* (2009) used eigenfunctions; Heydari & Avazzadeh (2018) solved this problem by a computational method based on Legendre functions; and Nemati & Yousefi (2016) used the Ritz method for solving 2D-FOCPs.

In this paper, an efficient method is considered for solving 2D-FOCPs, where the fractional derivative is defined in Caputo sense. The general procedure for solving these systems based on expanding the state and control functions in terms of fractional-order Bernstein functions. Using this achievement, we expand the linear form of FOCP in terms of control points and subsequently convert the 2D-FOCP to an algebraic system of equations. In this way, we construct a quadratic programming problem subject to linear algebraic constraints corresponding to control problems normally viewed as nonlinear, and now the whole wonderful paraphernalia of quadratic programming can be employed to study them. To demonstrate the method, we consider the following 2D-FOCP:

$$\textit{minimize } J(u,\nu) = \int_{t_0}^{T}\int_{a}^{b} \boldsymbol{\zeta}\,(x,t,u(x,t),\nu(x,t))\,\mathrm{d}x\mathrm{d}t \tag{1.1}$$

subject to

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = a_2(x)\frac{\partial^2 u(x,t)}{\partial x^2} + a_1(x)\frac{\partial u(x,t)}{\partial x} + a_0(x)u(x,t) + \lambda\nu(x,t), \tag{1.2}$$

$$u(b,t) = h(t), \tag{1.3}$$

$$u(x,0) = u_0(x), \tag{1.4}$$

where $x \in \Omega = [a,b]$, $t \in [t_0,T]$ and $\lambda \in \mathbb{R}$. The coefficients $a_0(x), a_1(x)$ and $a_2(x)$ are given continuous functions. Furthermore, $\boldsymbol{\zeta}$, $h(\cdot)$ and $u_0(\cdot)$ are continuous functions in $\Omega \times [t_0,T]$, $[t_0,T]$ and Ω, respectively. The fractional derivative $\dfrac{\partial^\alpha u(x,t)}{\partial t^\alpha}$ is defined in the Caputo sense, where $0 < \alpha < 1$. Now we are going to find the optimal pair (u^*,ν^*) that satisfies (1.2)–(1.4) and minimizes the objective function $J(u,\nu)$ in (1.1). We demonstrate precisely of finding approximate optimal pair in further sections.

The rest of this paper is organized as follows: Section 2 introduces some essential properties of fractional calculus and Bernstein functions. In Section 3, we define fractional-order Bernstein functions and a novel operational matrix of Caputo derivatives. The main part of the paper is Section 4. In this section, by expanding the state $u(x,t)$ and control function $\nu(x,t)$, in terms of fractional-order of

Bernstein functions and employing of operational matrices, we convert the FOCP to a mathematical optimization problem in terms of some unknown control points. In the fifth section, the existence and uniqueness for the solution of problem (1.1)–(1.4) with the convergence of the proposed method are proved. The efficiency and applicability of the proposed method are investigated through some test examples in Section 6. Finally, the conclusion is presented in Section 7.

2. Preliminaries

In this section, we summarize some basic definitions and properties of Bernstein polynomials (BPs) and approximation of functions by BPs, and then some definitions in fractional calculus are presented.

2.1. *Some preliminaries in BPs*

Definition 2.1 The BPs of degree n over the interval $\Omega = [a, b]$ are defined as follows:

$$B_i^n(x) = \frac{\binom{n}{i}}{(b-a)^n}(x-a)^i(b-x)^{n-i},$$

for $i = 0, 1, 2, \ldots, n$, where $\binom{n}{i} = \dfrac{n!}{i!\,(n-i)!}$.

If one uses the binomial expansion of $(b-x)^{n-i}$, then the BPs can be formulated by monomial basis functions. Let

$$T_n(x) = [1 \quad x \quad x^2 \quad \ldots \quad x^n]^{\mathsf{T}}, \tag{2.1}$$

and

$$\phi_n(x) = \left[\ B_0^n(x) \ \ B_1^n(x) \ \ B_2^n(x) \ \ \ldots \ \ B_n^n(x)\ \right]^{\mathsf{T}}. \tag{2.2}$$

Then for each $x \in \Omega = [a, b]$,

$$\phi_n(x) = \psi_n^{\Omega} T_n(x),$$

where ψ_n^{Ω} is the $(n+1) \times (n+1)$ matrix that can be expressed by

$$\psi_n^{\Omega} = \Psi^{n+1} A^{n+1}, \tag{2.3}$$

where

$$\Psi^{n+1}(i+1, j+1) = \begin{cases} \dfrac{(-1)^{j-i}}{(b-a)^j}\binom{n}{i}\binom{n-i}{j-i}, & i \leqslant j, \\ 0, & i > j, \end{cases}$$

and

$$A^{n+1}(i+1,j+1) = \begin{cases} \binom{i}{j}(-a)^{i-j}, & j \leqslant i, \\ 0, & j > i, \end{cases}$$

for $i,j = 0, 1, \ldots, n$.

2.2. *Function approximation*

In this subsection, we approximate arbitrary real functions $f(x)$ and $f(x,t)$ in the interval $\Omega = [a,b]$ and $\bar{\Omega} = \Omega \times [t_0, T]$ by BPs, respectively.

LEMMA 2.1 (Kreyszig, 1978) Suppose that $H = L^2[\Omega]$ and that $\{B_0^n(x), B_1^n(x), B_2^n(x), \ldots, B_n^n(x)\} \subset H$. Define

$$Y = Span\{B_0^n(x), B_1^n(x), B_2^n(x), \ldots, B_n^n(x)\}.$$

Then the finite-dimensional subspace Y of the complete space $L^2[\Omega]$, is a complete basis for the Hilbert space H.

DEFINITION 2.2 Let $X = (\mathbb{X}, \| \ . \ \|)$ be a normed space and let Y be a subspace of X. Given a point $x \in X$, a point $\bar{y} \in Y$ is called a best approximation to x out of Y if $\bar{y}$ has the minimum distance from x. The problem of determining such a point is called a best approximation problem.

LEMMA 2.2 (Dehghan, Yousefi & Rashidi, 2013; Kreyszig, 1978) If $f(x) \in L^2[\Omega]$ and $Y = Span\{B_0^n(x), B_1^n(x), B_2^n(x), \ldots, B_n^n(x)\}$, then the best approximation of order n to the function $f(x)$ out of Y is unique and is given by $P_n(x)$, where

$$f(x) \simeq P_n(x) = \mathbf{C}\phi_n(x)$$

and

$$\mathbf{C} = \begin{bmatrix} c_0 & c_1 & c_2 & \ldots & c_n \end{bmatrix} \tag{2.4}$$

is the vector of constant coefficients that recall its entries as control points.

