The effect of purchasing strategy on the economic production quantity model subject to random machine breakdown

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Abstract: In this article, we develop an economic production quantity (EPQ) model subject to process deterioration, machine breakdown and two types of repair, and study the influence of purchasing strategy on the system's expected total cost, as an alternative option to holding safety stock. It is obvious that the shortages may occur due to the prolonged repair time. To avoid this issue, we assume that the manufacturer could purchase some quantities from an available supplier with fixed non-zero lead-time. This paper addresses the following question: how much production lot size and purchasing lot size are required to minimise the expected total cost? The model is developed under general machine breakdown and repairs time distributions. The optimality of the model under exponential failure and exponential repair times are studied. For the numerical example, we illustrate the outcome of the proposed model and perform a sensitivity analysis on parameters, which directly influence optimal decisions.

Keywords: economic production quantity; EPQ; process deterioration; machine breakdown; defective items; purchasing.

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1 Introduction

Considering the importance that manufacturers place on responding to customers' needs in today's world of trade, and organisations' efforts to keep up their credit and service level in the supply chain, process deterioration during the production run is one of the important challenges facing economic production quantity (EPQ) systems. Process deterioration is manifested as a decrease in production rate, the production of defective items or machine breakdown. In this situation, management is forced to deviate from production planning. Classic EPQ models ignore process deterioration and machine breakdown during the production run and implicitly assume that all items continue to be produced perfectly; hence the inconsistency between models and practical situations. Usually, production systems deteriorate with age or use undergoing fatigue and corrosion (Chakraborty and Giri, 2012). In such situations, it is usually assumed that the production process shifts from an 'in-control' state to an 'out-of-control' state at any random time and the machine starts to produce some defective items (Prakash et al., 2013). If inspections exist in such systems, production of defective items continues until the 'out-of-control' state is discovered by inspectors, otherwise, this situation continues until the end of the production run or machine breaks down. In real-life production systems, different approaches such as preventive maintenance, holding safety stock and after sales service are employed to deal with system unreliability. For the last three decades, the majority of researchers working on imperfect EPQ models have suggested keeping safety stock in reserve, in case of machine failure, particularly when shortage in the system results in a lost sale (see, e.g., Chakraborty and Giri, 2012; Giri et al., 2005b; Sana and Chaudhuri, 2010). However, sometimes the expensive costs of this strategy are not beneficial for the system due to high product holding costs. In such situations, it may be advantageous to use an external supplier to meet the demand while the machine is being repaired. The issue is not necessarily about the final product. It could be related to a standard piece of a final product manufactured by the main supplier in which he/she is responsible for supplying the piece according to a long-term contract. In this case, the supplier can purchase from smaller suppliers with lower reliability to prevent shortages in urgent situations. For example, items such as printed boxes, plastic bottles or computer chips, could be produced by other manufacturers after minor changes to their production facilities. Although buying items from an external supplier would result in some loss in the margins, it could compensate for part of the overhead cost, protect the supplier's reputation, and ensure future demand (Peymankar et al., 2018).

The purpose of this paper is

1 to generalise the model of (Peymankar et al., 2018) incorporating non-zero lead-time and preventive repair time

2 to study simultaneous effects of process deterioration, machine breakdown, maintenance and purchasing using a stochastic EPQ model.

Considering all of these issues bring the model closer to the real world situations; however, it will make the model more complicated.

The reminder of this paper is structured as follows. Section 2 presents an overview of the relevant literature; in Section 3, the problem state notation and the basic assumptions of the model are defined. This is followed by the development of the mathematical model under general failure and general repair time distributions. The case of exponential failure and exponential repair time distributions are studied and the solution approach for this case is proposed in Section 5. A numerical example is presented in Section 6 to determine the optimum values production and purchasing lot sizes and the sensitivity of the model parameters are examined. In the final section, we provide some concluding remarks and provide future research directions.

2 Literature review

The EPQ model was introduced in the early twentieth century, but researchers have only considered imperfect production systems since the 1980s. Porteus (1986), Rosenblatt and Lee (1986) and Bielecki and Kumar (1988) were some of the leading researchers who worked on imperfect production systems. When considering the imperfect EPQ model, Porteus (1986) assumed exponential process shift distribution. Rosenblatt and Lee (1986) made a similar assumption and showed that the optimal production run-length in imperfect EPQ systems is shorter than the run-length in classical EPQ models. They also showed that a smaller production lot contains fewer defective items thereby reducing the cost associated with them. Bielecki and Kumar (1988) analysed a particular case of unreliable production system and showed that zero-inventory policies resulted in optimality for the systems with uncertainties.

In the following, we have categorised imperfect production literature in two generic classifications and briefly studied some of the most important research in each category.

2.1 Imperfect production with process deterioration

Production of imperfect items due to process deterioration is one area of research in unreliable production systems. Lee and Rosenblatt (1987) studied joint production and inspection during a production run and demonstrated that monitoring the process is an effective procedure for decreasing process deterioration cost. Lee and Rosenblatt (1989) further developed a model by considering warranty cost for defective items sold to costumers and found the warranty cost was much higher than the rework cost. Similarly, Lee and Park (1991) considered the difference between rework and warranty cost in their model. Khouja and Mehrez (1994) extended an EPQ model, assuming flexible production rate, exponential shift distribution with the mean as an increasing function of the production rate. They formulated model for optimising production rate and production lot size.

Tseng (1996) studied the influence of preventive maintenance policy instead of inspection policy in unreliable systems. Hayek and Salameh (2001) derived an optimal operating policy for an imperfect EPQ model by invoking rework capability. In their model, they assumed that all of the defective items were reparable. Jamal et al. (2004) considered the optimal manufacturing lot size employing two different policies for the rework process. In the first policy, rework is completed within the same production cycle, while in the second one; the rework is done after N cycles. Numerical results showed that in contrast to the first policy, the second one is very sensitive to the changes in production rate. Ben-Daya et al., 2008) investigated the effects of production rate deterioration on lot sizing decisions. They assumed that the production line is subjected to a random shift from the in-control state to out-of-control state. Sana (2010) considered an imperfect EPQ model and assumed that percentage of defective items is nonlinearly dependent on production rate and production run-time. Sarkar et al. (2010) developed an imperfect production system for stock dependent demand. Lin et al. (2011) studied the impact of inspection errors, imperfect maintenance and minimal repairs on an imperfect production system. Singh et al. (2012) presented an imperfect production system under allowable shortage. They assumed two type of production process in a production cycle. One is 'in-control' state producing good quality items and second one is 'out-of-control' state producing some percentage of defective items. Pal et al. (2013) developed an imperfect EPQ model with rework capability for selecting optimal product reliability and optimal lot size to maximise the expected average profit. They assumed that the unit production cost is a function of production lot size and product reliability. In recent studies, Karimi-Nasab and Sabri-Laghaie (2014) and Al-Salamah (2016) developed EPQ models for the case when both the production and inspection processes are imperfect. Shih and Wang (2016) developed a model, which provides thorough quality control and might result in repair and salvage. Huang et al. (2016) extended Ben-Daya et al. (2008) model by assuming that the expected shift time could be controllable by investment in resources. They also allowed shortages in the model, when the shifted production rate is smaller than the demand rate. Sarkar et al. (2018) developed an imperfect production system to obtain the optimal production run and inspection policy. They considered two types' inspection errors to make the model more realistic. Due to inspection error, they assumed that the non-inspected defective items are passes to customers with free minimal repair warranty.

2.2 Imperfect production with machine breakdown

Unexpected machine breakdown is a very common incident in a production environment and regarded as a critical reliability factor. This unreliability in EPQ systems is another area of research for motivated researchers to model. A literature review with this scope shows that in most studies two control policies are employed to deal with random machine breakdown. These policies were primarily introduced by Groenevelt et al. (1992). Under the first policy, when a breakdown occurs, the interrupted lot is aborted and a new one is started, once all available inventory is depleted. This is known as the no resumption-NR policy. The second policy is takes effect when the cost of resuming the production run after a failure, is substantially lower than the setup cost. In this policy, the production of the interrupted lot will be immediately resumed after the breakdown is fixed. This is known as the abort-resume A/R policy.

