Noname manuscript No.
(will be inserted by the editor)

Adjustable robust balanced hub location problem with uncertain transportation cost

Reza Rahmati · **Hossein Neghabi**

Received: date / Accepted: date

Abstract In this paper, an adjustable robust optimization with a polyhedral uncertainty set is used to deal with uncertain transportation cost in an uncapacitated multiple allocation balanced hub location problem. Adjustable robust optimization is modeled as two-stage or multi-stage problems in which decisions are determined in two or multi separated stages. In two-stage robust optimization, first, the location of hubs is determined in the absence of uncertain parameters, then the second stage decision determined flows path in the presence of uncertainty. Two new mathematical models are proposed for this problem with mixed-integer linear and non-linear structures. Benders decomposition algorithm with stronger cut (Pareto-optimal cut) is used to solve proposed models. Adjustable robust models and accelerated Benders decomposition algorithms are analyzed using well known AP data set with different levels of uncertainty. Also a size reduction method is introduced to solve medium and large instances with good solution quality and shorter computation time. The numerical experiment shows the superiority of the Pareto-optimal cut Benders decomposition algorithm comparing with a classic one. Also, the mixed-integer non-linear model has better results in CPU time and the gap in comparison with the linear integer one. Flow balancing affects hub configuration with a decreasing number of hub facilities. Also by increasing the uncertainty budget, more hubs are established and with increasing discount factor, number of hub facilities are decreased.

Keywords Balanced hub location · Benders decomposition algorithm · Adjustable robust optimization · Uncertain transportation cost · Pareto-optimal cut

R. Rahmati · H. Neghabi
Department of Industrial Engineering, Ferdowsi University of Mashhad, Iran
E-mail: reza.rahmati@mail.um.ac.ir

H. Neghabi
E-mail: hosseinneghabi@um.ac.ir

1 Introduction

Hub location is an important research area in location problems that received much attention in recent decades. Transportation and telecommunication are important applications for hub location problems such as public and air transportation and delivery cargo system. This problem seeks to find the best location for establishing hub facilities and determine the optimal pathway for sending commodities. In the hub location problem, some facilities are considered as a connection point (hubs) between each non-hub nodes. Hence, commodities are transmitted through one or two hubs in this network. With the economy of scale property, the transportation cost of the hub network is decreased. In other words, flows transportation through direct paths between origin-destination nodes raises the cost, because the vehicle that carries flows between origin-destination nodes may be empty on the return path and imposes an extra cost to the network. A hub as a connection point makes it possible to collect scattered flows and to send together or conversely. The firms can reduce their distribution network costs by getting the correct decision for the location of hub facilities in the network. According to this strategic decision, irreparable aftereffects reduced with a good decision. Therefore, this decision affects the performance of the system significantly.

Hub facilities manage commodities combinations according to their destinations. The commodities are entered and collected to each established hubs from each origin nodes. Commodities are transfer from origin to destination according to the nearest path that reduce transportation cost, in other words, origin node chose nearest hub and path to sending commodity. The large volume of commodities may be entered to a hub according to lack of capacity constraint for hubs. The planning and managing of commodities become difficult with flows congestion. Hence with balancing flow in hub facilities, this disadvantage is decreased. In addition, with commodity flow balancing, some of the strategic and operational planning and designs such as designing organizational charts, employing expert human resources, using the same equipment, utilizing similar maintenance planning for hub equipment, making decisions in logistic planning, and research and development planning can be done identically or similarly in hub facilities.

According to the real world, in hub location problem some parameters such as demands, hub establishment cost, transportation cost, distance factors and so on have an uncertain nature, so decide with the deterministic condition is not a good idea. Decide with the deterministic condition may impose an extra cost in the future with an incorrect decision because the value of parameters affects hub facilities established. Although when considering the uncertainty, a decision may need more budget compared to deterministic conditions, ignoring uncertainty in the model may be more costly in the future. Different conditions, such as government policies, customers' behavior, and shipping tariffs may be changed over time. On the other hand, the company has to raise its services price to compensate for imposed costs, which reduces customer satisfaction. Robust optimization and stochastic programming are two approaches used to deal with uncertainty. The probability distribution of uncertain parameters is known in stochastic programming and an optimal solution is obtained by considering probability distribution or several scenarios. Two-stage, multi-stage and chance constraints are different ways that considered in stochastic programming. In the two-stage stochastic programming, decisions are divided into two separate stages. Robust optimization is the other way to deal with uncertainty that uncertain parameters are defined by an interval uncertainty or discrete scenarios. In adjustable robust optimization approaches, first-stage decision variables are determined by focusing on the worst-case of second-stage scenarios. Static and adjustable robust optimization are two kinds of robust discrete optimization (Bertsimas et al. 2011). In R_{static} all decisions are taken before revealing of uncertainty, while in the $R_{adjustable}$ proposed by Ben-Tal et al. (2004) some decisions are taken after revealing of the uncertainty. So the solution obtained by $R_{adjustable}$ maybe less conservatism in comparison with R_{static} ($R_{adjustable} \leq R_{static}$).

The uncapacitated hub location problem with multiple allocation is formulate in this paper with considering flows balancing. In this case, two new models are proposed with mixed-integer linear and non-linear structures. Then, adjustable robust optimization is applied to the proposed models to dealing with uncertain transportation cost. Also classic and accelerated Benders decomposition algorithm are used for solving adjustable robust models. Furthermore, a size reduction method is introduced to solve medium and large instances with good quality and shorter computation time.

The remainder of the paper is organized as follows: Section 2 pays attention to the literature review of hub location problems and robust optimization, Section 3 introduces the mathematical models of balanced hub location problem, Section 4 introduces the adjustable robust optimization applied to the proposed models, Sections 5 and 6 introduces classic and accelerated Benders decomposition algorithms. In Section 7, computational experiments are done for analyzing the performance of accelerated Benders decomposition algorithm to solve proposed models, finally Section 8 concludes the paper and suggests future studies.

2 Literature review

Goldman (1969) introduced the idea of hub location problem. Also the hub location problem for air network background was proposed by O'Kelly (1986). O'kelly (1987) proposed a mathematical model for a one-stop hub location problem. Then Campbell (1994) introduced a model that the number of hubs is determined based on hub establishment and transportation costs. Farahani et al. (2013) provided a study about the solution method and application of the hub location problem until 2012. Contreras et al. (2011) presented a hub location problem with unlimited capacity for hubs and used two-stage stochastic programming to deal with uncertain demands and transportation costs. They used the Benders decomposition algorithm augmented with a sample average approximation method to obtain optimal solution considering of a proper number of scenarios. Alumur et al. (2012) proposed a single and a multiple allocation hub location problems with uncertain demands and hub establishment cost. They presented three formulations to deal with the uncertainty. In the first model, hub establishment cost had an uncertain nature and the objective was minimizing the worst-case regret over all scenarios. The second model was formulated as two-stage stochastic programming with uncertain demands, which Contreras et al. (2011) proved that the problem was equivalent to the model with an expected value of demands. The third problem was modeled as robust-stochastic with both uncertainties in demands and hub establishment cost. Bertsimas et al. (2013) and Lorca et al. (2016) proposed a two-stage and a multistage robust model for the unit commitment problem, respectively. Zeng and Zhao (2013) used adjustable robust optimization to deal with uncertainty in a location transportation problem. They developed a decomposition algorithm to generate new columns and rows in each iteration and decrease the number of iterations and computational time.

Shahabi and Unnikrishnan (2014) developed a robust optimization for single and multiple allocation hub location problems with uncertain demands. The value of the uncertain parameter belongs to an ellipsoidal interval uncertainty. They proposed a mixed-integer non-linear model and transformed it into a conic quadratic problem with a relaxation strategy. They showed that more hubs are established by the robust model comparing with the deterministic one. Ghaffari-Nasab et al. (2015) introduced a robust optimization model for a capacitated single and multiple allocation hub location problems with uncertain demands. Habibzadeh Boukani et al. (2016) proposed a robust formulation for single and multiple allocation hub location problems with uncertain hub establishment cost and capacity. They defined several discrete scenarios and obtained the objective value for each scenario to minimize the worst-case. Merakli and Yaman (2016) proposed a robust model for an uncapacitated p-hab median problem with uncertain demands. Uncertain demands were modeled in two different ways named as hose and hybrid models. In the hose model, the only information about demands was the upper limit on the total flow adjacent to each node, while the latter model comprised both lower and upper limits on each origin/destination node. They used the Benders decomposition algorithm for large-scale instances and analyzed the effect of uncertainty on the model. Zetina et al. (2017) developed a model for an uncapacitated hub location problem and used robust optimization to deal with uncertain demands and transportation costs. The authors used the branch and cut algorithm for a case of uncertain demands and transportation costs.

Merakli and Yaman (2017) presented a multiple allocation hub location problem with capacity constraint and hose demands uncertainty. The authors used two kinds of Benders decomposition algorithm. Talbi and Todosijevi (2017) introduced a robust optimization for an uncapacitated multiple allocation hub location problems. They presented a new way to analyze the robustness of solution in the presence of uncertainties and used a variable neighborhood search algorithm. de Sa et al. (2018) proposed a model for a robust multiple allocation incomplete hub location problems with uncertain demands and hub establishment cost. Furthermore, Benders decomposition and hybrid heuristic approaches were used to solve large scale instances. Tikani et al. (2018) proposed integrated hub location and revenue management problem in the airline industry. Their models try to maximize the revenue of transportation network and minimize hub establishment costs. Karimi and Setak (2018) addressed flow shipment scheduling in the hub location-routing problem in which the hub network is not fully interconnected. In their model, the aim is to allocate each node to the established hub(s) and to schedule the departure time from the nodes.