The vector $\mathbf{C}$, defined in (2.4), is completely dependent to $f(x)$. In fact, $\mathbf{C}$ can be achieved by

$$\mathbf{C} = \langle f(x), \phi_n(x) \rangle Q_n^{-1}, \tag{2.5}$$

where

$$\begin{aligned} \langle f(x), \phi_n(x) \rangle &= \int_{\Omega} f(x)\phi_n^{\mathsf{T}}(x)\mathrm{d}x \\ &= \begin{bmatrix} \langle f(x), B_0^n(x) \rangle & \langle f(x), B_1^n(x) \rangle & \langle f(x), B_2^n(x) \rangle & \ldots & \langle f(x), B_n^n(x) \rangle \end{bmatrix}. \end{aligned}$$

The entries of the $(n+1)\times(n+1)$ matrix Q_n are defined as follows:

$$\begin{aligned}Q_n = \langle \phi_n(x), \phi_n(x)\rangle &= \int_{\Omega} \phi_n(x)\phi_n^{\mathsf T}(x)\mathrm{d}x = \int_{\Omega} (\psi_n^{\Omega} T_n(x))(\psi_n^{\Omega} T_n(x))^{\mathsf T}\mathrm{d}x \\ &= \psi_n^{\Omega}\int_{\Omega} T_n(x)(T_n(x))^{\mathsf T}\mathrm{d}x\, \psi_n^{\Omega\,\mathsf T} = \psi_n^{\Omega} G_{\Omega,n}\psi_n^{\Omega\,\mathsf T}\end{aligned} \tag{2.6}$$

and $G_{\Omega,n}$ is the $(n+1)\times(n+1)$ Hilbert matrix as follows:

$$G_{\Omega,n} = \begin{bmatrix} b-a & \frac{b^2-a^2}{2} & \frac{b^3-a^3}{3} & \cdots & \frac{b^{n+1}-a^{n+1}}{n+1} \\ \frac{b^2-a^2}{2} & \frac{b^3-a^3}{3} & \frac{b^4-a^4}{4} & \cdots & \frac{b^{n+2}-a^{n+2}}{n+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{b^{n+1}-a^{n+1}}{n+1} & \frac{b^{n+2}-a^{n+2}}{n+2} & \frac{b^{n+3}-a^{n+3}}{n+3} & \cdots & \frac{b^{2n+1}-a^{2n+1}}{2n+1}\end{bmatrix},$$

where ψ_n^{Ω} is defined in (2.3).

DEFINITION 2.3 For the product of two BPs of degree $n, m \in \mathbb{N}\cup\{0\}$, we define

$${}_i^nB_j^m(x,t) = B_i^n(x)B_j^m(t),$$

where $t \in [t_0, T]$ and $x \in \Omega = [a,b]$.

LEMMA 2.3 The set

$$\mathrm{Y} = Span\left\{ {}_0^nB_0^m(x,t),\ {}_0^nB_1^m(x,t),\dots,\ {}_0^nB_m^m(x,t),\ {}_1^nB_0^m(x,t),\ {}_1^nB_1^m(x,t),\dots,\ {}_n^nB_m^m(x,t)\right\}$$

is a complete basis for the Hilbert space $L^2[\bar{\Omega}]$, where $\bar{\Omega} = \Omega\times[t_0,T]$.

Proof. Since Y is a finite subset of $L^2[\bar{\Omega}]$, then it is complete. For more details, see Kreyszig (1978). □

LEMMA 2.4 (Kreyszig, 1978) If

$$\mathrm{Y} = Span\{ {}_0^nB_0^m(x,t),\ {}_0^nB_1^m(x,t),\dots,\ {}_0^nB_m^m(x,t),\ {}_1^nB_0^m(x,t),\ {}_1^nB_1^m(x,t),\dots,\ {}_n^nB_m^m(x,t)\}$$

and $f(x,t)\in L^2[\bar{\Omega}]$, then $P_{n,m}(x,t)$ is the best approximation of $f(x,t)$ out of Y, where

$$f(x,t) \simeq P_{n,m}(x,t) = \sum_{i=0}^{n}\sum_{j=0}^{m} f_{i,j}B_i^n(x)B_j^m(t) = \phi_n^T(x)\mathbf{F}\phi_m(t),$$

in which $\mathbf{F}$ is the $(n+1)\times(m+1)$ matrix such that

$$\mathbf{F} = Q_n^{-1}\langle \phi_n(x), \langle f(x,t), \phi_m(t)\rangle\rangle Q_m^{-1}$$

and Q_n is defined in (2.6).

2.3. *Some preliminaries in fractional calculus*

DEFINITION 2.4 The Riemann–Liouville fractional integral of order $\alpha > 0$ is defined as follows:

$$ {}_aI_x^\alpha \big(f(x)\big) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-\tau)^{\alpha-1} f(\tau)\mathrm{d}\tau, \qquad x > a, $$

where $f(x) \in L_1[a,b]$. Also, $\Gamma(\alpha) = \int_0^\infty \tau^{(\alpha-1)} e^\tau \mathrm{d}\tau$ is the well-known gamma function.

DEFINITION 2.5 The Caputo fractional derivative of order $\alpha > 0$ of the function $f(x) \in C^n[a,b]$ is defined as follows:

$$ {}_aD_x^\alpha f(x) = {}_aI_x^{n-\alpha} D^n f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-\tau)^{n-\alpha-1} f^{(n)}(\tau)\mathrm{d}\tau, \qquad x > a, $$

where n is a nonnegative integer number and $n = [\alpha] + 1$. For $\alpha \in \mathbb{N}$, the Caputo derivative operator is an usual derivative operator of integer order.

COROLLARY 2.1 For $\alpha > 0$ and the constant C, the Caputo fractional derivative is zero. In other words, $D^\alpha C = 0$.