Due to complexity of the models in A/R policy when repair times have random distributions, NR policy has been more interest by researchers. Abboud (1997) presented an unreliable EPQ system showing that machine failure follows a Poisson distribution. Moini and Murthy (2000) presented two kinds of repair strategy and assumed that the possibility of machine failure is different after each strategy. Boone et al. (2000) investigated the simultaneous effects of producing defective items and stoppage occurrence due to the machine breakdowns under an NR inventory control policy. Abboud (2001) developed an imperfect EPQ model by assuming that the shortage in the system is partially backlogged. Considering the discrete time through which the times to failure and repair times are distributed geometrically, he modelled the inventory-production system as a Markov chain and developed an efficient algorithm to compute the cost function. Giri et al. (2005b) investigated optimal lot size in an unreliable two-stage production-inventory system in which the machine in the first stage is failure-prone while at the second stage is failure-free. Chiu et al. (2007) developed an EPQ model with scrap, rework and machine breakdown under the NR policy. They proved the convexity of the model under a special condition. Chiu et al. (2009) investigated an imperfect EPQ model under an A/R inventory control policy with the rework capability for all defective items. Peter Chiu et al. (2010) developed Chiu et al. (2009) model which assumed that a portion of defective items can be reworked and the rest are scrap. Prakash et al. (2013) presented a production-inventory model with discrete random machine breakdown and discrete stochastic repair times. They assumed that the demand rate follows a discrete stochastic distribution.

The accumulation of safety stock and using of preventive maintenace are two strategies taken by bussinesses to avoid shortages. The literature on imperfect production systems shows that researchers analyse the performance of a system by using either one of two strategies or simultaneously. Cheung and Hausman (1997) presented the joint implementation of preventive maintenance and safety stocks in an unreliable production environment. They assumed that production rate is equivalent to the demand rate in a normal production phase. Dohi et al. (2001) revised the model theoretically and developed it by adding different restrictive assumptions. The joint implementation of the safety stock and age-based preventive maintenance for an unreliable production system was also investigated by Gharbi et al. (2007). Giri et al. (2005a) presented an unreliable production-inventory system, which considered exponential failure time while repair time occurs under an NR inventory control policy. In their model the production cost and failure rate depend on production rate, and they formulated the problem with and without a policy of holding safety stock. Chakraborty et al. (2008) developed model by concurrently considering the effects of process deterioration, machine breakdown and repairs (corrective and preventive) on lot sizing decisions. Assuming that non-confirming items are not detectable during the production run time, they fixed the warranty cost for the sold defective items. Furthermore, Chakraborty et al. (2009) analysed the lot size problem with the process deterioration and machine breakdown under inspection schedule. Sana and Chaudhuri (2010) investigated the joint determination of preventive maintenance and safety stock on an EPQ system under two A/R and NR policies for optimising production rate and lot size. Chakraborty and Giri (2012) developed an imperfect economic manufacturing quantity model under process deterioration, machine breakdown, preventive and corrective repairs, and buffer stock. They suggested a computational algorithm to define the optimal safety stock and production run time, which in turn minimised the expected cost per unit time in the steady state condition.

The effects of machine breakdown on an EPQ system with deteriorating items were examined by Lin and Gong (2006). They formulated the problem through random machine failure and fixed repair time under the NR inventory control policy. This model was extended by Widyadana and Wee (2011) with the randomisation of repair time. They solved the model where the repair time has both uniform and exponential distributions. Zhang et al. (2014) used a dynamic method for the production of lot sizing with machine failures in which the average cost is minimised instead of expected one. Paul et al. (2015) considered a single stage production-inventory system with random disruption. Their model maximises the total profit during the recovery time window by generating a revised plan after occurrence of disruption. Taleizadeh et al. (2017) developed a single-vendor/single-buyer model under the NR policy with random machine breakdown, multiple shipments and keeping safety stock capability. They assumed both batch lot size and distance between two shipments are identical and the buyer pays transportation cost. Öztürk (2018) investigate optimal production run time on an EPQ system under A/R machine breakdown policy with inspection and rework capability.

According to the survey, although there is a good amount of research related to imperfect production and specially machine breakdown case, there is only one study (Peymankar et al., 2018) which proposed external supply strategy in machine breakdown case. Peymankar et al. (2018) have assumed that in the case of breakdown, the manufacturer has the option to fulfil the demand using an external supplier, but in their model, the supplier lead-time was zero. In this paper, we assumed that the manufacturer could purchase the required items from an external supplier with fixed non-zero lead-time. This assumption brings the model closer to reality. Moreover, process deterioration and machine breakdown as well as preventive and corrective maintenance have been considered.

3 Problem notation and assumption

3.1 Notations

We use the following notations to develop the proposed model:

X Non-negative random variable denoting time to machine breakdown

$F_x(t)$ Cumulative distribution function with probability density function $f_x(t)$

D Constant demand rate (units/time)

$P > D$ Constant production rate (units/time)

l_1 Random variable denoting corrective maintenance time

$G_1(l_1), g_1(l_1)$ Cumulative distribution function and probability density function of l_1

l_2 Random variable denoting preventive maintenance time

$G_2(l_2), g_2(l_2)$ Cumulative distribution function and probability density function of l_2

c_0 Fixed setup cost for each production run time

c_0' Fixed ordering cost for purchased item from an external supplier

θ Reliability of external supplier

L Fixed lead-time of external supplier

T Expected total cycle time

c_h Inventory holding cost per unit item per unit time

c_P Production cost per unit item for the manufacturer

c' Unit purchase price from external supplier

c_r Corrective repair cost per unit time

c_m Preventive repair cost per unit time ($c_m < c_r$)

c_s Shortage cost per unit item

c_w Cost of rework per unit defective item

α Constant rate at which the defective items are produced in the 'out-of-control' state

N Random variable denoting the number of defective items produced during a production run

Q Production lot size per cycle (decision variable)

Q' Order quantity lot size to external supplier per cycle (decision variable)

3.2 Assumptions

We make the following assumptions to develop the proposed model:

1 The planning horizon is infinite.

2 The problem concerns single stage manufacturing producing a single type of products.

3 Setup time is negligible and equals to zero.

4 The demand rate of product and production rate are known constants and the production rate is greater than the demand rate.

5 The system suffers increasing wear and deteriorates with age and use. The system stochastic breakdown occurs when the deterioration level reaches the failure threshold.

6 If a machine breakdown occurs during a production phase, corrective repair is started immediately; thereafter, the machine is restored to its initial working condition.

7 If a machine breakdown does not occur during the production run, a preventive maintenance action renews the production system at the end of each production run.

8 Shortages may occur due to longer corrective/preventive repair times. All shortages will be lost.

9 Defective items cannot be identified in a production run (there is no inspection policy), so these items are reworked with minimal repair at the end of the period, and all of them are acceptable after being reworked.

4 Formulation of the model

We consider a single-machine single-item production system, which may shift from an 'in-control' state to an 'out-of-control' state and start to produce defective items or may breakdown, at any random time during the production run. If a machine breakdown occurs, corrective maintenance starts immediately; otherwise, the preventive maintenance starts after the production runs at time *Q/P*. During the corrective and preventive maintenance, the accumulated on-hand inventory decreases at a constant *D* to satisfy the demand in machine idle time (in the case of no machine breakdown $t_{1P} = Q / D - Q / P$ and in the case of machine breakdown $t_{1C} = Pt / D - t$). If the maintenance activity is completed before the finish of the accumulated on-hand inventory, a new production cycle is started as the inventory level decreases to zero; otherwise, shortages occur and they are not delivered after machine repair. To avoid shortages and because of the high costs of holding safety stock, we assume that the manufacturer has the option to fulfil the demand using an external supplier with a determined service level. It is also assumed that the ordered quantity is delivered after the known constant *L* unit times, so the manufacturer must order the desired quantity *L* unit times before the finish of the on-hand inventory. For preventive maintenance conditions, order point is $Q / D - L$; and for corrective maintenance conditions, order point is $Pt / D - L$, in which *t* denotes the time until the machine breaks down. It is worth noting that purchasing from an external supplier is acceptable when the time to use up the accumulated on-hand inventory is greater than the lead-time *L*. For preventive maintenance conditions, the necessary items are ordered if $t_{1P} > L$ and for corrective maintenance conditions, the necessary items are ordered if $t_{1C} > L$. Based on this assumption, we have two different formulations for the problem that will be described in Sections 4.1 and 4.2.