Ghaffarinasab (2018) developed p-hub median problem and used a robust optimization to deal with uncertain demands. Three different models were proposed: hose, hybrid, and budget of uncertainty models. An efficient tabu search algorithm based matheuristic was applied to solve the proposed models. de S et al. (2018) applied a Benders decomposition algorithm for an incomplete hub location problem with service time requirement. They used a robust optimization approach to deal with uncertain travel times. Rahmati and Bashiri (2018) presented robust optimization for hub location problems with uncertain demands, hub establishment cost and inter hub flow discount factor. voki and Stanimirovi (2019) introduced a new problem for single allocation hub location problem with pricing strategy. Their objective function seeks to find best hub facilities and spoke topology that maximize profit. They show that their proposed matheuristic approach performs better in comparison with commercial solver. Lozkins et al. (2019) proposed hub location problem which demands have uncertain nature. They addresses robust optimization for hub location problem with a set of demands scenarios. Also Benders decomposition algorithm was used for solving their proposed model. Li et al. (2020) considered flows and hub establishment cost as two source of uncertainty in robust hub location problem.

Table 1 Review of hub location problems based on robust optimization

Author(s)	Uncertain approach		Uncertain parameter						Solution approach
	SR	AR	w	c	f	α	cap	t	
Alumur et al. (2012)	✓		✓		✓				Commercial solver
Shahabi and Unnikrishnan (2014)	✓		✓						Commercial solver
Ghaffari-Nasab et al. (2015)	✓		✓						Commercial solver
Habibzadeh Boukani et al. (2016)	✓				✓		✓		Commercial solver
Merakli and Yaman (2016)	✓		✓						Commercial solver
Zetina et al. (2017)	✓		✓	✓					Branch and cut
Merakli and Yaman (2017)	✓		✓						Benders decomposition
Talbi and Todosijevi (2017)	✓		✓						VNS
de Sa et al. (2018)	✓		✓		✓				Benders decomposition
Ghaffarinasab (2018)	✓		✓						Tabu search
de S et al. (2018)	✓							✓	Benders decomposition
Rahmati and Bashiri (2018)	✓					✓			Commercial solver
Lozkins et al. (2019)	✓		✓						Benders decomposition
Li et al. (2020)	✓		✓	✓					Commercial solver
This research		✓		✓					Benders decomposition

SR = Static Robust, AR = Adjustable Robust, w = Demands, c = Transportation cost, f = Hub establishment cost, α = Discount factor, cap = Capacity of hubs, t = Time.

Table 1 shows a brief review of the hub location problems based on robust optimization. It is concluded that there is no research in hub location problem with the adjustable robust optimization method. In other words, static robust optimization with one stage decisions are considered in most researches in the literature. Also according to our knowledge, no research considers balance flows in hub facilities.

3 Balanced hub location problem

In this section, the deterministic model of the balanced hub location problem is introduced. N and H are the set of nodes and potential hubs, respectively and $H \subset N$. w_{ij} is the demands originated at node $i \in N$ and destined to node $j \in N$, f_k is the hub establishment cost, d_{ij} is distances or transportation cost between node $i \in N$ and node $j \in N$, χ is the collecting cost per unit, α is the inter hub flow discount factor and δ is the transfer cost per unit. c_{ij}^{kl} determines the transportation cost from origin node $i \in N$ to the destination node $j \in N$ through the hubs $k \in H$ and $l \in H$, respectively and is calculated by $c_{ij}^{kl} = \chi d_{ik} + \alpha d_{kl} + \delta d_{lj}$. M is a positive big value and is equal to total flows in this paper, Pe is penalty cost for flow unbalancing. z_k is a binary variable and is equal to one when a hub is established in node $k \in H$, otherwise, it gets zero value. x_{ij}^{kl} is the fraction of flows originated at node $i \in N$ and destined to node $j \in N$ using hubs $k \in H$ and $l \in H$. v_k is total flows that entered to each hub. p_{kl} is the flow difference between two hubs that established. λ_{kl}^1 and λ_{kl}^2 are binary variables that used for formulate problem and are equal to one when flows transfer through one and two hubs , respectively, otherwise have zero values.

3.1 Mixed-integer linear programming formulation

In this section, the mathematical model (mixed-integer linear) for the balanced hub location problem is presented. It is assumed that hub facilities have no capacity constraint, and flows between each origin-destination nodes can be transferred by different pathways (multiple allocation property). The mathematical model of the balanced hub location problem can be formulated as follows:

$$\min \sum_{k \in H} f_k z_k + \sum_{i \in N} \sum_{k \in H} \sum_{l \in H} \sum_{j \in N} w_{ij} c_{ij}^{kl} x_{ij}^{kl} + \sum_{k \in H} \sum_{l \in H, k<l} Pe \times p_{kl} \tag{1}$$

Subject to:

$$\sum_{k \in H} \sum_{l \in H} x_{ij}^{kl} = 1 \quad \forall i \in N, j \in N \tag{2}$$

$$\sum_{l \in H} x_{ij}^{kl} + \sum_{l \in H, l \neq k} x_{ij}^{lk} \leq z_k \quad \forall i \in N, j \in N, k \in H \tag{3}$$

$$v_k = \sum_{i \in N} \sum_{j \in N} \sum_{l \in H, l \neq k} w_{ij} x_{ij}^{kl} \quad \forall k \in H \tag{4}$$

$$z_k + z_l = \lambda_{kl}^1 + 2\lambda_{kl}^2 \quad \forall k \in H, l \in H, k < l \tag{5}$$

$$p_{kl} \geq v_k - v_l - M(1 - \lambda_{kl}^2) \quad \forall k \in H, l \in H, k < l \tag{6}$$

$$p_{kl} \geq v_l - v_k - M(1 - \lambda_{kl}^2) \quad \forall k \in H, l \in H, k < l \tag{7}$$

$$z_k \in \{0, 1\} \quad \forall k \in H \tag{8}$$

$$\lambda_{kl}^1, \lambda_{kl}^2 \in \{0, 1\} \quad \forall k \in H, l \in H \tag{9}$$

$$x_{ij}^{kl} \geq 0 \quad \forall i \in N, j \in N, k \in H, l \in H \tag{10}$$

$$p_{kl} \geq 0 \quad \forall k \in H, l \in H \tag{11}$$

$$v_k \geq 0 \quad \forall k \in H \tag{12}$$

Equation (1) is the objective function and minimize total cost. The first term of Equation (1) is hub establishment cost, the second term is transportation cost and the last one is penalty cost for unbalancing flows. Constraints (2) ensures that demands are fully transmitted. Constraints (3) prohibit creating a connection between non-hub nodes. In other words, flows should be transmitted through one or two hubs. According to the constraints (4), the flows that entered to each established hubs are calculated. Constraints (5), (6) and (7) are flow unbalancing constraints. In constraints (5), the values of λ^1_{kl} and λ^2_{kl} are determined based on the number of established hubs that used through origin-destinations. Constraints (6) and (7), calculated flows difference between each established hubs. In other words, a penalty cost is added to the objective function when two hub flows are dissimilar. When λ^2_{kl} is equal to zero, constraints (6) and (7) be redundant. Constraints (8) to (12) determine the domains of decision variables.

3.2 Mixed-integer non-linear programming formulation

Balanced hub location problem can be formulated as a non-linear mathematical programming without big value of M. Hence, constraints (5) and (9) are removed from MIP model and constraints (6) and (7) can be rewrite as constraints (13) and (14), respectively.

$$p_{kl} \geq (v_k - v_l) z_k z_l \quad \forall k \in H, l \in H, k < l \tag{13}$$

$$p_{kl} \geq (v_l - v_k) z_k z_l \quad \forall k \in H, l \in H, k < l \tag{14}$$

4 Adjustable robust optimization

In the previous models, its assumed that all parameters are known in the planning time. While, in real condition some parameters have uncertain nature. These uncertainties are taken into account by adjustable robust optimization. In adjustable or two-stage robust optimization, decisions are divided into two separate states (Ben-Tal et al. 2004). The location of hub facilities is taken in the first-stage without revealing of uncertainties and allocation decisions are taken in the second-stage in presence of uncertainties.

There are three types of uncertainty sets nominated by box, ellipsoidal, and polyhedral uncertainty sets in robust optimization. In the box uncertainty set, all uncertain parameters have the maximum deviation from their nominal values; hence this uncertainty set has excessive conservatism, which increases costs (Soyster 1973). Ellipsoidal uncertainty set has a non-linear structure (2-norm formulation) that all parameters are not at their worst-case values and consequently have less conservatism in comparison with the box one (Ben-Tal et al. 2009). Finally, in the polyhedral uncertainty set proposed by Bertsimas and Sim (2003), the risk-averse and conservatism level is controlled by an uncertainty budget. The high and low value for uncertainty budget increases and decreases the costs and conservatism level, respectively. Polyhedral uncertainty set is used in this paper and the level of conservatism is controlled by an uncertainty budget. Transportation cost is assumed taken a value in interval of $[d^n_{ij} - d^d_{ij}, d^n_{ij} + d^d_{ij}]$. d^n_{ij} and d^d_{ij}, respectively, are nominal and deviation values of transportation cost. Γ is a parameter that denotes the level of conservatism.