LEMMA 2.5 Let $f(x) = x^\beta$, $x \in [a,b]$ and let $\beta, \alpha > 0$, $\alpha \notin \mathbb{N}$. Then

$$ {}_aD_x^\alpha x^\beta = $$
$$ \begin{cases} 0, & \beta \in \mathbb{N}_0,\ \beta < \lceil \alpha \rceil \text{ or } \beta \notin \mathbb{N}_0,\ \beta \leqslant \lfloor \alpha \rfloor, \\ \dfrac{\Gamma(\beta+1)}{\Gamma(p)\Gamma(q)} \big(B(p,q) - B(\frac{a}{x};p,q)\big) x^{\beta-\alpha}, & \beta \in \mathbb{N}_0,\ \beta \geqslant \lceil \alpha \rceil \text{ or } \beta \notin \mathbb{N}_0,\ \beta > \lfloor \alpha \rfloor, \end{cases} $$

where the ceiling function $\lceil \alpha \rceil$ and the floor function $\lfloor \alpha \rfloor$ indicate the smallest integer greater than or equal to α and the largest integer less than or equal to α, respectively. Also, $B(p,q)$ and $B(x;p,q)$ are a beta function and incomplete beta function, respectively. Moreover, $\mathbb{N}_0 = \mathbb{N} \bigcup \{0\}$, $p = \beta - n + 1$, $q = n - \alpha$ and $n = [\alpha] + 1$.

Proof. By using Definition 2.5, we have

$$ \begin{aligned} {}_aD_x^\alpha x^\beta &= \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-\tau)^{n-\alpha-1} \left(\beta(\beta-1)(\beta-2)\cdots(\beta-n+1)\right) \tau^{\beta-n}\mathrm{d}\tau \\ &= \frac{1}{\Gamma(n-\alpha)} \times \frac{\Gamma(\beta+1)}{\Gamma(\beta-n+1)} \int_a^x (x-\tau)^{n-\alpha-1}\tau^{\beta-n}\mathrm{d}\tau \\ &= \frac{\Gamma(\beta+1)}{\Gamma(n-\alpha)\Gamma(\beta-n+1)} x^{\beta-\alpha-1} \int_a^x (1-\frac{\tau}{x})^{n-\alpha-1} (\frac{\tau}{x})^{\beta-n}\mathrm{d}\tau, \end{aligned} $$

where $n = [\alpha] + 1$. Using the variable change $\frac{\tau}{x} = y$, then we easily obtain

$$ {}_aD_x^\alpha x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(n-\alpha)\Gamma(\beta-n+1)} x^{\beta-\alpha} \left[\int_0^1 (1-y)^{n-\alpha-1} y^{\beta-n}\mathrm{d}y - \int_0^{\frac{a}{x}} (1-y)^{n-\alpha-1} y^{\beta-n}\mathrm{d}y \right]. $$

By considering $p = \beta - n + 1,\ q = n - \alpha$, and also with respect to the definition of beta function and incomplete beta function, the proof completes. □

THEOREM 2.1 (Odibat & Shawagfeh, 2007) (Generalized Taylor's formula) Let $0 < \alpha \leq 1$ and let ${}_aD_x^{k\alpha} f(x) \in C[a, b]$ for $k = 0, 1, \ldots, N + 1$. Then one can generalize Taylor's formula by Caputo fractional derivatives as follows:

$$f(x) = \sum_{k=0}^{N} \frac{({}_aD_x^{k\alpha} f)(a)}{\Gamma(k\alpha + 1)}(x - a)^{k\alpha} + \frac{({}_aD_x^{(N+1)\alpha} f)(\eta)}{\Gamma((N + 1)\alpha + 1)}(x - a)^{(N+1)\alpha},$$

where $a < \eta \leqslant x$ for all$x \in (a, b]$. If $\alpha = 1$, then classical Taylor's formula is obtained.

3. Fractional Bernstein functions and operational matrix of fractional order derivative

We need a suitable technique for solving fractional problem (1.1)–(1.4). To obtain this aim, we use Bernstein functions of fractional order. Assume that $\beta > 0$. Then by using Definition 2.1, we establish the fractional Bernstein function of order β as follows:

$$F^{\beta}B_i^n(x) = \frac{\binom{n}{i}}{(b - a)^n}(x^{\beta} - a)^i (b - x^{\beta})^{n-i}, \tag{3.1}$$

for $i = 0, 1, 2, \ldots, n$.

Now, as an extension of vectors (2.1) and (2.2), we define

$$F^{\beta}T_n(x) = [1 \quad x^{\beta} \quad x^{2\beta} \quad \ldots \quad x^{n\beta}]^{\mathsf{T}}, \tag{3.2}$$

and

$$F^{\beta}\phi_n(x) = \left[\ F^{\beta}B_0^n(x)\ \ F^{\beta}B_1^n(x)\ \ F^{\beta}B_2^n(x)\ \ \ldots\ \ F^{\beta}B_n^n(x)\ \right]^{\mathsf{T}}. \tag{3.3}$$

LEMMA 3.1 Let $F^{\beta}T_n(x)$ and $F^{\beta}\phi_n(x)$ be defined as (3.2) and (3.3), respectively. Then for each $x \in \Omega$, we have

$$F^{\beta}\phi_n(x) = \psi_n^{\Omega} F^{\beta}T_n(x), \tag{3.4}$$

where ψ_n^{Ω} is the $(n + 1) \times (n + 1)$ matrix defined in (2.3).

Now, to compute Caputo fractional derivatives for an arbitrary function $f(x)$, define a new operational fractional derivative matrix of order $\alpha > 0$ by using fractional BPs.

THEOREM 3.1 Let $F^{\beta}\phi_n(x)$ be a vector of fractional Bernstein functions on $\Omega = [a, b]$ as (3.3) and let $\beta > 0$. Then for each $\alpha > 0$, we have

$${}_aD_x^{\alpha} F^{\beta}\phi_n(x) = D_a^{\alpha,\beta} F^{\beta}\phi_n(x),$$

while $D_a^{\alpha,\beta} = \psi_n^{\Omega} K_{n\beta}^{\alpha} H^{\alpha}$, where $K_{n\beta}^{\alpha}$ and H^{α} are $(n+1)\times(n+1)$ matrices, defined, respectively, as follows:

$$K_{n\beta}^{\alpha}(i+1,j+1) = \begin{cases} \dfrac{\Gamma(i\beta+1)}{\Gamma(p_i)\Gamma(q_i)}\left(B(p_i,q_i) - B(\tfrac{a}{x};p_i,q_i)\right), & i\beta \in \mathbb{N}_0,\ i\beta \geqslant \lceil\alpha\rceil \ or\ i\beta \notin \mathbb{N}_0,\ i\beta > \lfloor\alpha\rfloor,\ i=j, \\ 0, & otherwise, \end{cases}$$

for $i,j = 0,1,\ldots,n,\ p_i = i\beta - [\alpha]$ and $q_i = [\alpha] - \alpha + 1$, and

$$H^{\alpha} = \left[H^{\alpha}(1), H^{\alpha}(2), \ldots, H^{\alpha}(n+1)\right],$$

where $H^{\alpha}(i+1) = \langle x^{i\beta-\alpha}, F^{\beta}\phi_n(x)\rangle (Q_n^{\beta})^{-1}$, $i = 0,1,\ldots,n$, and $Q_n^{\beta} = \langle F^{\beta}\phi_n(x), F^{\beta}\phi_n(x)\rangle$. The matrix $D_a^{\alpha,\beta}$ is called the operational derivative matrix for the fractional Caputo derivative of order α.