As mentioned, when the machine shifts to the 'out-of-control' state, the process produces defective items and continues to do so until the entire lot is produced or machine breakdown occurs. Sana and Chaudhuri (2010) used the two-state Markov chain and showed that the expected number of defective items during the production run is

$$E\left(N(Q)/X=t\right]=\begin{cases} pt-\dfrac{1-\alpha}{\alpha}\left(1-e^{pt\ln(1-\alpha)}\right) & \text{if } \ t\leq Q/p \\ Q-\dfrac{1-\alpha}{\alpha}\left(1-e^{Q\ln(1-\alpha)}\right) & \text{if } \ t\geq Q/p \end{cases} \tag{1}$$

and where α is the constant rate of defective items in the 'out-of-control' state. Therefore, the total expected number of defective items per cycle is

$$\begin{aligned} E\left(N(Q)\right)=&\int_0^{Q/p}\left[pt-\frac{1-\alpha}{\alpha}\left(1-e^{pt\ln(1-\alpha)}\right)\right]f_x(t)dt \\ &+\int_{Q/p}^{\infty}\left[Q-\frac{1-\alpha}{\alpha}\left(1-e^{Q\ln(1-\alpha)}\right)\right]f_x(t)dt \end{aligned} \tag{2}$$

These items will be reworked at the end of the period. Hence, the expected rework cost for a complete production run is

$$ERC(Q)=c_w E\left(N(Q)\right) \tag{3}$$

4.1 Condition 1 ($t_{1P} < L$)

In this condition, the supplier lead-time is greater than the time to use up the on-hand inventory, therefore; the manufacturer cannot order the necessary items. So the following cases may be occurred (see Figure 1 and Figure 2).

We define the interval of time between two successive production start times as a cycle. We consider the time from machine failure until its repair, the expected length of one cycle is

Figure 1 Condition 1: configuration of the model when no machine breakdown occurs, (a) no shortage occurs during preventive repair (b) shortages occur during preventive repair

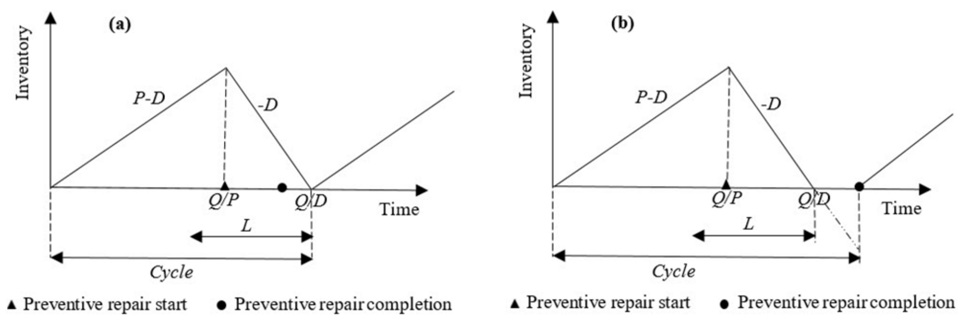
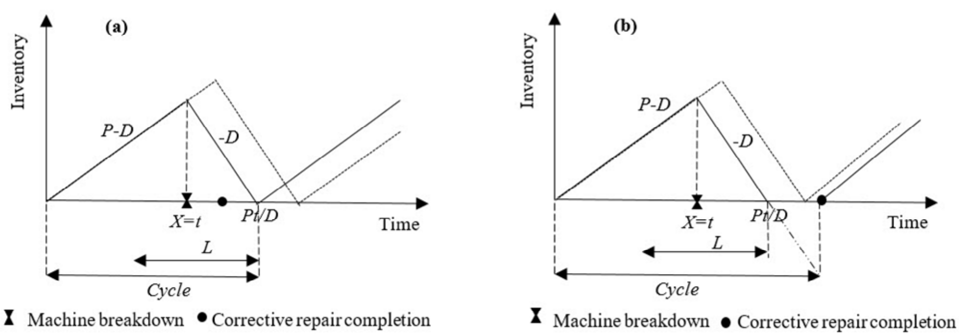


Figure 2 Condition 1: configuration of the model when machine break down occurs, (a) no shortage occurs during corrective repair (b) shortages occur during corrective repair



$$T(Q,0) = \int_0^{Q/P} k\left(t, l_1, G_1(l_1)\right) f_x(t)dt + \int_{Q/P}^{\infty} k\left(Q/P, l_2, G_2(l_2)\right) f_x(t)dt \tag{4}$$

where $k(t, l_1, G_1(l_1))$ is the expected duration of the cycle when $X = t \le Q / P$, and $k(Q / P, l_2, G_2(l_2))$ is the expected duration when $X = t \ge Q / P$;

$$k\left(t, l_1, G_1(l_1)\right) = \int_0^{(P-D)t/D} \frac{Pt}{D} dG_1(l_1) + \int_{(P-D)t/D}^{\infty} (t + l_1) dG_1(l_1)$$

$$k\left(Q/P, l_2, G_2(l_2)\right) = \int_0^{(P-D)Q/(PD)} \frac{Q}{D} dG_2(l_2) + \int_{P-D)Q/(PD)}^{\infty} (Q/P + l_2) dG_2(l_2)$$

The expected total cost of the system in this condition includes the set up cost, production cost, maintenance cost (corrective and preventive), inventory holding cost, cost of shortages and the rework cost for the defective items.

$$\begin{aligned} S(Q,0) &= c_0 + c_P\left\{P\int_0^{Q/P} tf_x(t)dt + Q\int_{Q/P}^{\infty} f_x(t)dt\right\} + c_r\int_0^{Q/P}\int_0^{\infty} l_1 dG_1(l_1)f_x(t)dt \\ &+ c_m\int_{Q/P}^{\infty}\int_0^{\infty} l_2 dG_2(l_2)f_x(t)dt + c_h\left\{\begin{aligned}&\int_0^{Q/P}\frac{(P-D)Pt^2}{2D}f_x(t)dt \\ &+\int_{Q/P}^{\infty}\frac{(P-D)Q^2}{2PD}f_x(t)dt\end{aligned}\right\} \\ &+ c_s\left\{\begin{aligned}&\int_0^{Q/P}\int_{\frac{(P-D)t}{D}}^{\infty} D\left(l_1 - \frac{(P-D)t}{D}\right)dG_1(l_1)f_x(t)dt \\ &+\int_{Q/P}^{\infty}\int_{\frac{(P-D)Q}{PD}}^{\infty} D\left(l_2 - \frac{(P-D)Q}{PD}\right)dG_2(l_2)f_x(t)dt\end{aligned}\right\} + c_w E(N(Q)) \end{aligned} \tag{5}$$

Now, based on the renewal reward theorem (Ross, 2014), the expected cost per unit time is given by

$$C(Q,0) = \frac{S(Q,0)}{T(Q,0)} \tag{6}$$

4.2 Condition 2 ($t_{1P} > L$)

In this condition, the manufacturer can obtain from an external supplier if the on-hand inventory cannot cover demand while repairs are being conducted. However, given the supplier lead-time is not zero, when no machine breakdown occurs, the order point is $Q / D - L$. In the case of machine break down, the possibility of ordering depends on the time to breakdown. If $t_{1C} < L$ the manufacturer gives no order; otherwise $t_{1C} > L$, and the desired items are ordered when the repair time passes from the $Pt / D - L$ point. Depending on the reliability of the supplier, the manufacture can obtain the desired items with probability of θ or with the probability of $1 - \theta$ if the system faces shortage and loses demand until the machine is repaired. All of these cases are shown in Figure 3 and Figure 4.

For this condition the expected length of one cycle is

$$\begin{aligned} T(Q,Q') &= \int_0^{LD/(P-D)} k_1(t,l_1,G_1(l_1))f_x(t)dt + \int_{LD/(P-D)}^{Q/P} k_2(t,l_1,G_1(l_1))f_x(t)dt \\ &+ \int_{Q/P}^{\infty} k(Q/P,l_2,G_2(l_2))f_x(t)dt \end{aligned} \tag{7}$$

where $k_1(t, l_1, G_1(l_1))$ is expected duration of the cycle when $X = t \le Q / P$ and $t_{1C} < L$, $k_2(t, l_1, G_1(l_1))$ is expected duration when $X = t \le Q / P$ and $t_{1C} > L$, and $k(Q / P, l_2, G_2(l_2))$ is expected duration when $X = t \ge Q / P$;

$$k_1\left(t,l_1,G_1\left(l_1\right)\right)=\int_0^{(P-D)t/D}\frac{Pt}{D}dG_1\left(l_1\right)+\int_{(P-D)t/D}^{\infty}\left(t+l_1\right)dG_1\left(l_1\right)$$

$$\begin{aligned}k_2\left(t.l_1,G_1\left(l_1\right)\right)=&\int_0^{\frac{(P-D)t}{D}-L}\frac{Pt}{D}dG_1\left(l_1\right)+(1-\theta)\left\{\begin{array}{l}\int_{\frac{(P-D)t}{D}-L}^{\frac{(P-D)t}{D}}\frac{Pt}{D}dG_1\left(l_1\right)\\+\int_{\frac{(P-D)t}{D}}^{\infty}\left(t+l_1\right)dG_1\left(l_1\right)\end{array}\right\}\\&+\theta\left\{\int_{\frac{(P-D)t}{D}-L}^{\frac{Pt+Q'}{D}-t}\left(\frac{Pt+Q'}{D}\right)dG_1\left(l_1\right)+\int_{\frac{Pt+Q'}{D}-t}^{\infty}\left(t+l_1\right)dG_1\left(l_1\right)\right\}\end{aligned}$$