4.1 Adjustable robust balanced hub location problem(ARBHLP)

The mathematical model of adjustable robust balanced hub location problem with uncertain transportation cost is as follows:

$$\min_z \sum_k f_k z_k + \max_{d\in D} \min_{(x,p,v)\in \Upsilon(z,d)} \sum_{i\in N}\sum_{k\in H}\sum_{l\in H}\sum_{j\in N} w_{ij} c^{kl}_{ij} x^{kl}_{ij} + \sum_{k\in H}\sum_{l\in H, k<l} Pe \times p_{kl} \tag{15}$$

Subject to:

$$z_k + z_l = \lambda^1_{kl} + 2\lambda^2_{kl} \quad , (z_k, z_l, \lambda^1_{kl}, \lambda^2_{kl}) \in S_1 \tag{16}$$

where $\Upsilon(z,d) = \{(x,p,v) \in S_2 : (2)-(4), (6), (7), (10)-(12)\}$ with $S_1 \subseteq R^n_+$ and $S_2 \subseteq R^m_+$. In this model, z_k, λ^1_{kl} and λ^2_{kl} are the first-stage decision variables, while second-stage decision variables consist of x^{kl}_{ij}, v_k and p_{kl}. Hence, first-stage decision variables are minimized according to the worst-case of second-stage decision variables.

4.2 Adjustable robust non-linear balanced hub location problem(ARNBHLP)

The mathematical model of adjustable robust balanced hub location problem with non-linear formulation is like to MIP formulation. The model of Section 4.1 is considered for non-linear formulation, but with some changing. In other words, constraints (16) is removed and constraints (6) and (7) replaced by constraints (13) and (14), respectively.

The objective function of the adjustable robust mixed-integer linear and non-linear balanced hub location problems have min max-min terms that makes the problems hard to solve. Hence, these models can not be solved directly without implementing decomposition algorithms, such as the Benders decomposition algorithm. Also ARNBHLP problem have non-linear structure in some constraints. That's why, a decomposition algorithm should be used to solve adjustable robust problems, decomposition algorithm can be used to simplify the proposed models. In the next section, Benders decomposition algorithm is applied to solve adjustable robust balanced hub location problems.

5 Benders decomposition algorithm

Benders decomposition algorithm was proposed by Benders (1962). In this algorithm, original model is divided into two separate problems named as master and sub problems. Sub problem is a linear model, so positive variables shall be existing in this problem. Master problem consist of hard variables (e.g. Binary or integer variables).

5.1 Benders decomposition algorithm for ARBHLP

In sub problem, $\bar{z}_k$ and $\bar{\lambda}^2_{kl}$ are solutions that obtained from master problem. Sub problem of proposed model with second-stage decision variables can be written as follows:

$$\max_{d\in D} \min \sum_{i\in N}\sum_{k\in H}\sum_{l\in H}\sum_{j\in N} w_{ij}c^{kl}_{ij}x^{kl}_{ij} + \sum_{k\in H}\sum_{l\in H, k<l} Pe \times p_{kl} \tag{17}$$

Subject to:

$$\sum_{k\in H}\sum_{l\in H} x^{kl}_{ij} = 1 \quad \forall i \in N, j \in N \tag{18}$$

$$\sum_{l\in H} x^{kl}_{ij} + \sum_{l\in H, l\neq k} x^{lk}_{ij} \leq \bar{z}_k \quad \forall i \in N, j \in N, k \in H \tag{19}$$

$$v_k = \sum_{i\in N}\sum_{j\in N}\sum_{l\in H} w_{ij}x^{kl}_{ij} \quad \forall k \in H \tag{20}$$

$$p_{kl} \geq v_k - v_l - M(1-\bar{\lambda}^2_{kl}) \quad \forall k \in H, l \in H, k < l \tag{21}$$

$$p_{kl} \geq v_l - v_k - M(1-\bar{\lambda}^2_{kl}) \quad \forall k \in H, l \in H, k < l \tag{22}$$

$$x^{kl}_{ij} \geq 0 \quad \forall i \in N, j \in N, k \in H, l \in H \tag{23}$$

$$p_{kl} \geq 0 \quad \forall k \in H, l \in H \tag{24}$$

$$v_k \geq 0 \quad \forall k \in H \tag{25}$$

The objective function (17) of sub problem have a max-min term, so with dual method can transform min form to max form. In other words, in the term of minimization, that consist of positive variables, strong duality theorem ensures the same objective value for primal and dual problems. y_{ij}, g^k_{ij}, r_k, h_{kl} and b_{kl} are dual variables of constraints (18), (19), (20), (21) and (22), respectively. The dual of sub problem can be written as follows:

$$\max_{d\in D} \sum_{i\in N}\sum_{j\in N} y_{ij} - \sum_{i\in N}\sum_{j\in N}\sum_{k\in H} g^k_{ij}\bar{z}_k + \sum_{k\in H}\sum_{l\in H, k<l} (h_{kl}+b_{kl})(-M(1-\bar{\lambda}^2_{kl})) \tag{26}$$

Subject to:

$$y_{ij}-g_{ijk}-g_{ijl}+w_{ij}r_k \leq w_{ij}(\chi d_{ik}+\alpha d_{kl}+\delta d_{lj}) \quad \forall i \in N, j \in N, k \in H, l \in H, k \neq l \tag{27}$$

$$y_{ij} - g_{ijk} + w_{ij}r_k \leq w_{ij}(\chi d_{ik} + \delta d_{kj}) \qquad \forall i \in N, j \in N, k \in H \tag{28}$$

$$h_{kl} + b_{kl} \leq Pe \qquad \forall k \in H, l \in H, k < l \tag{29}$$

$$\sum_{l \in H, l<k} (h_{lk} - b_{lk}) + \sum_{l \in H, l>k} (-h_{kl} + b_{kl}) - r_k \leq 0 \qquad \forall k \in H \tag{30}$$

$$g_{ij}^k \geq 0 \qquad \forall i \in N, j \in N, k \in H \tag{31}$$

$$h_{kl} \geq 0 \qquad \forall k \in H, l \in H \tag{32}$$

$$b_{kl} \geq 0 \qquad \forall k \in H, l \in H \tag{33}$$

In constraints (27) and (28), exactly two and one hubs are considered for transfer flows between origin-destination nodes, respectively. However, uncertain parameter (transportation cost) still not considered at this model. For considering uncertain transportation cost, constraints (27) and (28) that contain d_{ij}, with polyhedral uncertainty set, should be changed. Hence, a new positive decision variable (μ_{ij}) is considered at model for controlling amount of uncertain parameters that deviate from their nominal values. The worst-case value for uncertain parameters in primal model is equal to the best-case value for uncertain parameters in dual model (Jeyakumar and Li 2010). So in constraints (35) and (36), polyhedral uncertainty set is written for transportation cost parameter according to the best-case of right hand sides.

$$\max \sum_{i \in N} \sum_{j \in N} y_{ij} - \sum_{i \in N} \sum_{j \in N} \sum_{k \in H} g_{ij}^k \bar{z}_k + \sum_{k \in H} \sum_{l \in H, k<l} (h_{kl} + b_{kl})(-M(1 - \bar{\lambda}_{kl}^2)) \tag{34}$$

Subject to:

$$y_{ij} - g_{ij}^k + w_{ij}r_k \leq w_{ij} \begin{pmatrix} \chi(d_{ik}^n + d_{ik}^d \mu_{ik})+ \\ \alpha(d_{kl}^n + d_{kl}^d \mu_{kl})+ \\ \delta_{lj}(d_{lj}^n + d_{lj}^d \mu_{lj}) \end{pmatrix} \quad \forall i \in N, j \in N, (k \neq l) \in H \tag{35}$$

$$y_{ij} - g_{ijk} + w_{ij}r_k \leq w_{ij} \begin{pmatrix} \chi(d_{ik}^n + d_{ik}^d \mu_{ik})+ \\ \delta_{kj}(d_{kj}^n + d_{kj}^d \mu_{kj}) \end{pmatrix} \qquad \forall i \in N, j \in N, k \in H \tag{36}$$

$$\sum_{i \in N} \sum_{j \in N} \mu_{ij} \leq \Gamma \tag{37}$$

$$0 \leq \mu_{ij} \leq 1 \qquad \forall i \in N, j \in N \tag{38}$$

(29)-(33)

Constraint (37) is control maximum parameters that allowed have deviation from their nominal values according to the uncertainty budget (Γ). Master problem with first-stage decision variables can be written as follows:

$$\min \sum_{k \in H} f_k z_k + \eta \tag{39}$$

Subject to:

$$\eta \geq \begin{pmatrix} \sum_{i \in N} \sum_{j \in N} \bar{y}_{ij} - \sum_{i \in N} \sum_{j \in N} \sum_{k \in H} \bar{g}_{ij}^k z_k + \\ \sum_{k \in H} \sum_{l \in H, k<l} (\bar{h}_{kl} + \bar{b}_{kl})(-M(1 - \lambda_{kl}^2)) \end{pmatrix} \tag{40}$$

$$\sum_{k \in H} z_k \geq 1 \tag{41}$$

(5),(8), (9)

Constraints (40) is the optimality cut that be added in each iteration to the master problem according to the dual of sub problem solutions. Constraint (41) guarantees that at least one hub should be established. This constraint eliminate need of feasibility cut for the master problem.

