AKCE International Journal of Graphs and Combinatorics

ISSN: (Print) (Online) Journal homepage: https://www.tandfonline.com/loi/uakc20

End-regularity of generalized bicycle graph

A. Rajabi , A. Erfanian & A. Azimi

AKCE INTERNATIONAL JOURNAL OF GRAPHS AND COMBINATORICS
https://doi.org/10.1016/j.akcej.2019.11.002

End-regularity of generalized bicycle graph

A. Rajabi[a], A. Erfanian[b], and A. Azimi[c*]

[a]Department of Pure Mathematics, Ferdowsi University of Mashhad, Mashhad, Iran; [b]Department of Mathematics and Center of Excellence in Analysis on Algebraic Structures, Ferdowsi University of Mashhad, Mashhad, Iran; [c]Department of Mathematics, University of Neyshabur, Neyshabur, Iran

ABSTRACT

A graph G is called End-regular, if every endomorphism of G is regular as a monoid. In this article, we investigate End-regularity of bicycle graphs. Moreover, a generalization of bicycle graph is defined and gives a characterization of End-regular generalized bicycle graphs. Furthermore, properties End-idempotent-closed and End-orthodox have been considered for these graphs. Finally, it has been proved that when join of two generalized bicycle graphs is End-regular.

KEYWORDS

Bicycle graph; endomorphism; End-regular; End-orthodox; generalized bicycle graph; join two graph

2000 MATHEMATICS SUBJECT CLASSIFICATION

Primary 05C25; 05C60; Secondary 20M1; 20M19

1. Introduction

Let G and H be two simple graphs. A homomorphism between G and H is a function $\alpha : V(G) \rightarrow V(H)$ such that:

$$\{u, v\} \in E(G) \Rightarrow \{\alpha(u), \alpha(v)\} \in E(H).$$

If $G = H$ then a homomorphism between G and H is said to be an endomorphism. The set of all endomorphisms of G is denoted by $End(G)$. A monoid is a semi-group with an identity element. For $f, g \in End(G)$, the product of gf denoted by $gf(x) = g(f(x))$ for all $x \in V(G)$. The $End(G)$ forms a monoid. Recall that a element f in a monoid S is called regular if there exists $g \in S$ such that $fgf = f$ and g is called a pseudo-inverse of f. The monoid S is called regular if each element in S is regular. An element $e \in S$ is called an idempotent if $e = ee$. The monoid S is called idempotent-closed if it close under composition of its idempotents. Also, monoid S is called orthodox if it has both regular and idempotent-closed. A graph G is said to be End-regular if each element of monoid $End(G)$ is regular. The set of all idempotent endomorphisms of G denoted by $Idp(G)$. A graph G is said to be End-idempotent-closed if $Idp(G)$ is a monoid. A graph G which is End-regular and End-idempotent-closed called End-orthodox.

The concept of regularity first introduced by Von Neumann in 1936 in ring theory [12], after that some of authors try to answer this question that, what graphs have regular endomorphism monoid? (Such an open problem was raised by Marki in [7]). Respond to this question is difficult generally. Hence, researchers try to answer this question for specific graphs. For instance, End-regularity of connected bipartite graphs explores in [15] and End-regular split graphs obtained in [6]. Also, it is known End-regularity n-prism graphs, unicyclic graphs, book graphs, generalized wheel graphs, and complement of a path (see [8–11, 13] and [3] for more details).

In this article, first we study the End-regularity of bicycle graphs. Then a generalization of bicycle graphs has been introduced and classifies all generalized bicycle graphs which all endomorphisms are regular. Next, we show that generalized bicycle graphs are End-idempotent-closed if and only if they have two cycles of different lengths where at least one of lengths is odd. Also, there is no End-orthodox generalized bicycle graph. In Section 2, we investigate End-regularity of join two generalized bicycle graphs. Recall that join of two graphs G and H is a graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{\{x, y\} | x \in V(G), y \in V(H)\}$. It is usually denoted by $G + H$.

Let $f \in End(G)$ and I_f be endomorphism image of G under f that is a subgraph of G with $V(I_f) = f(V(G))$ and $f(a)$ and $f(b)$ are adjacent if there exists $c \in f^{-1}(f(a))$ and $d \in f^{-1}(f(b))$ such that c and d are adjacent in G. For a fixed $f \in End(G)$, define a relation ρ_f given by $(a, b) \in \rho_f$ if and only if $f(a) = f(b)$ for every $a, b \in V(G)$. It is easily to see that ρ_f is an equivalent relation and for every $a \in V(G)$, the related equivalence class of a will denote by $[a]_{\rho_f}$.

2. End-regular generalized bicycle graphs

We start this section by definition of bicycle graphs.

CONTACT A. Rajabi az.ra.ma20@gmail.com Department of Pure Mathematics, Ferdowsi University of Mashhad, Mashhad, Iran.
*A. Azimi is now affiliated with the Ferdowsi University of Mashhad, Mashhad, Iran.

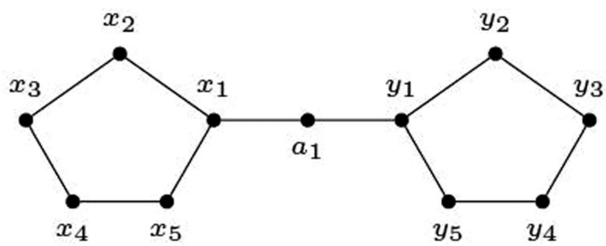

Figure 1. B (5, 5, 2).

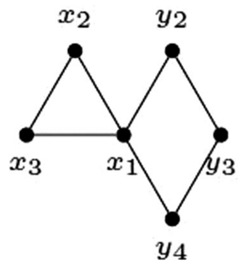

Figure 2. B (3, 4, 0).