$$\begin{aligned}k\left(Q/P,l_2,G_2\left(l_2\right)\right)=&\int_0^{\frac{(P-D)Q}{PD}-L}\frac{Q}{D}dG_2\left(l_2\right)+(1-\theta)\left\{\begin{array}{l}\int_{\frac{(P-D)Q}{PD}-L}^{\frac{(P-D)Q}{PD}}\frac{Q}{D}dG_2\left(l_2\right)\\+\int_{\frac{(P-D)Q}{PD}}^{\infty}\left(Q/P+l_2\right)dG_2\left(l_2\right)\end{array}\right\}\\&+\theta\left\{\begin{array}{l}\int_{\frac{(P-D)Q}{PD}-L}^{\frac{Q+Q'}{D}-\frac{Q}{P}}\left(\frac{Q+Q'}{D}\right)dG_2\left(l_2\right)\\+\int_{\frac{Q+Q'}{D}-\frac{Q}{P}}^{\infty}\left(Q/P+l_2\right)dG_2\left(l_2\right)\end{array}\right\}\end{aligned}$$

The expected total cost of the system in this condition includes the set up cost, production cost, purchasing cost, fixed ordering cost, maintenance cost (corrective and preventive), inventory holding cost, cost of shortages and the rework cost for the defective items.

$$\begin{aligned}S\left(Q,Q'\right)=&c_0+c_P\left\{P\int_0^{Q/P}tf_x(t)dt+Q\int_{Q/P}^{\infty}f_x(t)dt\right\}\\&+c'\theta Q'\left\{\int_{\frac{LD}{P-D}}^{Q/P}\int_{\frac{(P-D)t}{D}-L}^{\infty}dG_1(l_1)f_x(t)dt+\int_{Q/P}^{\infty}\int_{\frac{(P-D)Q}{PD}-L}^{\infty}dG_2\left(l_2\right)f_x(t)dt\right\}\\&+\theta c_0'\left\{\int_{\frac{LD}{P-D}}^{Q/P}\int_{\frac{(P-D)t}{D}-L}^{\infty}dG_1\left(l_1\right)f_x(t)dt+\int_{Q/P}^{\infty}\int_{\frac{(P-D)Q}{PD}-L}^{\infty}dG_2\left(l_2\right)f_x(t)dt\right\}\\&+c_r\int_0^{Q/P}\int_0^{\infty}l_1dG_1(l_1)f_x(t)dt+c_m\int_{Q/P}^{\infty}\int_0^{\infty}l_2dG_2(l_2)f_x(t)dt\\&+c_h\left\{\begin{array}{l}\int_0^{\frac{LD}{P-D}}\frac{(P-D)Pt^2}{2D}f_x(t)dt+\int_{\frac{LD}{P-D}}^{Q/P}\sigma\left(t,l_1,G_1\left(l_1\right)\right)f_x(t)dt\\+\int_{Q/P}^{\infty}\sigma\left(Q/P,l_2,G_2\left(l_2\right)\right)f_x(t)dt\end{array}\right\}\end{aligned}$$

$$
\begin{aligned}
&+c_s\left\{\begin{aligned}&\int_0^{\frac{LD}{P-D}}\gamma_1\left(t,l_1,G_1\left(l_1\right)\right)f_x(t)dt+\int_{\frac{LD}{P-D}}^{Q/P}\gamma_2\left(t,l_1,G_1\left(l_1\right)\right)f_x(t)dt\\&+\int_{Q/P}^{\infty}\gamma\left(Q/P,l_2,G_2\left(l_2\right)\right)f_x(t)dt\end{aligned}\right\}\\
&+c_wE(N(Q))
\end{aligned}
\tag{8}
$$

where $\gamma_1(t, l_1, G_1(l_1))$ is expected shortage when $X = t \leq Q / P$ and $t_{1C} < L$, $\sigma(t, l_1, G_1(l_1))$ and $\gamma_2(t, l_1, G_1(l_1))$ are the expected inventory and shortage, respectively, when $X = t \leq Q / P$ and $t_{1C} > L$, $\sigma(Q / P, l_2, G_2(l_2))$ and $\gamma_2(Q / P, l_2, G_2(l_2))$ are the expected inventory and shortage, respectively, when $X = t \geq Q / P$;

$$
\gamma_1\left(t,l_1,G_1\left(l_1\right)\right)=\int_{\frac{(P-D)t}{D}}^{\infty}D\left(l_1-\frac{(P-D)t}{D}\right)dG_1\left(l_1\right)
$$

$$
\begin{aligned}
\gamma_2\left(t,l_1,G_1\left(l_1\right)\right)=&(1-\theta)\int_{\frac{(P-D)t}{D}}^{\infty}D\left(l_1-\frac{(P-D)t}{D}\right)dG_1\left(l_1\right)\\
&+\theta\int_{\frac{(P-D)t}{D}}^{\infty}D\left(l_1-\frac{(P-D)t}{D}-\frac{Q'}{D}\right)dG_1\left(l_1\right)
\end{aligned}
$$

$$
\begin{aligned}
\gamma\left(Q/P,l_2,G_2\left(l_2\right)\right)=&(1-\theta)\int_{\frac{(P-D)Q}{PD}}^{\infty}D\left(l_2-\frac{(P-D)Q}{PD}\right)dG_2\left(l_2\right)\\
&+\theta\int_{\frac{(P-D)Q}{PD}}^{\infty}D\left(l_2-\frac{(P-D)Q}{PD}-\frac{Q'}{D}\right)dG_2\left(l_2\right)
\end{aligned}
$$

$$
\begin{aligned}
\delta\left(t,l_1,G_1\left(l_1\right)\right)=&\int_0^{\frac{(P-D)t}{D}-L}\frac{(P-D)Pt^2}{2D}dG_1\left(l_1\right)\\
&+(1-\theta)\int_{\frac{(P-D)t}{D}-L}^{\infty}\frac{(P-D)Pt^2}{2D}dG_1\left(l_1\right)\\
&+\theta\int_{\frac{(P-D)t}{D}-L}^{\infty}\left(\frac{(P-D)Pt^2}{2D}+\frac{Q'^2}{2D}\right)dG_1\left(l_1\right)
\end{aligned}
$$

$$
\begin{aligned}
\delta\left(Q/P,l_2,G_2\left(l_2\right)\right)=&\int_0^{\frac{(P-D)Q}{PD}-L}\frac{(P-D)Q^2}{2PD}dG_2\left(l_2\right)\\
&+(1-\theta)\int_{\frac{(P-D)Q}{PD}-L}^{\infty}\frac{(P-D)Q^2}{2PD}dG_2\left(l_2\right)\\
&+\theta\int_{\frac{(P-D)Q}{PD}-L}^{\infty}\left(\frac{(P-D)Q^2}{2PD}+\frac{Q'^2}{2D}\right)dG_2\left(l_2\right)
\end{aligned}
$$

Now, based on the renewal reward theorem, the expected cost per unit time is given by

$$
C\left(Q,Q'\right)=\frac{S\left(Q,Q'\right)}{T\left(Q,Q'\right)}
\tag{9}
$$

In order to find the optimal solution for the problem we should solve both objective functions that are $C(Q, 0)$ subject to $0 \leq Q \leq L.P.D / (P - D)$ and $C(Q, Q')$ in case that $Q' \geq 0$ and $Q > L.P.D / (P - D)$, then choose the optimal value of Q and Q' based on the objective function which has the lower expected cost. Due to the complexity of the objective functions, it is difficult to analyse the model with general failure and general repair time distributions. In the following, we consider the model in both conditions under exponential failure and repair time distributions.