Proof. To prove the theorem, we use (3.4) and some properties of the fractional Caputo derivative as follows:

$${}_aD_x^{\alpha} F^{\beta}\phi_n(x) = {}_aD_x^{\alpha}(\psi_n^{\Omega} F^{\beta} T_n(x)) = \psi_n^{\Omega}\ {}_aD_x^{\alpha} F^{\beta} T_n(x).$$

Now from (3.2), we obtain

$${}_aD_x^{\alpha} F^{\beta} T_n(x) = \left[\, {}_aD_x^{\alpha} 1 \quad {}_aD_x^{\alpha} x^{\beta} \quad {}_aD_x^{\alpha} x^{2\beta} \quad \ldots \quad {}_aD_x^{\alpha} x^{n\beta} \,\right]^{\mathsf{T}},$$

while each element of the above vector can be found by Lemma 2.5 as follows:

$${}_aD_x^{\alpha} x^{i\beta} = \begin{cases} 0, & i\beta \in \mathbb{N}_0,\ i\beta < \lceil\alpha\rceil \ or\ i\beta \notin \mathbb{N}_0,\ i\beta \leqslant \lfloor\alpha\rfloor, \\ \dfrac{\Gamma(i\beta+1)}{\Gamma(p_i)\Gamma(q_i)}\left(B(p_i,q_i) - B(\tfrac{a}{x};p_i,q_i)\right) x^{i\beta-\alpha}, & i\beta \in \mathbb{N}_0,\ i\beta \geqslant \lceil\alpha\rceil \ or\ i\beta \notin \mathbb{N}_0,\ i\beta > \lfloor\alpha\rfloor, \end{cases}$$

for $i = 1,2,\ldots,n$, where $p_i = i\beta - [\alpha]$ and $q_i = [\alpha] - \alpha + 1$. We define the diagonal matrix $K_{n\beta}^{\alpha}$ as

$$K_{n\beta}^{\alpha} = \begin{bmatrix} {}_aD_x^{\alpha} 1 & 0 & 0 & \ldots & 0 \\ 0 & {}_aD_x^{\alpha} x^{\beta} & 0 & \ldots & 0 \\ 0 & 0 & {}_aD_x^{\alpha} x^{2\beta} & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & {}_aD_x^{\alpha} x^{n\beta} \end{bmatrix}.$$

Let

$$x^{i\beta-\alpha} = \sum_{h=0}^{n} c_h^i F^{\beta} B_h^n(x) = \mathbf{C}_i F^{\beta}\phi_n(x), \tag{3.5}$$

where

$$\mathbf{C}_i = \left[c_0^i \;\; c_1^i \;\; c_2^i \;\; \ldots \;\; c_n^i\right].$$

By employing the inner product of $F^\beta \phi_n(x)$ on both sides of (3.5) in the interval $\Omega = [a, b]$, we have

$$\langle x^{i\beta-\alpha}, F^\beta \phi_n(x)\rangle = \mathbf{C}_i \langle F^\beta \phi_n(x), F^\beta \phi_n(x)\rangle;$$

hence,

$$\langle x^{i\beta-\alpha}, F^\beta \phi_n(x)\rangle = \mathbf{C}_i Q_n^\beta,$$

where $Q_n^\beta = \langle F^\beta \phi_n(x), F^\beta \phi_n(x)\rangle$. If one assumes $H^\alpha(i+1) = \mathbf{C}_i = \langle x^{i\beta-\alpha}, F^\beta \phi_n(x)\rangle (Q_n^\beta)^{-1}$, then the theorem is proved. □

REMARK 3.1 One can find Q_n^β in Theorem 3.1 as follows:

$$\begin{aligned} Q_n^\beta = \langle F^\beta \phi_n(x), F^\beta \phi_n(x)\rangle &= \int_a^b F^\beta \phi_n(x)(F^\beta \phi_n(x))^{\mathsf{T}} \mathrm{d}x \\ &= \int_a^b (\psi_n^\Omega F^\beta T_n(x))(\psi_n^\Omega F^\beta T_n(x))^{\mathsf{T}} \mathrm{d}x \\ &= \psi_n^\Omega \int_a^b F^\beta T_n(x)(F^\beta T_n(x))^{\mathsf{T}} \mathrm{d}x\, (\psi_n^\Omega)^{\mathsf{T}} = \psi_n^\Omega G_{\Omega,n}^\beta (\psi_n^\Omega)^{\mathsf{T}}, \end{aligned} \tag{3.6}$$

where $G_{\Omega,n}^\beta$ is the following symmetric matrix:

$$G_{\Omega,n}^\beta = \begin{bmatrix} b-a & \frac{b^{\beta+1}-a^{\beta+1}}{\beta+1} & \frac{b^{2\beta+1}-a^{2\beta+1}}{2\beta+1} & \cdots & \frac{b^{n\beta+1}-a^{n\beta+1}}{n\beta+1} \\ \frac{b^{\beta+1}-a^{\beta+1}}{\beta+1} & \frac{b^{2\beta+1}-a^{2\beta+1}}{2\beta+1} & \frac{b^{3\beta+1}-a^{3\beta+1}}{3\beta+1} & \cdots & \frac{b^{(n+1)\beta+1}-a^{(n+1)\beta+1}}{(n+1)\beta+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{b^{n\beta+1}-a^{n\beta+1}}{n\beta+1} & \frac{b^{(n+1)\beta+1}-a^{(n+1)\beta+1}}{(n+1)\beta+1} & \frac{b^{(n+2)\beta+1}-a^{(n+2)\beta+1}}{(n+2)\beta+1} & \cdots & \frac{b^{2n\beta+1}-a^{2n\beta+1}}{2n\beta+1} \end{bmatrix}.$$

REMARK 3.2 Suppose that $f(x) \in C^m[a, b]$; then for each $k \in \mathbb{N}$, $k \leqslant m$, and $k \leqslant n$, there exists an $(n+1) \times (n+1)$ matrix D_Ω such that

$$f^{(k)}(x) \simeq P_n^{(k)}(x) = \mathbf{C}\left(D_\Omega\right)^k \phi_n(x),$$

where $\mathbf{C}$ is as (2.5) and $D_\Omega = \psi_n^\Omega \Lambda \mathcal{V}$, with

$$\Lambda_{i+1,j+1} = \begin{cases} i, & i = j+1, \\ 0 & \textit{otherwise}, \end{cases}$$

for $i = 0, \ldots, n$ and $j = 0, \ldots, n-1$ and $\mathcal{V}$ can be expressed by

$$\mathcal{V}_k = \psi_{n,k}^{\Omega^{-1}}, \qquad k = 1, 2, \ldots, n,$$

where $\psi_{n,k}^{\Omega^{-1}}$ is the kth row of $\psi_n^{\Omega^{-1}}$. For details, see Karimi *et al.* (2016).