Definition 2.1. We define a bicycle graph to be a graph considering either of two cycles with exactly one vertex in common, or two vertex-disjoint cycles and a path joining them having only its end-vertices in common with the cycles. In other words, a bicycle graph consists vertex-disjoint cycles C_m, C_n, and a path P_r of length r joining them. We denote bicycle graph by $B(m, n, r)$, with $V(C_m) = \{x_1, ..., x_m\}, V(C_n) = \{y_1, ..., y_n\}$ and $V(P_r) = \{a_1, ..., a_{r-1}\}$ such that $x_1 \sim a_1$ (i.e. x_i is adjacent with a_1) and $a_{r-1} \sim y_1$. If r = 0, x_1 and y_1 are coincided, then we put instead y_1, x_1 (see Figures 1 and 2).

In this article, we explore End-regularity of a bicycle graph. For that, we need to remind some known results as in ([4, 8, 14, 15] and [1]) which are necessary for our proofs. The following lemma gives us a necessary and sufficient condition for End-regularity of connected bipartite graph.

Lemma 2.2. (Wilkeit [15], Theorem 3.4). *Let G be a connected bipartite graph. Then G is End-regular if and only if G is one of the following graphs.*

(1) *Complete bipartite graph;*
(2) *Tree T with diameter 3;*
(3) *Cycle C_6 and C_8;*
(4) *Path with five vertices.*

We call graph G an 8-graph if there exists two cycle subgraphs C_m, C_n with $C_m \cup C_n = G$ and $C_m \cap C_n = P_r$ for some $r \geq 0$. We denote this 8-graph by $C_{m,n}; P_r$. It is obvious that if $r = 0$, $B(m, n, r)$ and $C_{m,n}; P_r$ are isomorphic.

End-regularity of 8-graphs has been determined in [14] by the following theorem.

Theorem 2.3 ([14], Theorem 3.2.10). *Exactly the following 8-graphs are End-regular.*

- $C_{2n+1,2n+1}; P_r$ *for every* $r \geq 0$ *and* $n \geq 1$,
- $C_{2n+1,4}; P_1$ *for every* $n \geq 1$,
- $C_{4,4}; P_2 = K_{2,3}$.

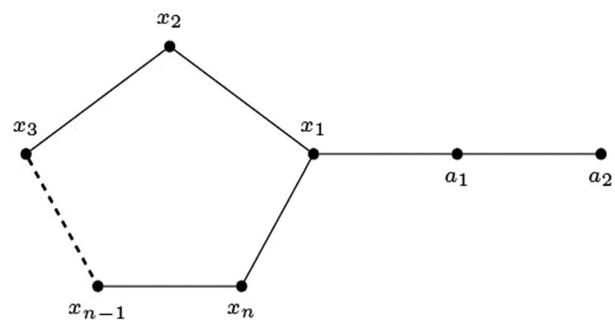

Figure 3. $U(n,3)$.

We put $U(n, 3)$ for an unicyclic graph that have girth n obtained by appending a pendent vertex of P_3 with a vertex of C_n (see Figure 3). Ma [8] proved that this graph is not End-regular for $n \geq 3$.

Lemma 2.4 (Ma [8], Lemma 2.4). *Let $n \geq 3$, then $U(n, 3)$ is not End-regular.*

Recall that a subgraph H of G is a retract if there is a homomorphism f from G to H such that the restriction of f to H is the identity map. The following lemma comes from [15].

Lemma 2.5. (Wilkeit [15]). *Let H be a retract of G. If H is not End-regular, then G cannot be End-regular.*

The following lemma gives a relation between a regular endomorphism and its pseudo-inverse.

Lemma 2.6 (Ma [8], Lemma 2.3). *Let f be a regular endomorphism of a graph G with a pseudo-inverse g, then $g(x) \in f^{-1}(x)$ for any $x \in f(V(G))$.*

Now, we state the following lemma which plays an important role to prove regularity of an endomorphism.

Lemma 2.7. (Li [4], Theorem 2.5). *Let G be a graph and $f \in End(G)$. Then f is regular if and only if there exists idempotents $g, h \in End(G)$ such that $I_f = I_g$ and $\rho_f = \rho_h$.*

The following lemma gives us a necessary and sufficient condition for regularity an endomorphism.

Lemma 2.8. (S. Fan [1], Lemma 2.2). *Let f be an endomorphism of the graph G. Then the following are equivalent:*

(i) *f is regular.*
(ii) *I_f is a retract and there exists a retract A such that $f|_{V(A)}$ is an isomorphism from A to I_f.*

Now, we start our discussions on End-regularity of the bicycle graphs.

Theorem 2.9. *If at least one of m and n is an even number, then $B(m, n, r)$ is not End-regular.*

Proof. Let both n and m are even numbers, then $B(m, n, r)$ has no odd cycles and is a bipartite graph, so by Lemma 2.2, $B(m, n, r)$ is not End-regular. Now, let m be an odd number and n be an even number. We define the following endomorphism.

$$f(x)=\begin{cases} x & x=x_i \\ a_1 & x=a_i(i \text{ is odd}), x=y_i(i \text{ odd and } r \text{ even}), x=y_i(i \text{ even and } r \text{ odd}) \\ a_2 & x=a_i(i \text{ is even}), x=y_i(i,r \text{ even}), x=y_i(i,r \text{ odd}) \end{cases}$$

f is a retraction of $B(m,n,r)$ to $U(2k+1,3)$, so $B(m,n,r)$ is not End-regular by Lemmas 2.4 and 2.5. □