Figure 3 Condition 2: configuration of the model when no machine breakdown occurs, (a) repair operation done before order point (b) orders are not satisfied with probability $1 - \theta$, but operation done before finishing on-hand inventory (c) orders are not satisfied and shortages occur due to prolonged repair time (d) orders are satisfied with probability θ and repair operation done before finishing purchasing items (e) orders are satisfied, but shortages occur due to prolonged repair time

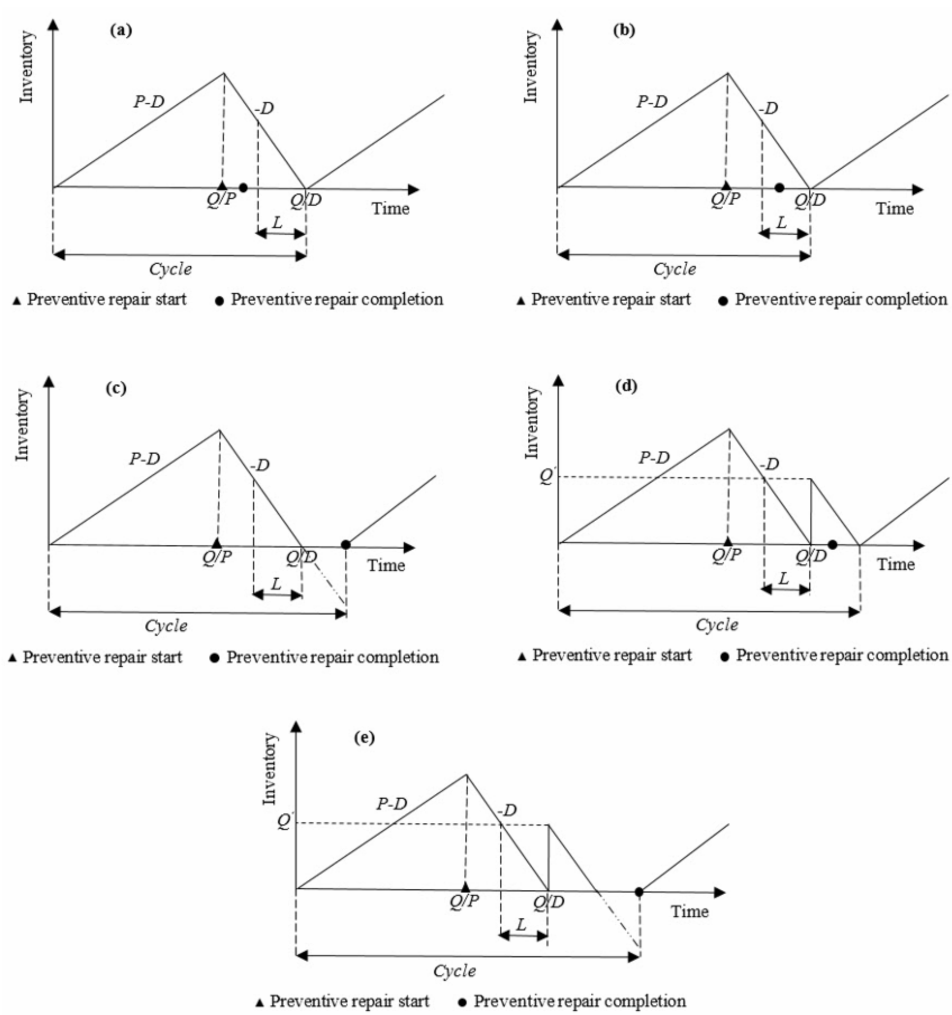
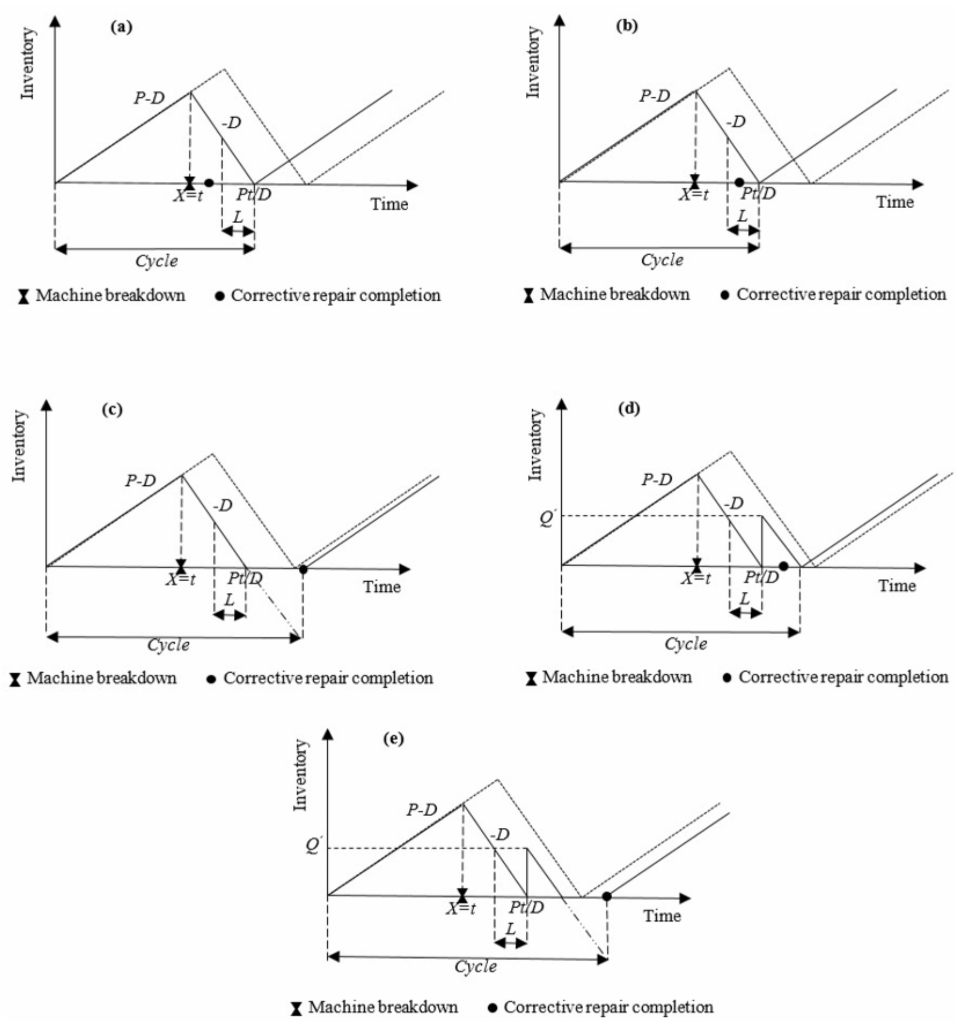


Figure 4 Condition 2: configuration of the model when machine breakdown occurs, (a) repair operation done before order point (b) orders are not satisfied with probability $1 - \theta$, but operation done before finishing on-hand inventory (c) orders are not satisfied and shortages occur due to prolonged repair time (d) orders are satisfied with probability θ and repair operation done before finishing purchasing items (e) orders are satisfied, but shortages occur due to prolonged repair time



5 The model with exponential failure and exponential repair times

5.1 Condition 1

Suppose that the failure time distribution $F_x(t) = 1 - e^{-\lambda t}$, preventive repair time distribution $G_1(l_1) = 1 - e^{\mu_1 l_1}$ and corrective repair time distribution $G_2(l_2) = 1 - e^{\mu_2 l_2}$.

Then, the mean time length of a cycle and the expected total cost per cycle for *condition 1* should be obtained from equations (4) and (5) as

$$T(Q,0)=\frac{P}{D}\left(\frac{1}{\lambda}-\left(\frac{1}{\lambda}+\frac{Q}{P}\right)e^{-\lambda Q/P}\right)+\frac{\lambda}{\mu_1\left(\mu_1\left(\frac{P-D}{D}\right)+\lambda\right)}\left(1-e^{-\left(\mu_1\left(\frac{P-D}{D}\right)+\lambda\right)Q/P}\right)$$
$$+\left(\frac{Q}{D}+\frac{1}{\mu_2}\left(1-e^{-\mu_2\left(\frac{P-D}{D}\right)\frac{Q}{P}}\right)\right)e^{-\lambda Q/P} \tag{10}$$

and

$$S(Q,0)=c_0+\frac{c_P.P}{\lambda}(1-e^{-\lambda Q/P})+\frac{c_r}{\mu_1}(1-e^{-\lambda Q/P})+\frac{c_m}{\mu_2}e^{-\lambda Q/P}$$
$$+c_h\left\{\frac{(P-D)P}{2D}\left(\frac{2}{\lambda^2}-\left(\frac{2}{\lambda^2}+\frac{2Q}{\lambda P}\right)e^{-\lambda Q/P}\right)\right\}$$
$$+c_s\left\{\frac{\lambda D}{\mu_1\left(\mu_1\left(\frac{P-D}{P}\right)+\lambda\right)}\left(1-e^{-\left(\mu_1\left(\frac{P-D}{P}\right)+\lambda\right)Q/P}\right)+\frac{D}{\mu_2}e^{-\left(\mu_2\left(\frac{P-D}{D}\right)+\lambda\right)\frac{Q}{P}}\right\} \tag{11}$$
$$+c_w\left\{\frac{P}{\lambda}(1-e^{-\lambda Q/P})+\frac{(1-\alpha)P\ln(1-\alpha)}{\alpha\left(\lambda-P\ln(1-\alpha)\right)}\left(1-e^{-(\lambda-P\ln(1-\alpha))}\right)\right\}$$

In order to determine the optimal production lot size in *condition 1*, the following propositions are proposed here.