4. Approximate analytical solution of fractional optimal control problems

In this section, we develop fractional Bernstein functions for solving the FOCP (1.1)–(1.4). To achieve this goal, by approximating the optimal admissible pairs of control and trajectory by fractional Bernstein functions, we convert the mentioned FOCP to a mathematical programming problem subject to algebraic constraints. According to what discussed in the previous sections and considering FOCP (1.1)–(1.4), we assume the following:

$$\begin{aligned}
&u(x,t) \simeq \phi_n^{\mathsf{T}}(x)\mathbf{U}F^{\beta}\phi_m(t),\\
&v(x,t) \simeq \phi_n^{\mathsf{T}}(x)\mathbf{V}F^{\beta}\phi_m(t),\\
&u(b,t) \simeq \phi_n^{\mathsf{T}}(b)\mathbf{U}F^{\beta}\phi_m(t), \qquad u(x,0) \simeq \phi_n^{\mathsf{T}}(x)\mathbf{U}F^{\beta}\phi_m(0),\\
&h(t) \simeq \mathbf{H}F^{\beta}\phi_m(t), \qquad u_0(x) \simeq \phi_n(x)^{\mathsf{T}}\mathbf{U}_0,\\
&a_0(x) \simeq \mathbf{A}_0^{\mathsf{T}}\phi_n(x), \qquad a_1(x) \simeq \mathbf{A}_1^{\mathsf{T}}\phi_n(x), \qquad a_2(x) \simeq \mathbf{A}_2^{\mathsf{T}}\phi_n(x).
\end{aligned} \tag{4.1}$$

Hence,

$$\begin{aligned}
&\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}} \simeq \frac{\partial^{\alpha}}{\partial t^{\alpha}}\phi_n^{\mathsf{T}}(x)\mathbf{U}F^{\beta}\phi_m(t) = \phi_n^{\mathsf{T}}(x)\mathbf{U}D_a^{\alpha,\beta}F^{\beta}\phi_m(t),\\
&\frac{\partial u(x,t)}{\partial x} \simeq \frac{\partial}{\partial x}\phi_n^{\mathsf{T}}(x)\mathbf{U}F^{\beta}\phi_m(t) = \phi_n^{\mathsf{T}}(x){D_{\Omega}}^{\mathsf{T}}\mathbf{U}F^{\beta}\phi_m(t),\\
&\frac{\partial^2 u(x,t)}{\partial x^2} \simeq \phi_n^{\mathsf{T}}(x){D_{\Omega}^2}^{\mathsf{T}}\mathbf{U}F^{\beta}\phi_m(t).
\end{aligned}$$

Also, we can use the product operational matrices to find the following relations:

$$\begin{aligned}
&a_0(x)u(x,t) \simeq \mathbf{A}_0^{\mathsf{T}}\phi_n(x)\phi_n^{\mathsf{T}}(x)\mathbf{U}F^{\beta}\phi_m(t) \simeq \phi_n^{\mathsf{T}}(x)\bar{\mathbf{A}}_{\mathbf{0}}\mathbf{U}F^{\beta}\phi_m(t),\\
&a_1(x)\frac{\partial u(x,t)}{\partial x} \simeq \mathbf{A}_1^{\mathsf{T}}\phi_n(x)\phi_n^{\mathsf{T}}(x){D_{\Omega}}^{\mathsf{T}}\mathbf{U}F^{\beta}\phi_m(t) \simeq \phi_n^{\mathsf{T}}(x)\bar{\mathbf{A}}_1{D_{\Omega}}^{\mathsf{T}}\mathbf{U}F^{\beta}\phi_m(t),\\
&a_2(x)\frac{\partial^2 u(x,t)}{\partial x^2} \simeq \mathbf{A}_2^{\mathsf{T}}\phi_n(x)\phi_n^{\mathsf{T}}(x){D_{\Omega}^2}^{\mathsf{T}}\mathbf{U}F^{\beta}\phi_m(t) \simeq \phi_n^{\mathsf{T}}(x)\bar{\mathbf{A}}_2{D_{\Omega}^2}^{\mathsf{T}}\mathbf{U}F^{\beta}\phi_m(t),
\end{aligned} \tag{4.2}$$

where $\{\bar{\mathbf{A}}_{\mathbf{0}}\}_{i,j=0}^{n;n}$, $\{\bar{\mathbf{A}}_{\mathbf{1}}\}_{i,j=0}^{n;n}$, $\{\bar{\mathbf{A}}_{\mathbf{2}}\}_{i,j=0}^{n;n}$, $\mathbf{H} = [h_j]_{j=0}^{m}$ and $\mathbf{U}_0 = [u_{0i}]_{i=0}^{n}$ are given. We need to mention that since $a_0(x), a_1(x), a_2(x), h(t)$ and $u_0(x)$ in (1.2)–(1.4) are given, so $\bar{\mathbf{A}}_i, i = 0, 1, 2, \mathbf{H}$ and $\mathbf{U}_0$ are known. It is clear that the vectors $\mathbf{U} = [u_{i,j}]_{i,j=0}^{n;m}$ and $\mathbf{V} = [v_{i,j}]_{i,j=0}^{n;m}$ are, respectively, control points vectors of state and control functions that are needed to be found. By exerting (4.1) and (4.2) in (1.2)–(1.4), one can easily find that

$$\phi_n^{\mathsf{T}}(x)\mathbf{U}D_a^{\alpha,\beta}F^\beta\phi_m(t) = \phi_n^{\mathsf{T}}(x)\bar{\mathbf{A}_2}D_\Omega^2{}^{\mathsf{T}}\mathbf{U}F^\beta\phi_m(t) + \phi_n^{\mathsf{T}}(x)\bar{\mathbf{A}_1}D_\Omega{}^{\mathsf{T}}\mathbf{U}F^\beta\phi_m(t)$$
$$+ \phi_n^{\mathsf{T}}(x)\bar{\mathbf{A}_0}\mathbf{U}F^\beta\phi_m(t) + \lambda\phi_n^{\mathsf{T}}(x)\mathbf{V}F^\beta\phi_m(t),$$

or

$$\phi_n^{\mathsf{T}}(x)\mathbf{U}D_a^{\alpha,\beta}F^\beta\phi_m(t) - \phi_n^{\mathsf{T}}(x)\left(\bar{\mathbf{A}_2}D_\Omega^2{}^{\mathsf{T}}\mathbf{U} + \bar{\mathbf{A}_1}D_\Omega{}^{\mathsf{T}}\mathbf{U} + \bar{\mathbf{A}_0}\mathbf{U} + \lambda\mathbf{V}\right)F^\beta\phi_m(t) = 0.$$