Theorem 2.10. *Let m, n be odd numbers and $m \neq n$. Then $B(m,n,r)$ is not End-regular.*

Proof. Without loss of generality let $m<n$. We define $f,g \in End(B(m,n,r))$ by the following rules:

$$f(x)=\begin{cases} x & x=x_i, 1\le i\le m \\ a_1 & x=a_i, i \text{ odd} \\ a_2 & x=a_i, i \text{ even} \\ x_{i-1} & x=y_i, 2\le i\le m+1 \\ x_1 & x=y_i, i\ge m+2, i \text{ odd} \\ a_1 & x=y_1, x=y_i, i\ge m+2, i \text{ even} \end{cases} \quad g(x)=\begin{cases} x & x=x_i, 1\le i\le m \\ a_1 & x=a_i, i \text{ odd} \\ a_2 & x=a_i, i \text{ even} \\ x_i & x=y_i, 1\le i\le m \\ x_1 & x=y_i, i\ge m+1, i \text{ even} \\ a_1 & x=y_i, i\ge m+2, i \text{ odd} \end{cases}$$

If r is even then f is a retraction of $B(m,n,r)$ to $U(m,3)$ and if r is odd then g is a retraction of $B(m,n,r)$ to $U(m,3)$. So by Lemmas 2.4 and 2.5, we can see that $B(m,n,r)$ is not End-regular. □

Now, we are going to consider End-regularity of $B(m,n,r)$ whenever m and n are odd and $m=n$.

Theorem 2.11. *If $r\le 3$ and m is odd, then $B(m,m,r)$ is End-regular.*

Proof. We recall that $B(m,n,0)$ and $C_{m,n};P_0$ are isomorphic graphs. So, if $r=0$ then $B(m,m,r)$ is End-regular by Theorem 2.3. Put $C_n=C'_m$ and other labels are similar to Definition 2.1. Let $f\in End(B(m,m,r))$. Since homomorphic image of every odd cycle in a graph G is an odd cycle in G, then $f(C_m)=C_m$ or $f(C_m)=C'_m$ and $f(C'_m)=C_m$ or $f(C'_m)=C'_m$. Thus, we have the following four cases.

(*i*) $f(C_m)=C_m, f(C'_m)=C'_m$ (ii) $f(C_m)=C'_m, f(C'_m)=C_m$
(iii) $f(C_m)=C_m, f(C'_m)=C'_m$ (iv) $f(C_m)=C'_m, f(C'_m)=C'_m$

First, let $r=1$. If (*i*) or (*ii*) are true then f is an automorphism and so it is a regular endomorphism. Suppose that (*iii*) is true, Clearly I_f is a m-cycle. We define $g\in Idp(B(m,m,1))$ as the following.

$$g(x)=\begin{cases} x_i & x=x_i, 1\le i\le m \\ x_j & x=y_i, f(y_i)=f(x_j) \end{cases}$$

Note that in the definition f, g, if $i>m$, then will be considered as module m. We use this rule for all sections of the article, as well. We see that I_g is also a m-cycle, so $I_f=I_g$. Now, if $f(x_j)=f(y_i)$ then $g(x_j)=x_j=g(y_i)$. So $\rho_f=\rho_g$, and by Lemma 2.7 we have $B(m,m,1)$ is End-regular. If (*iv*) holds, then similar as the proof of case (*iii*) we can define $g\in Idp(B(m,m,1))$ and prove that $I_f=I_g$ and $\rho_f=\rho_g$. Hence, we omit more details and the proof follows.

Now, assume that $\mathbf{r=2}$ and $x_1\sim a_1\sim y_1$. In two cases (*i*) and (*ii*), we have $f(a_1)=a_1$, because a_1 is the only common neighbor of C_m and C'_m. Therefore, f is an automorphism and is a regular endomorphism. Let case (*iii*) be true. Then $f(a_1)\in C_m$, we define $g,h\in Idp(B(m,m,2))$ by the following rules.

$$g(x)=\begin{cases} x_i & x=x_i, y=y_i \\ x_2 & x=a_1 \end{cases} \quad h(x)=\begin{cases} x_i & x=x_i, 1\le i\le m \\ x_k & x=y_i, f(y_i)=f(x_k) \\ x_l & x=a_1, f(a_1)=f(x_l) \end{cases}$$

It is clear that $I_f=I_g$ and $\rho_f=\rho_h$. So f is regular by Lemma 2.7. If case (*iv*) is true then by a similar method as above we can see that f is regular.

Suppose that $r=3$ and $x_1\sim a_1\sim a_2\sim y_1$. If (*i*) is true $f(a_i)=a_i$ for $i=1$, 2 and if (*ii*) is true, $f(a_1)=a_2$ and $f(a_2)=a_1$, Therefore in these cases, f is an automorphism and is a regular endomorphism. Let (*iii*) be true. In this case, $Img(f)=\{C_m\}$ or $Img(f)=\{C_m,a_1\}$. If $Img(f)=\{C_m\}$ defined $g,h\in Idp(B(m,m,3))$ by the following rules.

$$g(x)=\begin{cases} x_i & x=x_i \\ x_2 & x=a_1 \\ x_1 & x=a_2 \\ x_{i+1} & x=y_i \end{cases} \quad h(x)=\begin{cases} x_i & x=x_i, 1\le i\le m \\ x_k & x=y_i, f(y_i)=f(x_k) \\ x_l & x=a_j, f(a_j)=f(x_l), j=1,2 \end{cases}$$

Clearly, $I_f=I_g$ and $\rho_f=\rho_h$. So f is regular by Lemma 2.7.

Let $Img(f)=\{C_m,a_1\}$ define $g,h\in Idp(B(m,m,3))$ by the following rules.