Property 1. $T(Q, 0)$ is a concave function of Q for all $Q > 0$.

Proof. To prove the concavity of function $T(Q, 0)$ with respect to Q it is sufficient to show $\frac{d^2T(Q,0)}{dQ^2}<0$. Now, with second order partial derivative of equation (10) with respect to Q, we get

$$\frac{dT^2(Q,0)}{dQ^2}=-\frac{\lambda}{P.D}e^{-\lambda Q/P}-\frac{\lambda\left(\mu_1.\frac{P-D}{D}+\lambda\right)}{\mu_1.P^2}e^{-\left(\mu_1.\frac{P-D}{D}+\lambda\right)Q/P}$$
$$-\frac{\left(\mu_2.\frac{P-D}{D}+\lambda\right)^2}{\mu_2.P^2}e^{-\left(\mu_2.\frac{P-D}{D}+\lambda\right)Q/P}+\frac{\lambda^2}{\mu_2.P^2}e^{-\lambda Q/P}$$

Given the range of parameters, it is obvious that $D<\frac{\mu_2.P}{\lambda}$, So, $\frac{d^2T(Q,0)}{dQ^2}<0$ and the proof is completed.

Proposition 1. Let $C(Q_2, 0) \le C(Q_1, 0)$ for two distinct values Q_1 and Q_2 of Q. Then $C(Q, 0)$ is a pseudo-convex function provided

$$S(Q_2,0) \geq S(Q_1,0) + (Q_2 - Q_1)\frac{dS(Q,0)}{dQ}\Big|_{Q=Q_1}$$

Proof. Since $T(Q, 0) > 0$ and $S(Q, 0) > 0$ for all $Q \geq 0$, therefore $C(Q_2, 0) \leq C(Q_1, 0)$ implies that

$$\frac{T(Q_2,0)}{T(Q_1,0)} \geq \frac{S(Q_2,0)}{S(Q_1,0)}.$$

Again, since $T(Q, 0)$ is concave from *property 1*, we have

$$T(Q_1,0) + (Q_2 - Q_1)\left[\frac{dT(Q,0)}{dQ}\right]_{Q=Q_1} \geq T(Q_2,0)$$

Given that

$$\frac{T(Q_2,0)}{T(Q_1,0)} - 1 \geq \frac{S(Q_2,0)}{S(Q_1,0)} - 1$$

we have

$$\frac{(Q_2 - Q_1)\left[\frac{dT(Q,0)}{dQ}\right]_{Q=Q_1}}{T(Q_1,0)} \geq \frac{S(Q_2,0) - S(Q_1,0)}{S(Q_1,0)} \tag{12}$$

By first order derivative of $C(Q, 0)$ with respect to Q at the point of Q_1

$$\left[\frac{dC(Q,0)}{dQ}\right]_{Q=Q_1} = \frac{T(Q_1,0)\left[\frac{dS(Q,0)}{dQ}\right]_{Q=Q_1} - S(Q_1,0)\left[\frac{dT(Q,0)}{dQ}\right]_{Q=Q_1}}{T^2(Q_1,0)}$$

By substituting equivalent

$$\frac{\left[\frac{dT(Q,0)}{dQ}\right]_{Q=Q_1}}{T(Q_1,0)}$$

in (12) we have

$$(Q_2 - Q_1).T(Q_1,0)\left[\frac{dC(Q,0)}{dQ}\right]_{Q=Q_1} \leq S(Q_1,0)$$

$$+(Q_2 - Q_1)\left[\frac{dS(Q,0)}{dQ}\right]_{Q=Q_1} - S(Q_2,0) \leq 0$$

If

$$S(Q_2,0) \geq S(Q_1,0) + (Q_2 - Q_1)\left[\frac{dS(Q,0)}{dQ}\right]_{Q=Q_1}$$

Also $C(Q_2, 0) \leq C(Q_1, 0)$ implies

$$\left(Q_2 - Q_1\right)\left[\frac{dC(Q,0)}{dQ}\right]_{Q=Q_1} \leq 0.$$

This proves that $C(Q, 0)$ is a pseudo-convex function provided

$$S\left(Q_2,0\right) \geq S\left(Q_1,0\right) + \left(Q_2 - Q_1\right)\frac{dS(Q,0)}{dQ}\Big|_{Q=Q_1}$$

satisfying $C(Q_2, 0) \leq C(Q_1, 0)$ (Bazaraa et al., 2006).

Proposition 2. Under proposition 1, there exists a unique Q^* which minimises $C(Q, 0)$.

Proof. The first derivative of $C(Q, 0)$ with respect to Q is

$$\varphi(Q) \equiv dC(Q,0)/dQ = U(Q)/O(Q)$$

where

$$U(Q) = \left\{\begin{array}{l} \left(c_p + \dfrac{c_r\lambda}{\mu_1.P} - \dfrac{c_m\lambda}{\mu_2.P}\right)e^{-\lambda Q/P} + c_h\left(-\dfrac{P-D}{\lambda D} + \dfrac{Q(P-D)}{P.D} + \dfrac{2}{\lambda P}\right)e^{-\lambda Q/P} \\ +c_s\left(\dfrac{\lambda D}{\mu_1.P}e^{-\left(\mu_1\left(\frac{P-D}{D}\right)+\lambda\right)Q/P} - \dfrac{D\left(\mu_2\left(\dfrac{P-D}{D}\right)+\lambda\right)}{\mu_2.P}e^{-\left(\mu_2\left(\frac{P-D}{D}\right)+\lambda\right)Q/P}\right) \\ +c_w.e^{-\lambda Q/P} \end{array}\right\}T(Q,0)$$

$$-\left\{\begin{array}{l} \left(\dfrac{1}{D} - \dfrac{\lambda}{\mu_2.P}\right)e^{-\lambda Q/P} + \dfrac{\lambda}{\mu_1.P}e^{-\left(\mu_1\left(\frac{P-D}{D}\right)+\lambda\right)Q/P} \\ + \dfrac{\mu_2\left(\dfrac{P-D}{D}\right)+\lambda}{\mu_2.P}e^{-\left(\mu_2\left(\frac{P-D}{D}\right)+\lambda\right)Q/P} \end{array}\right\}S(Q,0)$$

and

$$O(Q) = \left[T(Q,0)\right]^2$$

We have

$$\lim_{Q\to 0} U(Q) = -\left[c_0 + \frac{c_m}{\mu_2} + \frac{c_s D}{\mu_2} + c_w.G\right]\left[\frac{1}{D} + \frac{\lambda}{\mu_1 P} + \frac{P-D}{P.D}\right]$$

and

$$\lim_{Q\to 0} O(Q) = 0.$$

where

$$G = \frac{(1-\alpha)P\ln(1-\alpha)}{\alpha\left(\lambda - P\ln(1-\alpha)\right)}\left(1 - e^{-\left(\lambda - P\ln(1-\alpha)\right)}\right)$$

It is observed that $\varphi(Q) \to -\infty$ as $Q \to 0$. Moreover, $\varphi(Q) \to 0$ as $Q \to \infty$. So, given the pseudo-convexity of $C(Q, 0)$ under the *proposition 1*, there exist a unique root Q^* of $\varphi(Q) = 0$.

Now, we obtain the optimal production lot size by solving the equation $\varphi(Q) = 0$. If $Q^* < L.P.D / (P - D)$ then Q^* will be the optimal lot size, otherwise, $Q^* = L.P.D / (P - D)$.