5.2 Benders decomposition algorithm for ARNBHLP

Also in non-linear model, sub and master problems have some low changing in comparison of MIP model. Constraints (5) and (9) are removed from master problem and optimality cut is changed as constraint (42).

$$\eta \geq \sum_{i \in N} \sum_{j \in N} \bar{y}_{ij} - \sum_{i \in N} \sum_{j \in N} \sum_{k \in H} \bar{g}_{ij}^k z_k \tag{42}$$

The dual of sub problem for ARNBHLP model can be written as follows:

$$\max \sum_{i \in N} \sum_{j \in N} y_{ij} - \sum_{i \in N} \sum_{j \in N} \sum_{k \in H} g_{ij}^k \bar{z}_k \tag{43}$$

Subject to:

$$(h_{kl} + b_{kl}) \bar{z}_k \bar{z}_l \leq Pe \quad \forall k \in H, l \in H, k < l \tag{44}$$

$$\sum_{l \in H, l<k} \bar{z}_k \bar{z}_l (h_{lk} - b_{lk}) + \sum_{l \in H, l>k} \bar{z}_k \bar{z}_l (-h_{kl} + b_{kl}) - r_k \leq 0 \quad \forall k \in H \tag{45}$$

(31)-(33), (35)-(38)

Equation (43) is the objective function of dual sub problem for non-linear model. Also constraints (29) and (30) in dual of sub problem (MIP model) changed as constraints (44) and (45), respectively.

Considering the Benders decomposition algorithm, the binary variables in the master problem are fixed in dual sub problem and behave as parameters. Hence the constraints (13) and (14) act like linear constraints structure. Algorithm 1 shows the pseudo code of Benders decomposition algorithm. OFV(dsp) denotes the objective function value of the dual sub problem.

Algorithm 1: Benders decomposition algorithm

Data: LB= $-\infty$, UB=$+\infty$
while UB - $LB > \varepsilon$ **do**
  Step 1 : Solve master problem
  $\bar{z}_k \leftarrow z_k$ (linear and non-linear models)
  $\bar{\lambda}^2_{kl} \leftarrow \lambda^2_{kl}$ (linear model)
  $LB \leftarrow \sum_{k \in H} f_k \bar{z}_k + \eta$
  Step 2 : Solve dual of sub problem
  $\bar{y}_{ij} \leftarrow y_{ij}$
  $\bar{g}^k_{ij} \leftarrow g^k_{ij}$
  $\bar{h}_{kl} \leftarrow h_{kl}$
  $\bar{b}_{kl} \leftarrow b_{kl}$
  $UB \leftarrow OFV(dsp) + \sum_{k \in H} f_k \bar{z}_k$
end

6 Enhancement of Benders decomposition algorithm

The solution that obtained from dual of sub problem may be have alternative solutions with same objective function. Magnanti and Wong (1981) proposed a method to obtaining a solution that generate strong optimality cut. In other words, this method find solution that generate higher value for lower bound. A cut is called a Pareto-optimal cut if it is not dominated by any other cuts. A cut generated using a solution $(y^a_{ij}, g^{ka}_{ij}, h^a_{kl}, b^a_{kl})$ achieved from the dual sub problem is stronger than $(y^e_{ij}, g^{ke}_{ij}, h^e_{kl}, b^e_{kl})$ if and only if:

$$\begin{pmatrix} \sum_{i \in N} \sum_{j \in N} y^a_{ij} - \sum_{i \in N} \sum_{j \in N} \sum_{k \in H} g^{ka}_{ij} \bar{z}_k + \\ \sum_{k \in H} \sum_{l \in H, k<l} (h^a_{kl} + b^a_{kl})(-M(1 - \bar{\lambda}^2_{kl})) \end{pmatrix} \geq \begin{pmatrix} \sum_{i \in N} \sum_{j \in N} y^e_{ij} - \sum_{i \in N} \sum_{j \in N} \sum_{k \in H} g^{ke}_{ij} \bar{z}_k + \\ \sum_{k \in H} \sum_{l \in H, k<l} (h^e_{kl} + b^e_{kl})(-M(1 - \bar{\lambda}^2_{kl})) \end{pmatrix} \tag{46}$$

6.1 Pareto-optimal cut model for ARBHLP

The Pareto-optimal cut model of linear mathematical model can be written as follows:

$$\max \sum_{i \in N} \sum_{j \in N} y_{ij} - \sum_{i \in N} \sum_{j \in N} \sum_{k \in H} g^k_{ij} z^o_k + \sum_{k \in H} \sum_{l \in H, k<l} (h_{kl} + b_{kl})(-M(1 - \lambda^{2o}_{kl})) \tag{47}$$

Subject to:

$$\sum_{i \in N} \sum_{j \in N} y_{ij} - \sum_{i \in N} \sum_{j \in N} \sum_{k \in H} g^k_{ij} \bar{z}_k + \sum_{k \in H} \sum_{l \in H, k<l} (h_{kl} + b_{kl})(-M(1 - \bar{\lambda}^2_{kl})) = \bar{z}_{dsp} \tag{48}$$

$$(29)\text{-}(33),\ (35)\text{-}(38)$$

z_k^o and λ_{kl}^{2o} are core points and have a value between zero and one. Constraint (48) guarantees that the value of the dual sub problem should not be changed.

6.2 Pareto-optimal cut model for ARNBHLP

The following mathematical model should be solved after a dual sub problem to obtain a Pareto-optimal cut for the non-linear model.

$$\max \sum_{i\in N}\sum_{j\in N} y_{ij} - \sum_{i\in N}\sum_{j\in N}\sum_{k\in H} g_{ij}^k z_k^o \tag{49}$$

Subject to:

$$\sum_{i\in N}\sum_{j\in N} y_{ij} - \sum_{i\in N}\sum_{j\in N}\sum_{k\in H} g_{ij}^k \bar{z}_k = \bar{z}_{dsp} \tag{50}$$

$$(31)\text{-}(33),\ (35)\text{-}(38),\ (44)\text{-}(45)$$

z_k^o is core point and have a value between zero and one. According to the constraint (50), the dual sub problem value cannot be changed in the Pareto-optimal cut model. Hence the Pareto model seeks to find a solution that generates a stronger optimality cut for the master problem. A pseudo-code of the Pareto-optimal cut Benders decomposition algorithm is presented in Algorithm 2.

Algorithm 2: Pareto-optimal cut Benders decomposition algorithm

Data: LB= $-\infty$, UB=$+\infty$
while UB - $LB > \varepsilon$ **do**
  Step 1 : Solve master problem
  $\bar{z}_k \leftarrow z_k$ (linear and non-linear models)
  $\bar{\lambda}_{kl}^2 \leftarrow \lambda_{kl}^2$ (linear model)
  $LB \leftarrow \sum_{k\in H} f_k \bar{z}_k + \eta$
  Step 2 : Solve dual of sub problem
  Step 3 : Solve Pareto model
  $\bar{y}_{ij} \leftarrow y_{ij}$
  $\bar{g}_{ij}^k \leftarrow g_{ij}^k$
  $\bar{h}_{kl} \leftarrow h_{kl}$
  $\bar{b}_{kl} \leftarrow b_{kl}$
  $UB \leftarrow OFV(dsp) + \sum_{k\in H} f_k \bar{z}_k$
end

7 Computational experiment

In this section, the performance of the proposed models and Benders decomposition algorithm are analyzed. The well-known set of instances such as the AP data set (Australian Post) is used in this paper for analysis that used widely in hub location problems. This data set is consists of demands, distances (transportation cost) between each node. It is assumed that $\chi = \delta = 1$ and because of the economy of scale property, α have a value lower than χ and δ. The inter hub flow discount factor (α) is allowed to be 0.1, 0.2, ..., 0.8. The value of the uncertainty budget (Γ) is presented as a percentage in the further analysis ($\Gamma \in \{0.1, 0.2, ..., 1\}$). In other words, an uncertainty budget of 0.2 denotes that 20 percent of parameters are allowed to have deviation from their nominal values. As an example, for 10-node instance ($|N| \times |N-1| = 10 \times 9 = 90$ parameters) and uncertainty budget of 0.1, 9 transportation cost parameters are allowed to have deviation from their nominal values. In this paper, the fixed cost of establishing hub facilities introduced by Correia et al. (2018) is calculated according to Equations (51) and (52). The hub establishment cost factor is denoted by c as an input parameter. A high value for c increases hub establishment cost and decreases the number of established hubs. Therefore, this parameter should be tuned. Proposed adjustable robust balanced hub location problems were solved by GAMS software (CPLEX solver) and run in an Intel Core i7 with 3.7 GHz CPU and 32 GB of RAM.

$$o_k = \sum_{j \in N} w_{kj} \quad \forall k \in H \tag{51}$$

$$f_k = c \times \log(o_k) \quad \forall k \in H \tag{52}$$

In Section 7.1 both the hub location problem (HLP) and balanced hub location problem (BHLP) are compared with each other. The proper value of the core point in the Pareto-optimal cut Benders decomposition algorithm for each uncertainty budget is analyzed and presented in Section 7.2. Section 7.3 evaluates the performance of the accelerated Benders decomposition algorithm in comparison with the classic Benders decomposition algorithm, also the linear and the non-linear models are compared in different values of uncertainty budget (Γ) and discount factor (α). In Section 7.4, a size reduction method is introduced for solving medium and large instances with good quality and shorter computation time.