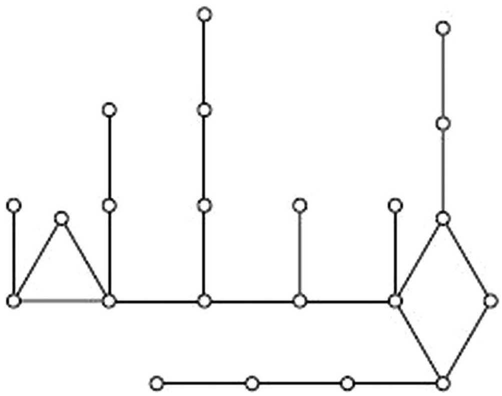

Figure 4. $GB\ (3, 4, 3, P_{2,0,1}, P_{1,2,0,3}, P_{3,1})$.

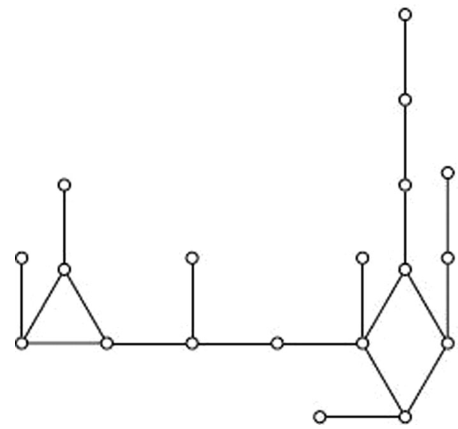

Figure 5. $GB\ (3,4,3, P_{0,1,1}, P_{1,3,2,1}, P_{1,0})$.

$$g(x) = \begin{cases} x_i & x = x_i \\ a_1 & x = a_1 \\ x_1 & x = a_2 \\ x_{i+1} & x = y_i \end{cases} \quad h(x) = \begin{cases} x_i & x = x_i, 1 \le i \le m \\ a_1 & x = a_1 \\ x_k & x = y_i, f(y_i) = f(x_k) \\ x_l & x = a_2, f(a_2) = f(x_l) \end{cases}$$

Clearly, $I_f = I_g$ and $\rho_f = \rho_h$. So f is regular by Lemma 2.7.

If case (iv) is true then by a similar method as above we can see that f is regular. □

Theorem 2.12. *$B(m, m, r)$ is not End-regular for all $r \ge 4$.*

Proof. We proceed the proof in the following two cases.

Case 1. $r \ge 4$ and even. The following endomorphism is a retraction of $B(m, m, r)$ to $U(m, 3)$.

$$g(x) = \begin{cases} x_i & x = x_i, x = y_i 1 \le i \le m \\ a_1 & x = a_i, i \text{ is odd} \\ a_2 & x = a_i, i \text{ is even} \end{cases}$$

Again in this case, $B(m, m, r)$ is not End-regular by Lemmas 2.4 and 2.5.

Case 2. $r \ge 5$ and odd. We define a retraction of $B(m, m, r)$ to $U(m, 3)$ by the following rules.

$$h(x) = \begin{cases} x_i & x = x_i, 1 \le i \le m \\ a_1 & x = a_i, i \text{ is odd} \\ a_2 & x = a_2 \\ x_1 & x = a_2, i \ne 2 \text{ is even} \\ x_{i+1} & x = y_i 1 \le i \le m \end{cases}$$

If $r \ge 5$ and odd then $B(m, m, r)$ is not End-regular by Lemmas 2.4 and 2.5. □

Hence, by Theorems 2.9–2.12, we can state necessary and sufficient conditions for bicycle graph $B(m, n, r)$ to be End-regular.

Corollary 2.13. *$B(m, n, r)$ is End-regular if and only if $m = n$ are odd numbers and $r \le 3$.*

Now, we are going to define generalized bicycle graphs and explore when they are End-regular.

Definition 2.14. Let $B(m, n, r)$ be a bicycle graph with vertex sets $V(C_m) \cup V(C_n) \cup V(P_r)$, where $V(C_m) = \{x_1, \ldots, x_m\}$, $V(C_n) = \{y_1, \ldots, y_n\}$ and $V(P_r) = \{x_1, a_1, \ldots, a_{r-1}, y_1\}$. If we append a pendant path of length at least zero to each of the above vertices then we have a generalized bicycle graph. We may denote precisely the generalized bicycle graph in terms of length of appending pendant paths by $GB(m, n, r, P_{l_1, \ldots, l_m}, P_{k_1, \ldots, k_n}, P_{t_1, \ldots, t_{r-1}})$, where for each i, j, s, $(1 \le i \le m, 1 \le j \le n, 1 \le s \le r - 1)$ l_i, k_j and t_s are lengths of appending paths to vertices x_i, y_j, and a_s, respectively. Sometimes the notation $GB(m, n, r, P_{l_1, \ldots, l_m}, P_{k_1, \ldots, k_n}, P_{t_1, \ldots, t_{r-1}})$ can be denoted as shorten notation $GB(m, n, r)$ whenever we assume arbitrary lengths for pendant paths. If for every i, j, and s, we have $l_i = k_j = t_s = 0$ then clearly the generalized bicycle graph coincide to the bicycle graph (see Figures 4 and 5).