5.2 Condition 2

Assuming that the mean time to failure and repair time follow the exponential distribution, for *condition 2*, from equations (7) and (8) we have

$$T(Q,Q') = \frac{P}{\lambda D}\left(1-e^{-\frac{\lambda Q}{P}}\right) + \frac{\lambda}{\mu_1\left(\mu_1\left(\frac{P-D}{D}\right)+\lambda\right)} \left\{ \begin{array}{l} 1-e^{-\left(\mu_1\left(\frac{P-D}{D}\right)+\lambda\right)\frac{LD}{P-D}} + \left(\theta e^{-\frac{\mu_1 Q'}{D}} + \frac{\mu_1\theta Q'}{D}e^{\mu_1 L} + 1-\theta\right) \\ \left(e^{-\left(\mu_1\left(\frac{P-D}{D}\right)+\lambda\right)\frac{LD}{P-D}} - e^{-\left(\mu_1\left(\frac{P-D}{D}\right)+\lambda\right)\frac{Q}{P}}\right) \end{array} \right\} + \frac{1}{\mu_2}\left(\theta e^{-\frac{\mu_2 Q'}{D}} + \frac{\mu_2\theta Q'}{D}e^{\mu_2 L} + 1-\theta\right)\left(e^{-\lambda\frac{Q}{P}} - e^{-\left(\mu_2\left(\frac{P-D}{D}\right)+\lambda\right)\frac{Q}{P}}\right) \tag{13}$$

and

$$S(Q,Q') = c_0 + \frac{c_P.P}{\lambda}\left(1-e^{-\lambda Q/P}\right) + \frac{c_r}{\mu_1}\left(1-e^{-\lambda Q/P}\right) + \frac{c_m}{\mu_2}e^{-\lambda Q/P} + \left(c'\theta Q' + \theta c_0'\right)\left\{ \begin{array}{l} \frac{\lambda e^{\mu_1 L}}{\mu_1\left(\frac{P-D}{D}\right)+\lambda}\left(e^{-\left(\mu_1\left(\frac{P-D}{D}\right)+\lambda\right)\frac{LD}{P-D}} - e^{-\left(\mu_1\left(\frac{P-D}{D}\right)+\lambda\right)\frac{Q}{P}}\right) \\ +e^{-\left(\mu_2\left(\frac{(P-D)Q}{PD}-L\right)+\lambda\frac{Q}{P}\right)} \end{array} \right\} + c_h\left\{ \begin{array}{l} \frac{(P-D)P}{2D}\left(\frac{2}{\lambda^2} - \left(\frac{2}{\lambda^2} + \frac{2Q}{\lambda P} + \frac{Q^2}{P^2}\right)e^{-\frac{\lambda Q}{P}}\right) \\ + \frac{\theta\lambda Q' e^{\mu_1 L}}{2D\left(\mu_1\left(\frac{P-D}{D}\right)+\lambda\right)}\left(e^{-\left(\mu_1\left(\frac{P-D}{D}\right)+\lambda\right)\frac{LD}{P-D}} - e^{-\left(\mu_1\left(\frac{P-D}{D}\right)+\lambda\right)\frac{Q}{P}}\right) \\ + \left(\frac{\theta Q'^2}{2D}e^{-\mu_2\left(\frac{(P-D)Q}{PD}-L\right)} + \frac{(P-D)Q^2}{2PD}\right)e^{-\frac{\lambda Q}{P}} \end{array} \right\}$$

$$+c_s\left\{\begin{array}{l}\dfrac{\lambda D}{\mu_1\left(\mu_1\left(\dfrac{P-D}{D}\right)+\lambda\right)}\left(1-e^{-\left(\mu_1\left(\frac{P-D}{D}\right)+\lambda\right)\frac{LD}{P-D}}\right)\\ +\dfrac{\lambda D\left(\dfrac{1}{\mu_1}-L-\dfrac{\theta Q'}{D}\right)e^{\mu_1 L}}{\mu_1\left(\dfrac{P-D}{D}\right)+\lambda}\left(e^{-\left(\mu_1\left(\frac{P-D}{D}\right)+\lambda\right)\frac{LD}{P-D}}-e^{-\left(\mu_1\left(\frac{P-D}{D}\right)+\lambda\right)\frac{Q}{P}}\right)\\ +D\left(\dfrac{1}{\mu_2}-L-\dfrac{\theta Q'}{D}\right)e^{-\left(\mu_2\left(\frac{(P-D)Q}{PD}-L\right)+\lambda\frac{Q}{P}\right)}\end{array}\right\} \quad (14)$$

$$+c_w\left\{\frac{P}{\lambda}(1-e^{-\lambda Q/P})+\frac{(1-\alpha)P\ln(1-\alpha)}{\alpha(\lambda-P\ln(1-\alpha))}\left(1-e^{-(\lambda-P\ln(1-\alpha))}\right)\right\}$$

Due to the complexity of the model, it is difficult to prove the convexity of $C(Q, Q')$ for any given parameters when Q and Q' are decision variables. We propose two approaches to find Q and Q' in this condition.

At the first approach, based on Giri et al. (2005a), we look forward for the local optimal solution. Let η be the Lagrange multiplier corresponding to constraint $Q > L.P.D / (P - D)$. By considering Kuhn-Tucker necessary conditions we have

$$T(Q,Q')\frac{\delta S(Q,Q')}{\delta Q'}-S(Q,Q')\frac{\delta T(Q,Q')}{\delta Q'}=0 \quad (15)$$

$$T(Q,Q')\frac{\delta S(Q,Q')}{\delta Q}-S(Q,Q')\frac{\delta T(Q,Q')}{\delta Q}-\eta.T^2(Q,Q')=0 \quad (16)$$

$$\eta.\left[Q-L.P.D/(P-D)\right]=0 \quad (17)$$

Obviously $\eta = 0$ since $Q > L.P.D / (P - D)$. So the optimal values of Q and Q' can obtained by solving the system of the nonlinear equations (15) and (16). it is worth noting the global optimality of the function is not guaranteed by this approach.

At the second approach, first we will discuss the convexity of the model when Q' is known. For doing this, the following property and proposition are presented.

Property 2. For any given $Q' > 0$, $T(Q, Q')$ is concave for all $Q > L.P.D / (P - D)$.

Proof. To proof the concavity of function $T(Q, Q')$ with respect to Q it is sufficient to show $\dfrac{\delta^2 T(Q,Q')}{\delta Q^2}<0$. Now, the first order partial derivative of $T(Q, Q')$ with respect to Q is

$$\frac{\delta T(Q,Q')}{\delta Q}=\frac{1}{D}e^{-\lambda Q/P}+\frac{\lambda}{\mu_1.P}K_1e^{-\left(\mu_1\left(\frac{P-D}{D}\right)+\lambda\right)Q/P}$$
$$+\frac{\left(\mu_2\left(\dfrac{P-D}{D}\right)+\lambda\right)}{\mu_2.P}K_2e^{-\left(\mu_2\left(\frac{P-D}{D}\right)+\lambda\right)Q/P}-\frac{\lambda}{\mu_2.P}K_2e^{-\lambda Q/P}$$

where

$$K_1 = \theta e^{-\frac{\mu_1 Q'}{D}} + \frac{\mu_1 \theta Q'}{D} e^{\mu_1 L} + 1 - \theta$$

$$K_2 = \theta e^{-\frac{\mu_2 Q'}{D}} + \frac{\mu_2 \theta Q'}{D} e^{\mu_2 L} + 1 - \theta$$

Again, differentiating we get

$$\frac{\delta^2 T(Q,Q')}{\delta Q^2} = -\frac{\lambda}{PD} e^{-\lambda Q/P} - \frac{\lambda\left(\mu_1\left(\frac{P-D}{D}\right)+\lambda\right)}{\mu_1 . P^2} K_1 e^{-\left(\mu_1\left(\frac{P-D}{D}\right)+\lambda\right)Q/P}$$
$$-\frac{\left(\mu_2\left(\frac{P-D}{D}\right)+\lambda\right)^2}{\mu_2 . P^2} K_2 e^{-\left(\mu_2\left(\frac{P-D}{D}\right)+\lambda\right)Q/P} + \frac{\lambda^2}{\mu_2 . P^2} K_2 e^{-\lambda Q/P}$$

Now we can conclude that

$$\frac{\delta^2 T(Q,Q')}{\delta Q^2} < 0$$

if

$$\frac{P\lambda\mu_2}{D.K_2} + \left(\mu_2\left(\frac{P-D}{D}\right)+\lambda\right)^2 e^{-\mu_2\left(\frac{P-D}{D}\right)} > \lambda^2.$$

So proof is completed just for specific range of parameters.

Proposition 3. Let $C(Q_2, Q') \leq C(Q_1, Q')$ for two distinct values of Q_1 and Q_2 of Q, given $Q > L.P.D / (P - D)$. Then $C(Q, Q')$ is a pseudo-convex function provided

$$S(Q_2,Q') \geq S(Q_1,Q') + (Q_2 - Q_1)\frac{\delta S(Q,Q')}{\delta Q}\Big|_{Q=Q_1}$$

Proof. Under the *property 2*, the proof of pseudo-convexity of $C(Q, Q')$ is the same as the method presented in *proposition 1*.

Now in this approach, we follow an exhaustive search algorithm in which at each step we enumerate Q' and then solve the equation (15) for known Q' to find Q knowing that $\eta = 0$. Best Q and Q' are reported based on minimum value of $C(Q, Q')$.