7.1 HLP and BHLP analysis

In this section, the hub location problem (HLP) and balanced hub location problem (BHLP) are compared with each other in deterministic conditions. Figures 1 and 2 shows hub networks for HLP and BHLP, respectively for AP 10-node instance. In Figures 1 and 2, square and circle shapes represent hubs and nodes, respectively. According to the Figure 1, the value of the commodities flow that entered to each established hubs are dissimilar. Flows that entered to hub 5 is much less than hub 2 and 8, while hub 8 has the largest flow value. In Figure 2, the value of the commodities flow that entered to each hub is similar and equal to 161.7.

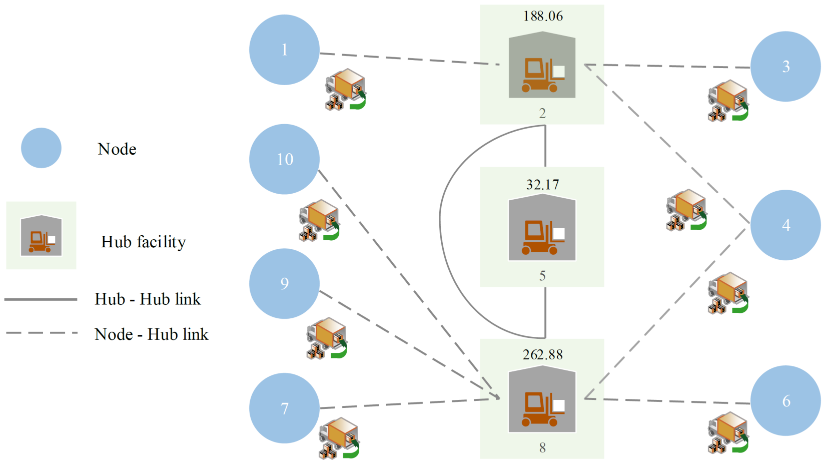


Fig. 1 Hub network of HLP for AP 10-node instance, $\alpha = 0.2$

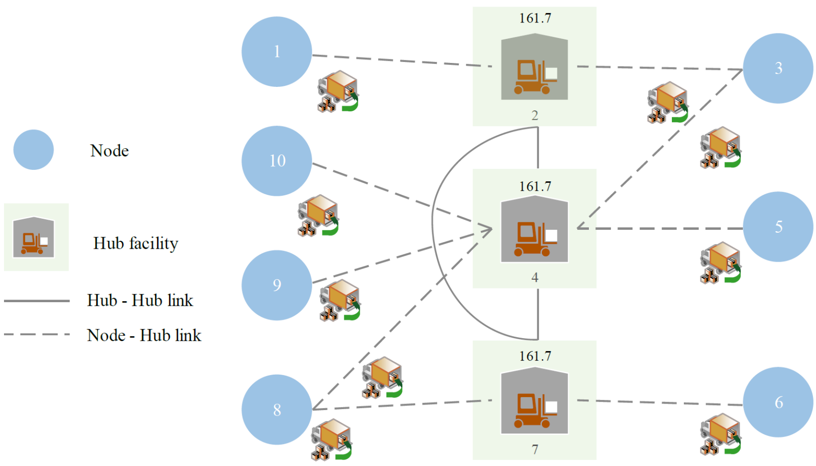


Fig. 2 Hub network of BHLP for AP 10-node instance, $\alpha = 0.2$

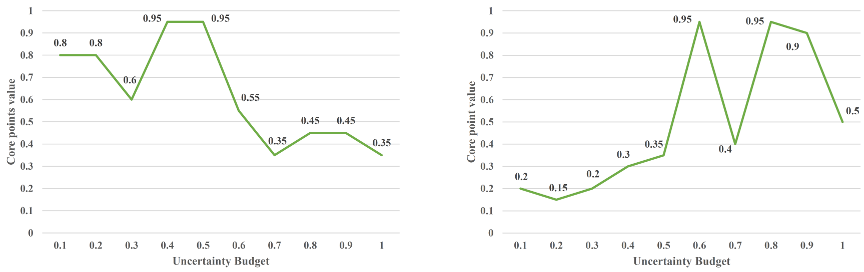
In Table 2, the objective function value (cost) and hub configuration of HLP and BHLP are compared with different discount factors from 0.1 to 0.8 and different penalty costs for AP 20-node instance. With increasing in discount factor α, the objective function value (cost) is increased, while the number of hub facilities are decreased. In the last column of Table 2, the check mark represents the full balancing of hub facilities, and the associated unbalancing penalty cost is zero. It should be noted that penalty cost values are only used in the BHLP model, so the HLP model results are the same for each penalty cost value.

Table 2 Comparison between HLP and BHLP for AP 20-node instance

α	HLP		Penalty	BHLP		
	Obj.	Hub configuration		Obj.	Hub configuration	Balance
0.1	48056.64	2,5,8,11,15,18	0.5	49875.14	2,8,11,15,18	
			1	51422.43	2,8,15,16,18	
			1.5	52032.35	2,12,14,18	
			2	52094.31	2,12,14,18	✓
0.2	51398.20	2,8,11,15,18	0.5	53135.81	2,8,15,18	
			1	54091.72	2,12,14,18	
			1.5	54422.64	2,12,14,18	
			2	54591.46	2,12,14,18	
			2.5	54637.66	2,12,14,18	✓
0.3	54099.75	2,8,15,18	0.5	55642.52	2,12,14,18	
			1	56126.77	2,12,14,18	
			1.5	56463.63	2,12,14,18	
			2	56697.62	2,12,14,18	
			2.5	56802.87	2,12,14,18	
			3	56815.87	2,12,14,18	✓
0.4	56403.86	2,8,14,18	0.5	57519.60	2,12,14,18	
			1	57951.05	2,12,14,18	
			1.5	58233.01	7,14,18	
			2	58328.89	7,14,18	✓
0.5	58202.14	2,9,12,18	0.5	58963.85	7,14,18	
			1	59349.60	7,14,18	
			1.5	59470.70	7,14,18	
			2	59477.55	7,14,18	✓
0.6	59578.45	2,9,18	0.5	60143.09	7,14,18	
			1	60431.23	8,18	✓
0.7	60602.26	7,14,18	0.5	61038.06	8,18	✓
0.8	61352.80	7,14,18	0.5	61413.56	8,18	
			1	61414.33	8,18	✓

7.2 Core point analysis

Performance of the Pareto-optimal cut Benders decomposition algorithm is dependent on the core point value. In other words, the core point value affects the number of iterations and computation time to achieve the optimal solution. Therefore in this section, an analysis is done to obtain a good core point value for further experiments. It should be noted that linear and non-linear models have two and one core points, respectively. For reducing the number of calculations, it is assumed that the two core points of the linear model take the same values in each optimization. Furthermore, the number of iterations and CPU time are obtained with different value of the core points $(z_k^o, \lambda_{kl}^{2o} \in \{0.05, 0.1, ..., 0.95\})$ and uncertainty budgets $(\Gamma \in \{0.1, 0.2, ..., 1\})$ for AP 10-node instance. Based on the number of iteration and computation time, the best core point value for linear and non-linear models has been illustrated in Figures 3a and 3b, respectively. The core point value of each uncertainty budget shown in these figures is used for other instances (AP 20-node and 40-node instances).



(a) (b)

Fig. 3: Best core points value for (a) linear and (b) non-linear models, AP 10-node instance and $\alpha = 0.2$

7.3 Performance evaluation of the Pareto-optimal cut and the classic Benders decomposition algorithms

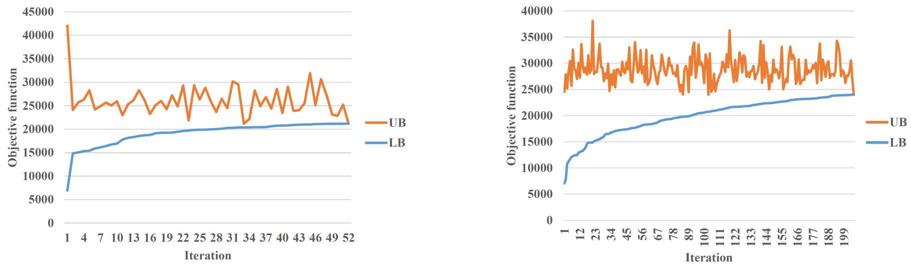
In this section, the performance of the Pareto-optimal cut and the classic Benders decomposition algorithms in solving the proposed mathematical models (linear and non-linear) are investigated under different values of the uncertainty budget ($\Gamma \in \{0.1, 0.2, ..., 1\}$) and discount factor ($\alpha \in \{0.2, ..., 0.4\}$). Also, the impacts of model parameters such as uncertainty budget (Γ) and discount factor (α) on hub network configuration and its objective function are analyzed. As illustrated in Table 3 for AP 10-node instance for ARBHLP (linear model) and ARNBHLP (non-linear model), it could be concluded that the Benders decomposition algorithm with Pareto-optimal cut performs better than the classic one considering the number of iterations to convergence. As an example, when Γ is equal to 1 the Pareto-optimal cut Benders decomposition algorithm in ARBHLP model converges to an optimal solution in 26 iterations while the classic algorithm can find the same solution in 187 iterations.