Theorem 2.15. *$GB(m, n, r)$ is End-regular if and only if $B(m, n, r)$ is End-regular and all pendant paths have length of at most one.*

Proof. First, let $B(m, n, r)$ is not End-regular. We know that every path retracts to one edge. For instance a pendant path appending to x_i can be retracted to an edge $\{x_i, x_{i+1}\}$. Similarly, each of pendant paths can retract to an edge of $B(m, n, r)$. So, $GB(m, n, r)$ retracts to $B(m, n, r)$ and therefore $GB(m, n, r)$ is not End-regular by Lemma 2.5. If $B(m, n, r)$ is End-regular and there exists a pendant path of length at least 2, then $m = n$ are odd numbers and $r \le 3$ by Corollary 2.13. We can find in $GB(m, m, r)$ a subgraph that is isomorphic with $U(m, 3)$. It is not difficult to define an endomorphism such that $GB(m, m, r)$ retracts to $U(m, 3)$. So $GB(m, m, r)$ is not End-regular by Lemma 2.5. Now let $B(m, n, r)$ is End-regular and all appending pendant paths have length at most 1. So, $m = n$ are odd numbers and $r \le 3$. Let v be a vertex of $B(m, n, r)$ and w_v be its pendant vertex in $GB(m, n, r)$. We recall that it is possible some vertices have no such pendant vertex. Suppose $f \in End(GB(m, m, r))$. It is obvious that $Im(f)$ contains at least a cycle of $GB(m, m, r)$. So I_f is an unicycle graph or a generalized bicycle graph. Hence, I_f is a retract and there exists a retract A such that $f|_{V(A)}$ is isomorphic to I_f. So f is regular by Lemma 2.8 and proof is complete. □

It is interesting to know whether the generalized bicycle graph is End-idempotent-closed or End-orthodox. In the following two theorems, we investigate these properties.

Theorem 2.16. *Generalized bicycle graph is End-idempotent-closed if and only if it has two cycles of different lengths where at least one of lengths is odd.*

Proof. Let G be a generalized bicycle graph. We consider the following four cases.
Case 1. G has two cycles with the same length. Let H and K be induced cycles by G. We may find $f, g \in Idp(G)$ such that f retract to H and g retract to K. It is not difficult to check that $fog \notin Idp(G)$.
Case 2. G has two different cycles of lengths even numbers. Let $\{x_1, x_2\}$ and $\{y_1, y_2\}$ be two edges from two cycle in G. We retract G to $\{x_1, x_2\}$ by f and retract G to $\{y_1, y_2\}$ by g. So, $f, g \in Idp(G)$ but $fog \notin Idp(G)$.
Case 3. G has two different cycles of odd lengths. Let C_m and $C_{m'}$ be induced cycles by G such that $m < m'$ and $V(C_m) = \{x_1, x_2, ..., x_m\}, V(C_{m'}) = \{y_1, y_2, ..., y_{m'}\}$. Since every odd cycle takes by an endomorphism to an odd cycle maybe with smaller length, so C_m in every arbitrary idempotent endomorphism is fix and $f(x_i) = x_i$ for each $x_i \in C_m$ and $f \in Idp(G)$. Also $C_{m'}$ is fix in each idempotent endomorphism or goes to C_m. Alternatively, remains edges G are fixed or going to C_m by an arbitrary idempotent endomorphism. So, clearly we can see that composition of every two idempotent endomorphisms again is an idempotent endomorphism. Hence, in this case, G is End-idempotent-closed.
Case 4. G has two cycles that one of length odd and another of length even. By a similar argument as in Case 3, it is easy to see that G is End-idempotent-closed. □

Thus, we can deduce the following corollary.

Corollary 2.17. *Generalized bicycle graph is not End-orthodox.*

3. End-regularity of join two generalized bicycle graphs

In this section, we investigate End-regularity of join two generalized bicycle graphs. The following lemma tells us if join of two graphs is End-regular then each of graphs should be End-regular.

Lemma 3.1. (Li [5], Corollary 3.2). *Let X and Y be two graphs. If $X+Y$ is End-regular, then both X and Y are End-regular.*

It is interesting to investigate the converse of the above lemma. In spite of the fact that the converse of lemma is not true in general, but it is valid in some special cases. In fact, we prove that if X and Y are two generalized bicycle graphs and they are End-regular, then $X+Y$ is End-regular as well. To prove this result, we need to remind some Lemmas and Theorems from [2] which are necessary here. We omit the proofs and one can refer to [2] for more details. Also, we remind that graph G is called K_3-free or triangle-free graph if it is an undirected graph in which no three vertices form a triangle of edges.

Lemma 3.2. (H. Hou and Luo [2], Lemma 2.2). *Let X and Y be two K_3-free graphs. If both of them are non-bipartite, then for any endomorphism f of $X+Y$, either $f(X) \subseteq X$ and $f(Y) \subseteq Y$, or $f(X) \subseteq Y$ and $f(Y) \subseteq X$.*

Lemma 3.3. (H. Hou and Luo [2], Lemma 2.3). *Let X and Y be two End-regular graphs. If for every $f \in End(X+Y)$, we have $f(X) \subseteq X$ and $f(Y) \subseteq Y$, then $X+Y$ is End-regular.*

Theorem 3.4. (H. Hou and Luo [2], Corollary 2.5). *Let X and Y be two K_3-free End-regular non-bipartite graphs. If $\chi(X) \neq \chi(Y)$, or X and Y have different odd girth, then $X+Y$ is End-regular.*

Theorem 3.5. (H. Hou and Luo [2], Theorem 2.9). *Let X be a K_3-free End-regular non-bipartite graph and Y be an End-regular graph which has at least one triangle. If $\chi(Y) < \chi(X)+1$, then $X+Y$ is End-regular.*

Now, by using the above results, we can deal with join of two End-regular generalized bicycle graphs as the following.