Hence, we obtain

$$\phi_n^{\mathsf{T}}(x)\left(\mathbf{U}D_a^{\alpha,\beta} - \mathbf{S}\mathbf{U} - \lambda\mathbf{V}\right)F^\beta\phi_m(t) = 0, \tag{4.3}$$

where $D_a^{\alpha,\beta} = [d_{i,j}]_{i,j=0}^{m;m}$ is defined in Theorem 3.1 and $\mathbf{S} = [s_{i,j}]_{i,j=0}^{n;n}$ is defined as follows:

$$= \bar{\mathbf{A}_2}D_\Omega^2{}^{\mathsf{T}} + \bar{\mathbf{A}_1}D_\Omega{}^{\mathsf{T}} + \bar{\mathbf{A}_0}.$$

Similarly, for initial and boundary conditions, we find the following relations:

$$\phi_n^{\mathsf{T}}(b)\mathbf{U}F^\beta\phi_m(t) = \mathbf{H}F^\beta\phi_m(t), \tag{4.4}$$

$$\phi_n^{\mathsf{T}}(x)\mathbf{U}F^\beta\phi_m(0) = \phi_n(x)^{\mathsf{T}}\mathbf{U}_0. \tag{4.5}$$

For the cost functional $J(u, v)$ in (1.1), we have

$$J(u(x,t), v(x,t)) \simeq J(\mathbf{U}, \mathbf{V}) = \int_{t_0}^{T}\int_a^b \boldsymbol{\zeta}\left(x, t, \phi_n^{\mathsf{T}}(x)\mathbf{U}F^\beta\phi_m(t), \phi_n^{\mathsf{T}}(x)\mathbf{V}F^\beta\phi_m(t)\right)\mathrm{d}x\mathrm{d}t. \tag{4.6}$$

Now the optimal control problem (1.2)–(1.4) with cost functional (1.1) can be approximated by the following optimization problem:

$$\begin{aligned} &\textit{minimize } J(\mathbf{U}, \mathbf{V}) \\ &\textit{s.t.} \\ &\textit{constraints} \quad (4.3) - (4.5) \\ &x \in [a,b] \quad \textit{and} \quad t \in [t_0, T] \end{aligned} \tag{4.7}$$

Now, by choosing $r \times s$ nodes as t_l's ($l = 0, \ldots, r$) and x_p's ($p = 0, \ldots, s$) on $[t_0, T]$ and $[a, b]$, respectively, the constraints (4.3)–(4.5) convert into an algebraic system as follows:

$$\sum_{i=0}^{n}\sum_{j=0}^{m}\left(\sum_{k=0}^{m} u_{i,k}d_{k,j} - \sum_{k=0}^{n} s_{i,k}u_{k,j} - \lambda v_{i,j}\right)B_i^n(x_p)F^\beta B_j^m(t_l) = 0, \quad l = 0, \ldots, r, \quad p = 0, \ldots, s,$$

$$\sum_{i=0}^{n}\sum_{j=0}^{m} u_{i,j}B_i^n(b)F^\beta B_j^n(t_l) = \sum_{j=0}^{m} h_j F^\beta B_j^m(t_l), \qquad l = 0, \ldots, r,$$

$$\sum_{i=0}^{n}\sum_{j=0}^{m} u_{i,j}B_i^n(x_p)F^\beta B_j^n(0) = \sum_{i=0}^{n} u_{0i}B_i^n(x_p), \qquad p = 0, \ldots, s. \tag{4.8}$$

These unknown control points $[u_{i,j}]_{i,j=0}^{n;m}$ and $[v_{i,j}]_{i,j=0}^{n;m}$ can be obtained by solving of quadratic programming problems (4.7)–(4.8). Then one can find the approximate values of vectors $\mathbf{U}$ and $\mathbf{V}$ as follows:

$$\mathbf{U} = \left[u_{i,j}\right]_{i,j=0}^{n;m} \qquad \textit{and} \qquad \mathbf{V} = \left[v_{i,j}\right]_{i,j=0}^{n;m}.$$

By substituting $\mathbf{U}$ and $\mathbf{V}$ in what follows, that is,

$$u(x,t) \simeq \phi_n^\intercal(x)\mathbf{U}F^\beta\phi_m(t)$$

and

$$v(x,t) \simeq \phi_n^\intercal(x)\mathbf{V}F^\beta\phi_m(t),$$

the approximate state and control functions will be obtained.

5. Convergence analysis

In this section, we prove the convergence of the approximate solution for the fractional differential equation (1.2)–(1.4). First we provide an error bound for this approximation.

THEOREM 5.1 Let $u(x,t) : \bar{\Omega} \longrightarrow \mathbb{R}$ be the solution of the fractional partial differential equations (1.2)–(1.4), where $\bar{\Omega} = [t_0, T] \times \Omega$, and assume ${}_aD_x^{k_1\alpha}u(x,t)$, ${}_{t_0}D_t^{k_2\alpha}u(x,t) \in C(\Omega)$, for $k_1 = 0, 1, \ldots, m$ and $k_2 = 0, 1, \ldots, n$, where $\alpha \in [0, 1]$. Define

$$\mathrm{Y} = \mathit{Span}\left\{ {}_0^nB_0^m(x,t),\ {}_0^nB_1^m(x,t), \ldots,\ {}_0^nB_m^m(x,t),\ {}_1^nB_0^m(x,t),\ {}_1^nB_1^m(x,t), \ldots,\ {}_n^nB_m^m(x,t)\right\}.$$

If $\phi_n^\intercal(x)\mathbf{U}F^\beta\phi_m(t)$ is the best approximation of $u(x,t)$ out of Y, then the error bound is as follows:

$$\| u(x,t) - \phi_n^\intercal(x)\mathbf{U}F^\beta\phi_m(t) \| \leqslant \frac{M}{\Gamma((N+1)\alpha + 1)}, \tag{5.1}$$

where

$$M = \max_{0<\eta\leqslant\xi\leqslant 1} \mid {}_0D_\xi^{(N+1)\alpha} u(a+\eta(b-a), t_0+\eta(T-t_0)) \mid,$$

and $N = n + m$.