By increasing the uncertainty budget value (Γ) in two models, the superiority of the Pareto-optimal cut Benders decomposition algorithm becomes more evident in CPU time (seconds) in comparison with the classic one. The results show the non-linear model's superiority compared to the linear one in both the number of iterations and CPU time.

Table 3 Comparison of the Pareto and classic Benders decomposition algorithm, AP 10-node instance

α	Γ	ARBHLP				ARNBHLP			
		Classic Benders		Pareto Benders		Classic Benders		Pareto Benders	
		# Iterations	CPU time (s)	# Iterations	CPU time (s)	# Iterations	CPU time (s)	# Iterations	CPU time (s)
0.2	0.1	102	25.98	69	24.93	53	6.46	29	13.51
	0.2	164	40.02	122	49.67	79	10.76	42	34.22
	0.3	191	48.59	147	61.42	100	13.38	53	44.39
	0.4	206	54.32	103	42.11	82	10.97	41	32.50
	0.5	191	47.45	76	31.11	81	9.52	22	12.67
	0.6	185	45.43	37	13.76	77	9.42	12	4.30
	0.7	185	41.00	25	9.30	76	9.40	8	3.48
	0.8	187	43.25	31	9.76	76	9.72	10	2.72
	0.9	187	44.04	27	7.70	76	8.99	16	3.91
	1	187	44.16	26	7.00	76	9.54	13	2.94
0.3	0.1	83	18.02	73	28.65	38	3.79	33	15.71
	0.2	135	30.69	123	50.77	67	8.10	60	44.43
	0.3	183	43.46	142	61.00	89	12.07	75	60.96
	0.4	166	42.29	110	45.72	84	12.62	74	55.06
	0.5	165	40.64	80	32.93	63	7.83	31	18.19
	0.6	168	45.43	43	17.55	65	7.88	14	4.60
	0.7	156	39.24	38	15.47	66	7.87	12	4.49
	0.8	167	36.57	29	10.33	65	7.49	9	2.45
	0.9	167	36.49	25	8.05	65	7.74	9	2.15
	1	167	36.50	24	7.68	65	8.03	8	1.81
0.4	0.1	75	15.27	58	21.29	42	5.35	33	15.33
	0.2	141	32.54	106	42.25	74	9.20	65	46.05
	0.3	168	41.73	141	60.46	89	12.09	51	42.02
	0.4	170	40.35	117	49.57	94	12.03	39	29.61
	0.5	165	43.12	85	34.28	80	10.19	28	15.62
	0.6	156	35.09	46	18.24	57	7.57	22	7.64
	0.7	149	35.06	26	9.60	65	7.63	12	4.69
	0.8	154	34.04	17	5.29	69	8.64	11	2.95
	0.9	155	34.12	12	3.23	69	7.99	10	2.95
	1	155	35.21	11	2.75	69	8.53	6	1.15
Average		161	38.34	65.63	26.06	71.7	9.02	28.27	17.75

Figures 4a and 4b shows the convergence trend for the Pareto-optimal cut and classic Benders decomposition algorithm, respectively in a case of uncertainty budget equals to 0.4. The value of the lower bound in each iteration must be greater than the value in the previous iterations. But upper bound does not have such behavior. In other words, upper bound can take different values in each iteration that may be greater or smaller than the upper bounds in previous iterations. Comparing two figures confirms the rapid convergence of the Pareto-optimal cut Benders decomposition algorithm to solve the proposed model compared with the classic one.



(a) (b)

Fig. 4: Convergence of the (a) Pareto and (b) classic Benders decomposition algorithm, $\alpha = 0.2$, $\Gamma = 0.4$

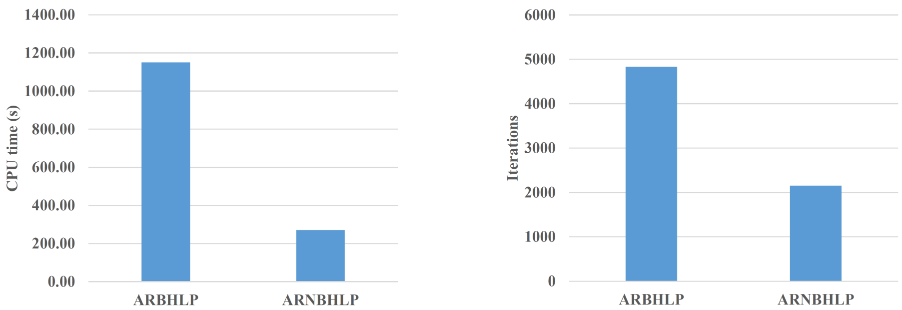
Table 4 represents the effects of different values of Γ and α on the objective function and hub configuration in AP 10-node instance. The discount factor has a direct impact on the objective function value, and the number of established hubs is decreased by increasing the discount factor (α). Furthermore, the uncertainty budget directly affects both the objective function value and the number of established hubs. In other words, the objective function increases with more established hubs in case of high values for uncertainty budget.

The total number of arcs between nodes is equal to $|N| \times |N-1|$, but all these arcs aren't considered in the hub location problem. For example, total arcs that considered in Figure 1 are equal to 11. So according to this discussion and Table 4, the result in the objective function and hub configuration are the same when the uncertainty budget is greater than 0.4 and increasing in uncertainty budget ($\Gamma > 0.4$) don't affect the optimal solution. In other words, the objective function value and optimal hub structure remain constant when the uncertainty budget is greater than 0.4. The total CPU time (seconds) and the total number of iterations are calculated according to the Table 3 results and illustrated in Figures 5a and 5b, respectively. ARNBHLP have much less number of iterations and CPU time (seconds) in comparison with ARBHLP.

Different uncertainty budget (Γ) and discount factor (α) for AP 20-node instances have been solved by the linear and non-linear models, and their results are presented in Tables 5 and 6, respectively. It should be noted that a maximum specific execution time (8 hours) is set for these instances and the results for all instances are reported according to this predetermined time.

Table 4 The effects of different Γ and α on objective function and hub configuration, AP 10-node instance

α	Γ	Objective function	Hub configuration
0.2	0.1	19959.49	2, 8
	0.2	22523.11	2, 8
	0.3	23832.85	2, 4, 7
	0.4-1	23991.99	2, 4, 7
0.3	0.1	20122.68	2, 8
	0.2	22705.59	2, 8
	0.3	24291.27	2, 4, 7
	0.4-1	24376.89	2, 8
0.4	0.1	20298.14	2, 8
	0.2	22877.02	2, 8
	0.3	24514.49	2, 8
	0.4-1	24720.00	2, 8



(a) (b)

Fig. 5: Comparison between ARBHLP and ARNBHLP according to the total summation of their (a) CPU time and (b) iterations, AP 10-node 30 instances

The results confirm that the Pareto-optimal cut Benders decomposition algorithm performs better than the classic one in both models (the linear and non-linear models). In the linear model, the average gap obtained by the classic Benders decomposition algorithm is equal to %60.25, while the corresponding gap in the Pareto model is equal to %46.18. Also, the Pareto-optimal cut Benders decomposition algorithm has fewer iterations compared to the classic one. Besides, the average gap in the non-linear model for the classic Benders decomposition algorithm is equal to %13.56, while the associated gap for the Pareto one is equal to %9.07. Furthermore, comparing the gap results obtained with linear and non-linear models reveal the appreciable performance of the non-linear model in comparison to the linear one. In other words, Tables 5 and 6 confirm that the non-linear model has better performance in comparison with the linear model in obtaining the lower bounds and upper bounds.

Table 5 Comparison of the Pareto-optimal cut Benders decomposition algorithm with classic one for AP 20-node instance, ARBHLP model

α	Γ	Classic				Pareto			
		LB	UB	% Gap	# Iteration	LB	UB	% Gap	# Iteration
0.2	0.1	44081.25	99725.75	55.80	2675	50457.20	95472.81	47.15	1517
	0.2	40900.54	98519.46	58.48	2678	46585.81	93376.21	50.11	1345
	0.3	39106.83	107666.59	63.68	2780	46668.47	97351.10	52.06	1287
	0.4	31518.14	98127.87	67.88	2925	58141.35	100520.92	42.16	1348
0.3	0.1	47753.81	101727.59	53.06	2665	53900.19	98570.87	45.32	1489
	0.2	43087.11	93447.37	53.89	2660	50968.39	93571.73	45.53	1516
	0.3	40834.29	113652.91	64.07	2762	51900.19	95570.87	45.69	1467
	0.4	34566.51	116115.86	70.23	3087	57582.83	105553.52	45.45	1423
0.4	0.1	50057.64	86206.77	41.93	2674	58778.17	87573.79	32.88	1811
	0.2	44419.48	115709.09	61.61	2627	53951.13	105382.51	48.80	1490
	0.3	41465.07	105166.43	60.57	2693	54087.82	102555.29	47.26	1434
	0.4	34361.85	121683.44	71.76	3111	56884.00	117845.32	51.73	1350
Average				60.25	2778.08			46.18	1456.42

LB = Lower bound, UB = Upper bound, Gap = ((UB-LB)/UB)