Theorem 3.6. *Let X and Y be two End-regular generalized bicycle graphs with different odd girth, then $X+Y$ is End-regular.*

Proof. Since X and Y are End-regular generalized bicycle graph with different odd girth so by Theorem 2.15, we can set $X = GB(m, m, r)$ and $Y = GB(m', m', r')$ such that m and m' are odd numbers and $m \neq m'$. First, suppose that m and m' are not three. Then $X+Y$ is End-regular by Theorem 3.4. Now, let one of m and m' be three. It is clear that $\chi(X) = \chi(Y) = 3$ and $\chi(Y) < \chi(X)+1$. So $X+Y$ is again End-regular by Theorem 3.5. □

Theorem 3.7. *If X and Y are two K_3-free End-regular generalized bicycle graphs with equal odd girth, then $X+Y$ is End-regular.*

Proof. Since X and Y are End-regular, so we may set $X = GB(m, m, r)$ and $Y = GB(m, m, r')$ by Theorem 2.15. Let $f \in End(X+Y)$. We prove that f is a regular endomorphism. If $f(X) \subseteq X$ and $f(Y) \subseteq Y$, f is regular by Theorem 3.3. Let $f(X) \nsubseteq X$ and $f(Y) \nsubseteq Y$. Since $m \neq 3$, then $f(X) \subseteq Y$ and $f(Y) \subseteq X$ by Theorem 3.2. Now, we consider the following 10 cases and we are going to introduce g and h with properties as in Theorem 2.7. We begin by recalling the notation of X and Y. Let C^1_m, C^2_m be cycles in X and C^3_m, C^4_m be cycles in Y with vertices $V(C^1_m) = \{x_1, ..., x_m\}, V(C^2_m) = \{y_1, ..., y_m\}$, $V(C^3_m) = \{x'_1, x'_2, ..., x'_m\}$ and $V(C^4_m) = \{y'_1, y'_2, ..., y'_m\}$. Also for each v in X and Y, w_v is pendant vertex of vertex v. Since endomorphism image of every odd cycle is an odd cycle and $f(X) \subseteq Y, f(Y) \subseteq X$ in each case, we may assume that $C^1_m, C^3_m \in Img(f)$ and this involves no loss of generality. In each of one the following cases, we just introduce $g, h \in Idp(X+Y)$ and left the proof that $I_f = I_g$ and $\rho_f = \rho_h$. So f will be regular by Theorem 2.7.

Case 1. $r = r' = 0$.

$$g(x) = \begin{cases} x & x \in Img(f) \\ x_{i+1} & x = y_i, w_{x_i}, x \notin Img(f) \\ y_{i+1} & x = w_{y_i}, x \notin Img(f), y_i \in Img(f) \\ x_i & x = w_{y_i}, x \notin Img(f), y_i \notin Img(f) \\ x'_{i+1} & x = y'_i, w_{x'_i}, x \notin Img(f) \\ y'_{i+1} & x = w_{y'_i}, x \notin Img(f), y'_i \in Img(f) \\ x'_i & x = w_{y'_i}, x \notin Img(f), y'_i \notin Img(f) \end{cases}$$

$$h(x) = \begin{cases} x & x \in Img(f) \\ x_j & x = y_i, x \notin Img(f), f(x) = f(x_j) \\ k & x = w_{x_i}, w_{y_i}, x \notin Img(f), f(x) = f(k), k \text{ is } x_j \text{ or } y_j \\ x'_j & x = y'_i, x \notin Img(f), f(x) = f(x'_j) \\ k' & x = w_{x'_i}, w_{y'_i}, x \notin Img(f), f(x) = f(k'), k' \text{ is } x'_j \text{ or } y'_j \end{cases}$$

Case 2. $r = r' = 1$. g, h are the same as in case 1. We note that in this case $x_1 \sim y_1$. As we observe in case 1, x_1 and y_1 were coincided.

Case 3. $r = r' = 2$. Let $V(P_2) = \{x_1, a_1, y_1\}$ in X and $V(P_2) = \{x'_1, a'_1, y'_1\}$ in Y. Now, we define $g, h \in Idp(X + Y)$ as the following form.

$$g(x) = \begin{cases} x & x \in Img(f) \\ x_i & x = y_i, x \notin Img(f) \\ x_2 & x = a_1, x \notin Img(f) \\ x_1 & x = w_{a_1}, x \notin Img(f) \\ x_{i+1} & x = w_{x_i}, x \notin Img(f) \\ y_{i+1} & x = w_{y_i}, x \notin Img(f), y_i \in Img(f) \\ x_{i+1} & x = w_{y_i}, x \notin Img(f), y_i \notin Img(f) \\ x'_i & x = y'_i, x \notin Img(f) \\ x'_2 & x = a'_1, x \notin Img(f) \\ x'_1 & x = w_{a'_1}, x \notin Img(f) \\ x'_{i+1} & x = w_{x'_i}, x \notin Img(f) \\ y'_{i+1} & x = w_{y'_i}, x \notin Img(f), y'_i \in Img(f) \\ x'_{i+1} & x = w_{y'_i}, x \notin Img(f), y'_i \notin Img(f) \end{cases}$$

$$h(x) = \begin{cases} x & x \in Img(f) \\ x_j & x = y_i, a_1, x \notin Img(f), f(x) = f(x_j) \\ k & x = w_{x_i}, w_{y_i}, w_{a_1}, x \notin Img(f), f(x) = f(k), k \text{ is } x_j \text{ or } y_j \text{ or } a_1 \\ x'_j & x = y'_i, a'_1, x \notin Img(f), f(x) = f(x'_j) \\ k' & x = w_{x'_i}, w_{y'_i}, w_{a'_1}, x \notin Img(f), f(x) = f(k'), k' \text{ is } x'_j \text{ or } y'_j \text{ or } a'_1 \end{cases}$$