Proof. We define a new function $f(\xi)\ :\ [0,1] \longrightarrow \mathbb{R}$ and use generalized Taylor's formula with multivariable, see Cheng (2018). To obtain this aim, let $f(\xi) = u(a+\xi(b-a), t_0+\xi(T-t_0)) = u(x,t)$, where $\xi \in [0,1]$, $x = a+\xi(b-a)$ and $t = t_0+\xi(T-t_0)$. We consider

$$f(\xi) = u(x,t) = P_{n,m}(x,t) + \frac{\left({}_0D_\xi^{(N+1)\alpha} f\right)(\eta)}{\Gamma((N+1)\alpha+1)}(\xi-0)^{(N+1)\alpha},$$

where

$$P_{n,m}(x,t) = \sum_{k=0}^{N} \frac{\left({}_0D_\xi^{k\alpha} f\right)(0)}{\Gamma(k\alpha+1)}(\xi-0)^{k\alpha}$$

and $0 < \eta \leq \xi$. Hence, we have

$$\mid f(\xi) - P_{n,m}(x,t) \mid = \mid \frac{\left({}_0D_\xi^{(N+1)\alpha} f\right)(\eta)}{\Gamma((N+1)\alpha+1)}(\xi-0)^{(N+1)\alpha} \mid .$$

Since $\phi_n^{\mathsf{T}}(x)\mathbf{U}F^{\beta}\phi_m(t)$ is the best approximation of f, we have

$$\parallel f(\xi) - \phi_n^{\mathsf{T}}(x)\mathbf{U}F^{\beta}\phi_m(t) \parallel \leqslant \parallel f(\xi) - P_{n,m}(x,t) \parallel .$$

In other words,

$$\parallel u(x,t) - \phi_n^{\mathsf{T}}(x)\mathbf{U}F^{\beta}\phi_m(t) \parallel \leqslant \mid \frac{\left({}_0D_\xi^{(N+1)\alpha} f\right)(\eta)}{\Gamma((N+1)\alpha+1)}(\xi-0)^{(N+1)\alpha} \mid .$$

Define M as

$$M = \max_{0<\eta\leqslant\xi\leqslant 1} \mid {}_0D_\xi^{(N+1)\alpha} f(\eta) \mid = \max_{0<\eta\leqslant\xi\leqslant 1} \mid {}_0D_\xi^{(N+1)\alpha} u(a+\eta(b-a), t_0+\eta(T-t_0)) \mid,$$

also we know that $(\xi-0)^{(N+1)\alpha} \leq 1$. This expression implies

$$\parallel u(x,t) - \phi_n^{\mathsf{T}}(x)\mathbf{U}F^{\beta}\phi_m(t) \parallel \leqslant \frac{M}{\Gamma((N+1)\alpha+1)}.$$

□

COROLLARY 5.1 The approximation solution $\phi_n^{\mathsf{T}}(x)\mathbf{U}F^{\beta}\phi_m(t)$ converges to the analytical solution $u(x,t)$ if the degrees of Bernstein approximation polynomial ${}_i^nB_j^m(x,t),\ i = 0,1,\ldots,n.\ j = 0,1,\ldots,m$ tends to infinity.

Proof. As can be seen from (5.1), when N tends to infinity, the approximation solution tends to the analytical solution. □

6. Numerical examples

In this section, to validate the accuracy of the presented method, two examples are considered. These test examples are solved by using the powerful *MATLAB R2018b* software on an *Intel Core i5-4200U*.

6.1. *Example 1*

Consider the following optimal control problem (see Rostamy *et al.*, 2013):

$$\textit{minimize}\ J(u,\nu)=\int_0^1 |\ \nu(t)\ |^2\,\mathrm{d}t+\eta\int_0^1 \|\ u(x,1)-u^d(x)\ \|^2\,\mathrm{d}x$$

subject to

$$\begin{aligned}
&\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}=\frac{\partial^2 u(x,t)}{\partial x^2}+u(x,t),\\
&u(0,t)=\nu(t),\\
&u(1,t)=1+\sin(1)+\sum_{k=1}^{\infty}\frac{t^{k\alpha}}{\Gamma(k\alpha+1)},\\
&u(x,0)=1+\sin(x),
\end{aligned}$$

where the function $\|\cdot\|$ indicates the Euclidean norm and η is a big number for penalty in the cost function. We are going to find the admissible control function $\nu(\cdot)$ such that the solution of the above partial differential equation corresponding to the given boundary conditions satisfies the following terminal condition:

$$u(x,1)=u^d(x),\qquad \textit{Lebesgue. a.e. for } x\in[0,1]$$

where

$$u^d(x)=1+\sin(x)+\sum_{k=1}^{\infty}\frac{1}{\Gamma(k\alpha+1)}$$

and $0<\alpha<1$. We solve this problem by different degrees of the BPs and fractional order of BPs to compare the results. In this example, without loss of generality, we assume $m=n$. The absolute error of the exact solution and approximate solution as shown in (6.1), for some degrees of BPs and FBPs are given in Table 1. As one can see in Table 1, when we use high-order degree polynomials, FBPs give more accurate results than the BPs. We divide the interval $[t_0,T]$ into two subintervals $[t_0,\frac{t_0+T}{2}]$ and $[\frac{t_0+T}{2},T]$ and used BPs and FBPs in each subinterval. A significant reduction occurs in the absolute error as shown in Table 2.