Table 6 Comparison of the Pareto-optimal cut Benders decomposition algorithm with classic one for AP 20-node instance, ARNBHLP model

α	Γ	Classic				Pareto			
		LB	UB	% Gap	# Iteration	LB	UB	% Gap	# Iteration
0.2	0.1	76163.61	89094.89	14.51	3757	76524.75	83761.25	8.64	1282
	0.2	82292.84	93570.39	12.05	3663	83382.37	89591.58	6.93	993
	0.3	84773.03	97064.55	12.66	3767	85329.23	91655.63	6.90	781
	0.4	85872.93	97098.52	11.56	3657	86143.75	94048.35	8.40	754
0.3	0.1	78640.27	84237.29	6.64	3792	78940.46	83147.29	5.06	1156
	0.2	85303.36	101253.07	15.75	3726	86423.75	100146.32	13.70	984
	0.3	89081.85	110922.20	19.69	3615	90147.76	104270.40	13.54	948
	0.4	91087.08	104756.51	13.05	3650	91417.43	99112.42	7.76	923
0.4	0.1	80631.25	91863.59	12.23	3816	80897.14	92141.43	12.20	1231
	0.2	87380.76	100250.75	12.84	3608	88993.45	101753.73	12.54	971
	0.3	91879.79	110864.27	17.12	3546	93423.73	99459.41	6.07	932
	0.4	95076.35	111288.84	14.57	3583	95981.32	103231.20	7.02	890
Average				13.56	3681.66			9.07	987.08

LB = Lower bound, UB = Upper bound, Gap = ((UB-LB)/UB)

7.4 Size reduction method

In the hub location problem, each node can send commodities flows through the network and can be chosen as a hub facility. Therefore, all nodes have the potential to be selected and operated as hubs. The assumption of considering all nodes as potential hubs increases the complexity of the models. It is correct that all nodes can be selected as hubs, but all of these nodes are not necessarily suitable for operating as hub facilities. In other words, the more suitable nodes are better to be considered as potential hubs. Hub establishment cost, demands, transportation cost, and uncertainty are the sources that affect the hub network. So in this section, a method is introduced to reduce the number of potential nodes that can be chosen as hubs. For this purpose, nodes are sorted via the biggest value of total input and output flows. Also, nodes are sorted from smallest to biggest value according to the fixed hub establishment costs. Then about 20 percent of the top of these sorted lists are chosen and considered as potential hub nodes. Furthermore, to complete the list of potential hubs, the deterministic model is solved according to the worst-case value for the transportation cost parameter, and the selected hubs are also added to the list. In other words, in the deterministic model, it is assumed that all parameters have a deviation from their nominal values. The solution obtained from this deterministic model (established hubs) is also added to the potential hub's list.

Table 7 shows that the full size and size reduction model are compared for the AP 10-node instance. It should be noted that in this section, the non-linear model and the Pareto-optimal cut Benders decomposition algorithm are used for analysis. The results show that the size reduction model obtained the optimal solution in much fewer iterations and CPU time than the full-size model.

Table 7 The impact of size reduction method for AP 10-node instance

α	Γ	Full size			Size reduction			
		Objective function	# Iterations	CPU time (s)	Objective function	# Iterations	CPU time (s)	% Gap
0.2	0.1	19959.49	29	13.51	19959.49	12	1.14	0
	0.2	22523.12	42	34.22	22523.12	12	1.05	0
	0.3	23832.85	53	44.39	23832.85	13	1.27	0
	0.4	23991.99	41	32.50	23991.99	8	0.98	0
0.3	0.1	20122.68	33	15.71	20122.68	12	1.03	0
	0.2	22705.59	60	44.43	22705.59	12	1.02	0
	0.3	24291.27	75	60.96	24291.27	14	1.28	0
	0.4	24376.89	74	55.06	24376.89	8	0.99	0
0.4	0.1	20298.15	33	15.33	20298.15	12	0.98	0
	0.2	22877.02	65	46.05	22877.02	12	1.16	0
	0.3	24514.49	51	42.02	24514.49	14	1.27	0
	0.4	24720.00	39	29.61	24720.00	10	1.09	0
Average			49.58	36.15		11.58	1.11	0

Tables 8 and 9 show the results for the full-size model and size reduction method for AP 20-node and AP 40-node instances, respectively. The results confirm that the size reduction method obtains a partially good solution in much less computation time than the full-size model. The results reported in Table 8 indicate the size reduction method obtains a high-quality solution close to the lower bound of the full-size model. Hence, it can be ensured that the solution obtained from the size reduction method is so close to the optimal solution. Furthermore, Table 9 indicates that the full-size model cannot obtain the right solution, and on average, has a %76.05 gap. Therefore, the lower bound obtained from the full-size model does not have enough validation to evaluate the size reduction solution.

Table 8 The impact of size reduction method for AP 20-node instance

α	Γ	Full size					Size reduction			
		LB	UB	% Gap	# Iterations	CPU time (s)	Objective function	# Iterations	CPU time (s)	Hub configuration
0.2	0.1	76524.75	83761.25	8.64	1282	28800	76981.98	299	115.49	2,8,15,17,18,19
	0.2	83382.37	89591.58	6.93	993	28800	84193.55	304	123.79	2,8,12,15,17,18,19
	0.3	85329.23	91655.63	6.90	781	28800	86849.14	367	126.95	2,8,11,15,17,18,19
	0.4	86143.75	94048.35	8.40	754	28800	87151.28	217	66.94	2,8,11,15,17,18,19
0.3	0.1	78940.46	83147.29	5.06	1156	28800	79288.66	358	130.84	2,8,15,17,18,19
	0.2	86423.75	100146.32	13.70	984	28800	87837.22	338	164.31	2,8,10,11,14,17,18
	0.3	90147.76	104270.40	13.54	948	28800	92096.68	354	160.98	2,8,12,15,17,18,19
	0.4	91417.43	99112.42	7.76	923	28800	92283.65	289	106.35	2,7,12,14,15,18,19
0.4	0.1	80897.14	92141.43	12.20	1231	28800	81381.25	378	142.02	2,8,14,17,18,19
	0.2	88993.45	101753.73	12.54	971	28800	90817.85	430	177.29	2,8,14,17,18,19
	0.3	93423.73	99459.41	6.07	932	28800	96052.08	410	146.65	2,8,12,14,17,18
	0.4	95981.32	103231.20	7.02	890	28800	96749.42	294	90.72	2,8,12,14,17,18
Average				9.07	987.08	28800		336.50	129.36	

Table 9 The impact of size reduction method for AP 40-node instance

α	Γ	Full size					Size reduction			
		LB	UB	% Gap	# Iterations	CPU time (s)	Objective function	# Iterations	CPU time (s)	Hub configuration
0.2	0.1	32456.78	134791.12	75.92	1169	28800	127486.51	823	3112.67	7,11,12,16,21,28,29,38
	0.2	37636.28	147908.23	74.55	808	28800	129787.44	349	629.68	7,11,12,16,21,28,29,38
	0.3	35379.42	158818.28	77.72	964	28800	131454.14	436	779.62	7,12,18,22,28,29,34,38
	0.4	30360.74	150953.51	79.89	1093	28800	133615.67	465	887.71	7,12,18,22,28,29,30,34,38
0.3	0.1	34567.42	131873.45	73.79	1059	28800	129973.23	822	3241.68	11,16,21,22,28,29,34
	0.2	39898.57	140913.85	71.69	823	28800	132121.42	365	895.27	11,12,8,22,28,30,34
	0.3	35175.64	151794.56	76.83	974	28800	136745.19	249	413.81	4,7,11,18,21,22,34,38
	0.4	30951.42	141084.03	78.06	1106	28800	138308.18	193	414.60	4,7,12,18,21,22,34,38
0.4	0.1	38456.46	149876.23	74.34	1204	28800	132483.42	651	2542.55	12,18,22,28,29
	0.2	43062.73	153158.69	71.88	870	28800	139513.47	256	686.41	12,18,22,28,29
	0.3	35522.93	172797.90	79.44	971	28800	145694.86	320	545.63	11,12,18,21,28,29,30
	0.4	37425.79	174236.45	78.52	986	28800	147395.29	258	440.41	4,12,16,18,22,28,29,34
Average				76.05	1002.25	28800		432.25	1215.84	

8 Conclusion

In this paper, two new models with mixed-integer linear and non-linear structures were proposed for uncapacitated multiple allocation hub location problem with balancing flows in established hubs. An adjustable robust optimization was used to deal with uncertain transportation cost in the proposed models. The location of hub facilities is taken in the first stage without revealing of uncertainties and allocation decisions are taken in the second stage in presence of uncertainty. Polyhedral uncertainty set was used in this paper and the level of conservatism was controlled by an uncertainty budget. An accelerated Benders decomposition algorithm was applied to solve proposed models. Furthermore, a size reduction method was proposed to solve larger instances of proposed models. Computational experiments showed that the Benders decomposition algorithm with Pareto-cuts had less iterations comparing with the classic Benders decomposition algorithm for all values of the uncertainty budgets. Also demand handling is more comfortable with balancing flows that entered to each hub facility. According to the results, number of hub facilities are increased when uncertainty is high. Also adjustable robust balanced hub location problem with mixed-integer non-linear structure had better result in CPU time and Benders iterations in comparison with linear one. In addition, results show that size reduction method had a good quality and speed. As a direction for the future study, modeling of a problem with time consistency in any period can be considered. Proposing of other algorithms to solve the problem can be considered as another direction for the future work.