Case 4. $r = r' = 3$. Let $V(P_3) = \{x_1, a_1, a_2, y_1\}$ in X and $V(P_3) = \{x'_1, a'_1, a'_2, y'_1\}$ in Y. Now, we define $g, h \in Idp(X + Y)$ as the following form.

$$g(x)=\begin{cases} x & x\in Img(f)\\ x_2 & x=a_1,w_{a_2},x\notin Img(f)\\ x_1 & x=a_2,w_{a_1},x\notin Img(f)\\ x_{i+1} & x=y_i,w_{x_i},x\notin Img(f)\\ y_1 & x=w_{a_2},x\notin Img(f),a_2\in Img(f)\\ y_{i+1} & x=w_{y_i},x\notin Img(f),y_i\in Img(f)\\ x_i & x=w_{y_i},x\notin Img(f),y_i\notin Img(f)\\ x'_2 & x=a'_1,w_{a'_2},x\notin Img(f)\\ x'_1 & x=a'_2,w_{a'_1},x\notin Img(f)\\ x'_{i+1} & x=y'_i,w_{x'_i},x\notin Img(f)\\ y'_1 & x=w_{a'_2},x\notin Img(f),a'_2\in Img(f)\\ y'_{i+1} & x=w_{y'_i},x\notin Img(f),y'_i\in Img(f)\\ x'_i & x=w_{y'_i},x\notin Img(f),y'_i\notin Img(f) \end{cases}$$

$$h(x)=\begin{cases} x & x\in Img(f)\\ x_j & x=y_i,a_1,a_2,x\notin Img(f),f(x)=f(x_j)\\ k & x=w_{a_1},w_{a_2},w_{x_i},w_{y_i},x\notin Img(f),f(x)=f(k),k \text{ is } a_j \text{ or } x_j \text{ or } y_j\\ x'_j & x=y'_i,a'_1,a'_2,x\notin Img(f),f(x)=f(x'_j)\\ k' & x=w_{a'_1},w_{a'_2},w_{x'_i},w_{y'_i},x\notin Img(f),f(x)=f(k'),k' \text{ is } a'_j \text{ or } x'_j \text{ or } y'_j \end{cases}$$

Case 5. $r=0$, $r'=1$. g, h are the same as in case 1.
Case 6. $r=0$, $r'=2$. g, h are as the following form.

$$g(x)=\begin{cases} x & x\in Img(f)\\ x_{i+1} & x=y_i,w_{x_i},x\notin Img(f)\\ y_{i+1} & x=w_{y_i},x\notin Img(f),y_i\in Img(f)\\ x_i & x=w_{y_i},x\notin Img(f),y_i\notin Img(f)\\ x'_i & x=y'_i,x\notin Img(f)\\ x'_2 & x=a'_1,x\notin Img(f)\\ x'_1 & x=w_{a'_1},x\notin Img(f)\\ x'_{i+1} & x=w_{x'_i},x\notin Img(f)\\ y'_{i+1} & x=w_{y'_i},x\notin Img(f),y'_i\in Img(f)\\ x'_{i+1} & x=w_{y'_i},x\notin Img(f),y'_i\notin Img(f) \end{cases}$$

$$h(x)=\begin{cases} x & x\in Img(f)\\ x_j & x=y_i,x\notin Img(f),f(x)=f(x_j)\\ k & x=w_{x_i},w_{y_i},x\notin Img(f),f(x)=f(k),k \text{ is } x_j \text{ or } y_j\\ x'_j & x=y'_i,a'_1,x\notin Img(f),f(x)=f(x'_j)\\ k' & x=w_{x'_i},w_{y'_i},w_{a'_1},x\notin Img(f),f(x)=f(k'),k' \text{ is } x'_j \text{ or } y'_j \text{ or } a'_1 \end{cases}$$

Case 7. $r=0$ and $r'=3$. g, h are as the following form.

$$g(x)=\begin{cases} x & x\in Img(f)\\ x_{i+1} & x=y_i,w_{x_i},x\notin Img(f)\\ y_{i+1} & x=w_{y_i},x\notin Img(f),y_i\in Img(f)\\ x_i & x=w_{y_i},x\notin Img(f),y_i\notin Img(f)\\ x'_2 & x=a'_1,w_{a'_2},x\notin Img(f)\\ x'_1 & x=a'_2,w_{a'_1},x\notin Img(f)\\ x'_{i+1} & x=y'_i,w_{x'_i},x\notin Img(f)\\ y'_1 & x=w_{a'_2},x\notin Img(f),a'_2\in Img(f)\\ y'_{i+1} & x=w_{y'_i},x\notin Img(f),y'_i\in Img(f)\\ x'_i & x=w_{y'_i},x\notin Img(f),y'_i\notin Img(f)\end{cases}$$

$$h(x)=\begin{cases} x & x\in Img(f)\\ x_j & x=y_i,x\notin Img(f),f(x)=f(x_j)\\ k & x=w_{x_i},w_{y_i},x\notin Img(f),f(x)=f(k),k \text{ is } x_j \text{ or } y_j\\ x'_j & x=y'_i,a'_1,a'_2,x\notin Img(f),f(x)=f(x'_j)\\ k' & x=w_{a'_1},w_{a'_2},w_{x'_i},w_{y'_i},x\notin Img(f),f(x)=f(k'),k' \text{ is } a'_j \text{ or } x'_j \text{ or } y'_j\end{cases}$$

Case 8. $r=1$ and $r'=2$. In this case g, h can be defined exactly similar to case 6.
Case 9. $r=1$ and $r'=3$. In this case g, h can be defined exactly similar to case 7.
Case 10. $r=2$ and $r'=3$. g, h are as the following form.