TABLE 1 *The absolute value of error for $\alpha = 0.9$ in Example 1*

Degree of Bezier polynomial (n)	3	4	5	8
Absolute error (e_n) when $\beta = 1$	8.50×10^{-3}	6.51×10^{-5}	2.91×10^{-5}	3.85×10^{-6}
Absolute error (e_n) when $\beta = 0.9$	9.70×10^{-3}	1.44×10^{-4}	1.75×10^{-6}	3.45×10^{-8}

TABLE 2 *The absolute value of error for $\alpha = 0.9$ when use two subintervals in Example 1*

Degree of Bezier polynomial (n)	3	4	5	8
Absolute error (e_n) when $\beta = 1$	6.90×10^{-3}	3.50×10^{-5}	7.91×10^{-7}	3.96×10^{-11}
Absolute error (e_n) when $\beta = 0.9$	2.00×10^{-3}	3.50×10^{-5}	7.91×10^{-7}	3.96×10^{-11}

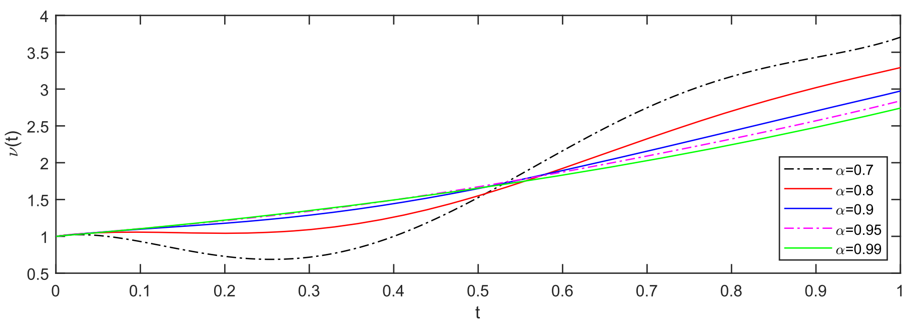


FIG. 1. Approximated graphs of $\nu(t)$ when $n = 5$, $m = 5$, $\beta = 1$.

Recall that the absolute error is defined as follows:

$$e_n = \max_{x \in [0,1]} \mid u(x,1) - \bar{u}(x,1) \mid, \tag{6.1}$$

where $\bar{u}(x,1)$ indicates the approximate solution of the equation at the final time $t = 1$ and $u(x,1) = u^d(x)$ is the tracking function.

Figure 1 shows the approximated graphs of the control function for $\alpha = 0.7, 0.8, 0.9, 0.95, 0.99$ in Example 1.

6.2. *Example 2*

For $0 < \alpha \leqslant 1$, consider the following 2D-FOCP:

$$\textit{minimize}\ J(u,\nu) = \int_0^1 \int_0^1 \frac{x}{2}\left(u^2(x,t) + \nu^2(x,t)\right) \mathrm{d}x\mathrm{d}t$$

subject to

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^2 u(x,t)}{\partial x^2} + \frac{1}{x}\frac{\partial u(x,t)}{\partial x} + \nu(x,t),$$

TABLE 3 *Comparison of the cost function J^* for $\alpha = 1$ and $\beta = 1$ in Example 2*

Degree of Bezier polynomial n, m	m=1 n=4	m=2 n=4	m=2 n=6	m=2 n=7	m=3 n=7	m=3 n=9	m=3 n=10
Ritz method (Mamehrashi & Yousefi, 2017)	$8.1044E-02$	$2.8790E-02$	$1.8283E-02$	$1.6484E-02$	$1.3027E-02$	$1.0405E-02$	$7.5690E-03$
GLJGR collocation method (Parand *et al.*, 2019)	$1.5864E-02$	$1.4089E-02$	$1.3968E-02$	$1.4015E-02$	$4.7307E-03$	$4.7063E-03$	$4.6844E-03$
Presented method	$4.0533E-03$	$6.0884E-04$	$2.7040E-04$	$2.3363E-04$	$6.2131E-05$	$4.2927E-05$	$3.9231E-05$

TABLE 4 *Comparison of the cost function J^* for $\alpha = 0.75$ and $\beta = 0.75$ in Example 2*

Degree of Bezier polynomial n, m	m=1 n=1	m=2 n=1	m=2 n=2	m=3 n=4	m=4 n=5	m=6 n=7
Ritz method (Nemati & Yousefi, 2016)	$2.6721E-01$	$2.5905E-01$	$1.1485E-01$	$3.7741E-02$	$2.5988E-02$	$1.5340E-02$
Presented method	$6.7533E-01$	$1.3967E-01$	$5.1643E-02$	$2.1416E-03$	$1.1821E-03$	$4.9500E-04$

with boundary conditions

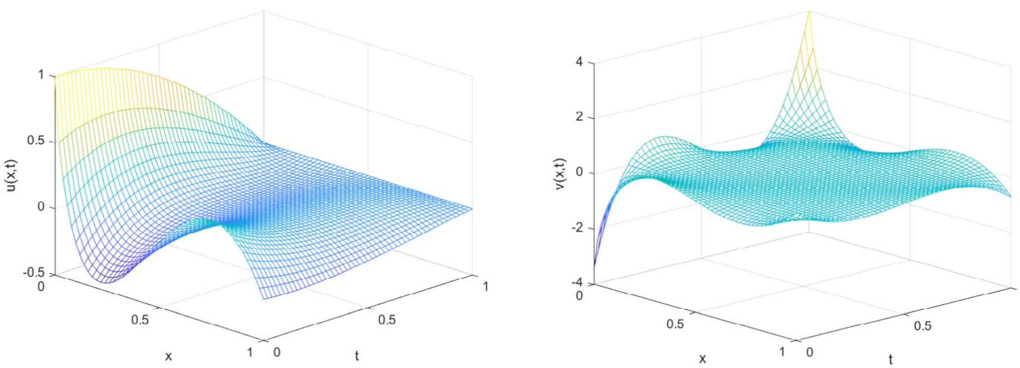
$$u(x,0) = 1 - x^2,$$
$$u(1,t) = 0.$$

This example has been solved by different methods in some papers. For $\alpha = 1$, Mamehrashi & Yousefi, 2017 used the Ritz method based upon the Legendre polynomial basis, and also Parand *et al.*, 2019 used the generalized Lagrangian Jacobi–Gauss–Radau collocation method. Here, we solve this example by the presented method and compare our results with the obtained results in the mentioned papers. As one can see, the cost function J^* acquired by the presented method is improved as shown in Table 3. Also we compare the results obtained by the Ritz method in Nemati & Yousefi (2016) with our obtained results for $\alpha = 0.75$ in Table 4.

Figure 2 shows the approximated graphs of state and control.

7. Conclusion

The aim of this paper was to determine a novel operational matrix to solve 2D-FOCPs. This study showed that the trajectories and control functions are expanded by fractional-order Bernstein functions with unknown control points. The operational matrix of fractional order derivative together with the collocation method are utilized to approximate the analytical solution of these kind of problems. The superior ability of this method is that it can introduce an operational matrix in an arbitrary range of real numbers without changing them into $[0, 1]$. The convergence of the method was proved. Also, two examples were expressed to show the validity and efficiently of this method.



(a) The graph of approximated state (b) The graphs of approximated control

FIG. 2. Approximated graphs of state and control functions in $\alpha = 0.75$ and $\beta = 0.75$ for Example 2.

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