References

Alumur SA, Nickel S, da Gama FS (2012) Hub location under uncertainty. Transportation Research Part B: Methodological 46(4):529 – 543, https://doi.org/10.1016/j.trb.2011.11.006, URL http://www.sciencedirect.com/science/article/pii/S019126151100172X

Ben-Tal A, Goryashko A, Guslitzer E, Nemirovski A (2004) Adjustable robust solutions of uncertain linear programs. Mathematical Programming 99(2):351–376, 10.1007/s10107-003-0454-y, URL https://doi.org/10.1007/s10107-003-0454-y

Ben-Tal A, El Ghaoui L, Nemirovski L (2009) Robust optimization. Princeton University Press

Benders JF (1962) Partitioning procedures for solving mixed-variables programming problems. Numerische Mathematik 4(1):238–252, 10.1007/BF01386316, URL https://doi.org/10.1007/BF01386316

Bertsimas D, Sim M (2003) Robust discrete optimization and network flows. Mathematical Programming 98(1):49–71, 10.1007/s10107-003-0396-4, URL https://doi.org/10.1007/s10107-003-0396-4

Bertsimas D, Brown DB, Caramanis C (2011) Theory and applications of robust optimization. SIAM Rev 53(3):464–501, 10.1137/080734510, URL http://dx.doi.org/10.1137/080734510

Bertsimas D, Litvinov E, Sun XA, Zhao J, Zheng T (2013) Adaptive robust optimization for the security constrained unit commitment problem. IEEE Transactions on Power Systems 28(1):52–63, 10.1109/TPWRS.2012.2205021

Campbell JF (1994) Integer programming formulations of discrete hub location problems. European Journal of Operational Research 72(2):387 – 405, https://doi.org/10.1016/0377-2217(94)90318-2, URL http://www.sciencedirect.com/science/article/pii/0377221794903182

Contreras I, Cordeau JF, Laporte G (2011) Stochastic uncapacitated hub location. European Journal of Operational Research 212(3):518 – 528, https://doi.org/10.1016/j.ejor.2011.02.018, URL http://www.sciencedirect.com/science/article/pii/S0377221711001494

Correia I, Nickel S, da Gama FS (2018) A stochastic multi-period capacitated multiple allocation hub location problem: Formulation and inequalities. Omega 74:122 – 134, https://doi.org/10.1016/j.omega.2017.01.011

Farahani RZ, Hekmatfar M, Arabani AB, Nikbakhsh E (2013) Hub location problems: A review of models, classification, solution techniques, and applications. Computers & Industrial Engineering 64(4):1096 – 1109, https://doi.org/10.1016/j.cie.2013.01.012, URL http://www.sciencedirect.com/science/article/pii/S0360835213000326

Ghaffari-Nasab N, Ghazanfari M, Teimoury E (2015) Robust optimization approach to the design of hub-and-spoke networks. The International Journal of Advanced Manufacturing Technology 76(5):1091–1110, 10.1007/s00170-014-6330-5, URL https://doi.org/10.1007/s00170-014-6330-5

Ghaffarinasab N (2018) An efficient matheuristic for the robust multiple allocation p-hub median problem under polyhedral demand uncertainty. Computers & Operations Research 97:31 – 47, https://doi.org/10.1016/j.cor.2018.04.021, URL http://www.sciencedirect.com/science/article/pii/S0305054818301126

Goldman AJ (1969) Optimal locations for centers in a network. Transportation Science 3(4):352–360, 10.1287/trsc.3.4.352, URL https://doi.org/10.1287/trsc.3.4.352, https://doi.org/10.1287/trsc.3.4.352

Habibzadeh Boukani F, Farhang Moghaddam B, Pishvaee MS (2016) Robust optimization approach to capacitated single and multiple allocation hub location problems. Computational and Applied Mathematics 35(1):45–60, 10.1007/s40314-014-0179-y, URL https://doi.org/10.1007/s40314-014-0179-y

Jeyakumar V, Li GY (2010) Strong duality in robust convex programming: Complete characterizations. SIAM Journal on Optimization 20(6):3384–3407, 10.1137/100791841, URL https://doi.org/10.1137/100791841, https://doi.org/10.1137/100791841

Karimi H, Setak M (2018) Flow shipment scheduling in an incomplete hub location-routing network design problem. Computational and Applied Mathematics 37, 10.1007/s40314-016-0368-y, URL https://doi.org/10.1007/s40314-016-0368-y

Li S, Fang C, Wu Y (2020) Robust hub location problem with flow-based set-up cost. IEEE Access 8:66178–66188

Lorca , Sun XA, Litvinov E, Zheng T (2016) Multistage adaptive robust optimization for the unit commitment problem. Operations Research 64(1):32–51, 10.1287/opre.2015.1456, URL https://doi.org/10.1287/opre.2015.1456, https://doi.org/10.1287/opre.2015.1456

Lozkins A, Krasilnikov M, Bure V (2019) Robust uncapacitated multiple allocation hub location problem under demand uncertainty: minimization of cost deviations. Journal of Industrial Engineering International 10.1007/s40092-019-00329-9, URL https://doi.org/10.1007/s40092-019-00329-9, https://doi.org/10.1007/s40092-019-00329-9

Magnanti T, Wong R (1981) Accelerating benders decomposition: Algorithmic enhancement and model selection criteria. Operations Research 29, 10.1287/opre.29.3.464

Merakli M, Yaman H (2016) Robust intermodal hub location under polyhedral demand uncertainty. Transportation Research Part B: Methodological 86:66 – 85, https://doi.org/10.1016/j.trb.2016.01.010, URL http://www.sciencedirect.com/science/article/pii/S0191261516000199

Merakli M, Yaman H (2017) A capacitated hub location problem under hose demand uncertainty. Computers & Operations Research 88:58 – 70, https://doi.org/10.1016/j.cor.2017.06.011, URL http://www.sciencedirect.com/science/article/pii/S0305054817301491

O'Kelly ME (1986) The location of interacting hub facilities. Transportation Science 20(2):92–106, 10.1287/trsc.20.2.92, URL https://doi.org/10.1287/trsc.20.2.92, https://doi.org/10.1287/trsc.20.2.92

O'kelly ME (1987) A quadratic integer program for the location of interacting hub facilities. European Journal of Operational Research 32(3):393 – 404, https://doi.org/10.1016/S0377-2217(87)80007-3, URL http://www.sciencedirect.com/science/article/pii/S0377221787800073

Rahmati R, Bashiri M (2018) Robust hub location problem with uncertain inter hub flow discount factor. Proceedings of the International Conference on Industrial Engineering and Operations Management 2018, URL https://www.scopus.com/inward/record.uri?eid=2-s2.0-85066947491&partnerID=40&md5=ff40f0fc810988915f2f54211837196a

de Sa EM, Morabito R, de Camargo RS (2018) Benders decomposition applied to a robust multiple allocation incomplete hub location problem. Computers & Operations Research 89:31 – 50, https://doi.org/10.1016/j.cor.2017.08.001, URL http://www.sciencedirect.com/science/article/pii/S0305054817302009

Shahabi M, Unnikrishnan A (2014) Robust hub network design problem. Transportation Research Part E: Logistics and Transportation Review 70:356 – 373, https://doi.org/10.1016/j.tre.2014.08.003, URL http://www.sciencedirect.com/science/article/pii/S1366554514001434

Soyster AL (1973) Technical noteconvex programming with set-inclusive constraints and applications to inexact linear programming. Operations Research 21(5):1154–1157, 10.1287/opre.21.5.1154, URL https://doi.org/10.1287/opre.21.5.1154

de S EM, Morabito R, de Camargo RS (2018) Efficient benders decomposition algorithms for the robust multiple allocation incomplete hub location problem with service time requirements. Expert Systems with Applications 93:50 – 61, https://doi.org/10.1016/j.eswa.2017.10.005, URL http://

www.sciencedirect.com/science/article/pii/S0957417417306802

Talbi EG, Todosijevi R (2017) The robust uncapacitated multiple allocation p-hub median problem. Computers & Industrial Engineering 110:322 – 332, https://doi.org/10.1016/j.cie.2017.06.017, URL http://www.sciencedirect.com/science/article/pii/S0360835217302693

Tikani H, Honarvar M, Mehrjerdi Y (2018) Developing an integrated hub location and revenue management model considering multi-classes of customers in the airline industry. Computational and Applied Mathematics 37, 10.1007/s40314-017-0512-3, URL https://doi.org/10.1007/s40314-017-0512-3

Zeng B, Zhao L (2013) Solving two-stage robust optimization problems using a column-and-constraint generation method. Operations Research Letters 41(5):457 – 461, https://doi.org/10.1016/j.orl.2013.05.003, URL http://www.sciencedirect.com/science/article/pii/S0167637713000618

Zetina CA, Contreras I, Cordeau JF, Nikbakhsh E (2017) Robust uncapacitated hub location. Transportation Research Part B: Methodological 106:393 – 410, https://doi.org/10.1016/j.trb.2017.06.008

voki D, Stanimirovi Z (2019) A single allocation hub location and pricing problem. Computational and Applied Mathematics 39(40), 10.1007/s40314-019-1025-z, URL https://doi.org/10.1007/s40314-019-1025-z