$$g(x)=\begin{cases} x & x\in Img(f)\\ x_i & x=y_i,x\notin Img(f)\\ x_2 & x=a_1,x\notin Img(f)\\ x_1 & x=w_{a_1},x\notin Img(f)\\ x_{i+1} & x=w_{x_i},x\notin Img(f)\\ y_{i+1} & x=w_{y_i},x\notin Img(f),y_i\in Img(f)\\ x_{i+1} & x=w_{y_i},x\notin Img(f),y_i\notin Img(f)\\ x'_2 & x=a'_1,w_{a'_2},x\notin Img(f)\\ x'_1 & x=a'_2,w_{a'_1},x\notin Img(f)\\ x'_{i+1} & x=y'_i,w_{x'_i},x\notin Img(f)\\ y'_1 & x=w_{a'_2},x\notin Img(f),a'_2\in Img(f)\\ y'_{i+1} & x=w_{y'_i},x\notin Img(f),y'_i\in Img(f)\\ x'_i & x=w_{y'_i},x\notin Img(f),y'_i\notin Img(f)\end{cases}$$

$$h(x)=\begin{cases} x & x\in Img(f)\\ x_j & x=y_i,a_1,x\notin Img(f),f(x)=f(x_j)\\ k & x=w_{x_i},w_{y_i},w_{a_1},x\notin Img(f),f(x)=f(k),k \text{ is } x_j \text{ or } y_j \text{ or } a_1\\ x'_j & x=y'_i,a'_1,a'_2,x\notin Img(f),f(x)=f(x'_j)\\ k' & x=w_{a'_1},w_{a'_2},w_{x'_i},w_{y'_i},x\notin Img(f),f(x)=f(k'),k' \text{ is } a'_j \text{ or } x'_j \text{ or } y'_j\end{cases}$$

□

Theorem 3.8. *If $X = GB(3,3,r)$ and $Y = GB(3,3,r')$ are two End-regular generalized bicycle graphs, then $X+Y$ is End-regular.*

Proof. Assume that $f \in End(X+Y)$. If $f(X) \subseteq X$ and $f(Y) \subseteq Y$, $X+Y$ is End-regular by Lemma 3.3. Let $f(X) \subseteq Y$ and $f(Y) \subseteq X$, So by a similar method as in the proof of Theorem 3.7 we can deduce that $X+Y$ is End-regular. Now, suppose that $f(X) \nsubseteq X, Y$ and $f(Y) \nsubseteq X, Y$. Since endomorphism image every triangle is again a triangle, at least one triangle in $X+Y$ must be in $Img(f)$. Let triangle H (with at least one vertex in X and one vertex in Y) be subset of $f(X)$. So, $f(X+Y) = H$ or $f(X+Y)$ contains H and other vertices of X or Y. In two cases, it is not difficult to see that I_f is a retract and found retract A such that $f|_{V(A)}$ be an isomorphism from A to I_f. Hence f is regular by Lemma 2.8. □

Finally, from two former theorems and Theorem 3.1, we have the following corollary.

Corollary 3.9. *Join two generalized bicycle graphs is End-regular if and only if each one is End-regular.*

Acknowledgments

The authors wish to thank the referee for some helpful comments and suggestions.

Disclosure statement

No potential conflict of interest was reported by the authors.

References

[1] Fan, S. (1996). On end-regular graphs. *Discrete Math.* 159(1–3): 95–102.

[2] Hou, H, Luo, Y. (2008). Graphs whose endomorphism monoids are regular. *Discrete Math.* 308(17): 3888–3896.

[3] Hou, H., Luo, Y, Cheng, Z. (2008). The endomorphism monoid of $\overline{P_n}$. *Eur. J. Combin.* 29(5): 1173–1185.

[4] Li, W. (1994). A regular endomorphism of a graph and its inverse. *Mathematik* 41(1): 189–198.

[5] Li, W. (2003). Graphs with regular monoid. *Discrete Math.* 265(1–3): 105–118.

[6] Li, W, Chen, J. (2001). Endomorphism-regularity of split graphs. *Eur. J. Combin.* 22(2): 207–216.

[7] Marki, L. (1988). Problems raised at the problem session of the Colloquium on semi-groups in Szeged. *Semi Group Forum* 37: 367–373.

[8] Ma, X., Wong, D, Zhou, J. (2016). Unicyclic graphs with regular endomorphism monoids. *J.Discrete Math. Algoritm. Appl.* 8(2). DOI: 10.1142/S1793830916500208

[9] Pipattanajinda, N, Arworn, S. (2010). Endo-regularity of cycle book graphs. *Thai J. Math.* 8(4): 99–104.

[10] Pipattanajinda, N., Knauer, U, Arworn, S. (2011). Endo-regularity of generalized wheel graphs. *Chamchuri J. Math.* 3: 45–57.

[11] Thomkeaw, J, Arworn, S. (2009). Endomorphism monoid of C_2n_{+1} book graphs. *Thai J. Math.* 7(2): 319–327.

[12] Von Neumann, J. (1936). On regular rings. *Proc. Natl. Acad. Sci. USA.* 22(12): 707–712.

[13] Wang, W, Hou, H. (2011). The endomorphism monoid of *N*-prism. *Int. Math. Forum* 6(49–52): 2461–2471.

[14] Wanichsombat, A. (2010). Algebraic structure of endomorphism monoids of finite graphs. PhD thesis. Oldenburg University, Oldenburg, Germany.

[15] Wilkeit, E. (1996). Graphs with regular endomorphism monoid. *Arch. Math.* 66(4): 344–352.