6 Numerical experiment

To find the optimal solution, the following dataset with appropriate units is considered:

$P = 600$, $D = 500$, $c_0 = 100$, $c_0' = 120$, $c_h = 4$, $c_p = 20$, $c' = 25$, $c_r = 80$, $c_m = 20$, $c_s = 40$, $c_w = 5$, $L = 0.8$, $\theta = 0.9$, $\alpha = 0.0005$, $\lambda = 0.6$, $\mu_1 = 0.7$ and $\mu_2 = 1.6$.

Based on the solution approaches in Section 5, we found the best values of Q and Q' using MATHEMATICA. We optimised $C(Q, 0)$ subject to $0 \leq 0 < 2{,}400$, and the optimal solution is obtained as $Q^* = 1{,}599.9$ and $C^* = 15{,}043.2$ as depicted in *Figure 5*. The second function $C(Q, Q')$ under constraints $Q > 2{,}400$ and $Q' \geq 0$ results in the best solution $Q^* = 7{,}896.9$, $Q'^* = 3{,}317.6$ and $C^* = 14{,}427$ (the result of both of the solution approaches were the same). The surface generated by $C(Q, Q')$ over the wide range of values of Q and Q' is shown in *Figure 6*. By comparing the optimal values of two conditions, the optimal values of $C(Q, Q')$ are selected.

Figure 5 Graphical representation of $C(Q, 0)$ (see online version for colours)

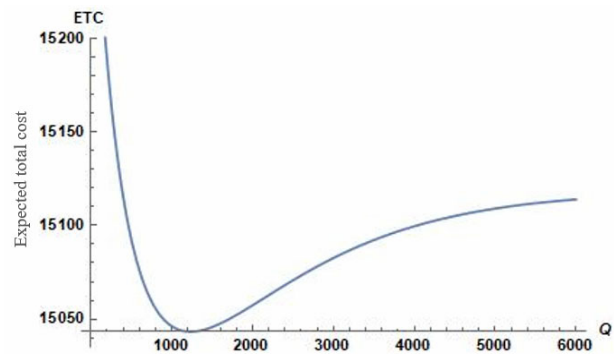
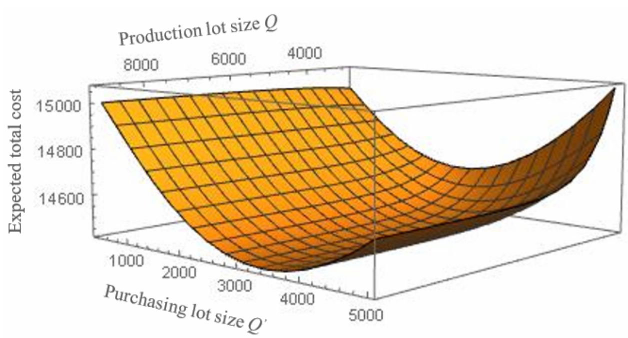


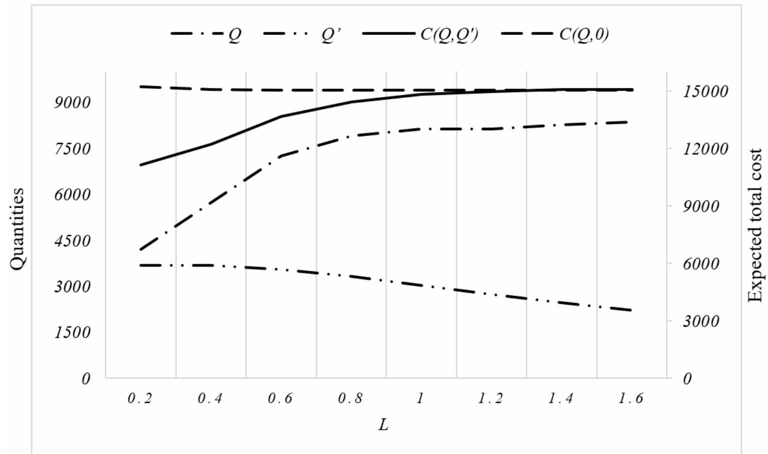
Figure 6 Graphical representation of $C(Q, Q')$ (see online version for colours)



6.1 Sensitivity analysis based on supplier parameters

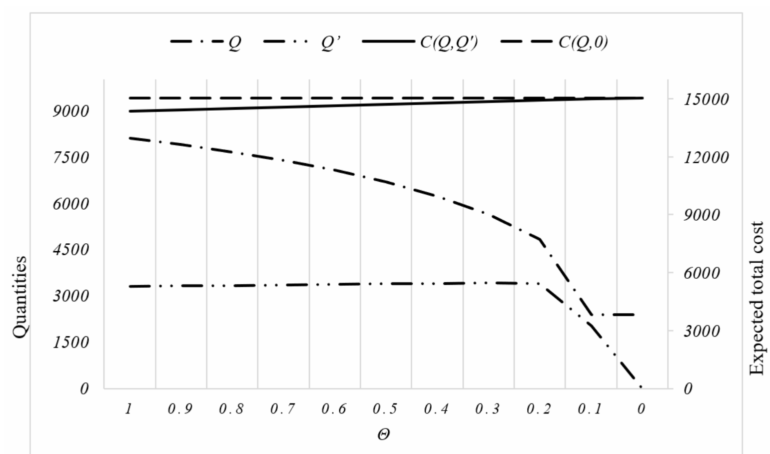
Now, we examine the sensitivity of optimal values for Q^*, Q'^* and C^* against changes in the parameters L and θ. As displayed in *Figure 7*, by decreasing lead-time, the optimal value of Q' increases, and the optimal value of Q is reduced as well as expected total cost. When the supplier lead-time rises, the model reduces the purchasing quantity, which increases the $C^*(Q, Q')$ so that for $L \geq 1.4$, $C^*(Q, 0)$ (15,043.2) would be less than $C^*(Q, Q')$ (15,054.1). This means that for high values of L, the model uses production as the basis for the business activity. This can be justified from the managerial point of view. Increasing the lead-time, increases the risk of facing lost sales, so the manufacturer prefers to have more production.

Figure 7 Variation of L effects on optimal value of Q^* and Q'^* and C^*



Sensitivity analysis shows that the supplier reliability can play important role when making inventory decision. As illustrated in *Figure 8*, when the supplier is more reliable, the decreases order quantity and increases production quantity. We also observe that by decreasing θ, the manufacturer increases the order quantity. This is because, in essence, the model is conservative. When the supplier is less reliable, the model suggests the manufacturer purchase larger quantities to confront against uncertainties; but for very small values of θ, the model reduces order quantity, so that when $\theta \to 0$ then $Q' \to 0$. In this situation, considering mean time to failure and repair time, by reducing production run as well as production quantity, the model moves to *condition 1* (as seen in Figure 8 when $\theta \to 0$ the expected total cost will be equal in both conditions).

Figure 8 Behaviour of Q^*, Q'^* and C^* by changing supplier reliability θ



A comparison of the model's performance, using different values of θ and L is shown in *Figure 9* and *Figure 10*. It should be noted that by increasing L, the purchasing lot size Q'^* and total cost $C^*(Q, Q')$ becomes less sensitive to changes in θ. Actually, when values of L are low, the reliability of the supplier is important to the manufacturer. For example when $L = 0.6$ and the value of θ changes from 0.3 to 1, the optimal value of Q' decreases by 6.6% and $C^*(Q, Q')$ decreases by 7.1%; and when $L = 1.6$ the changes are nearly null.

Figure 9 Variation of θ effects on Q'^* for different values of L

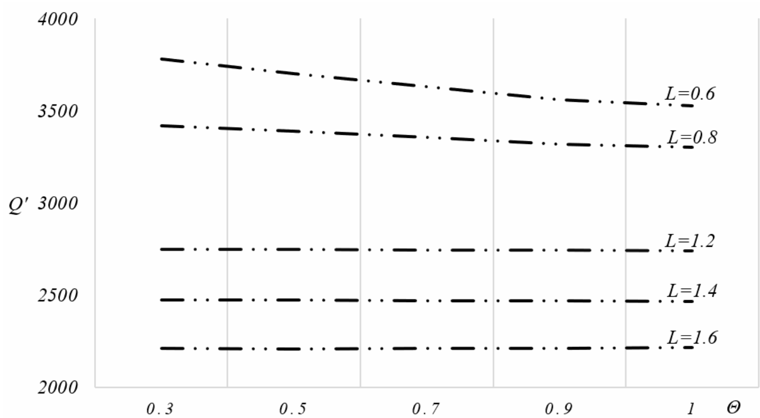
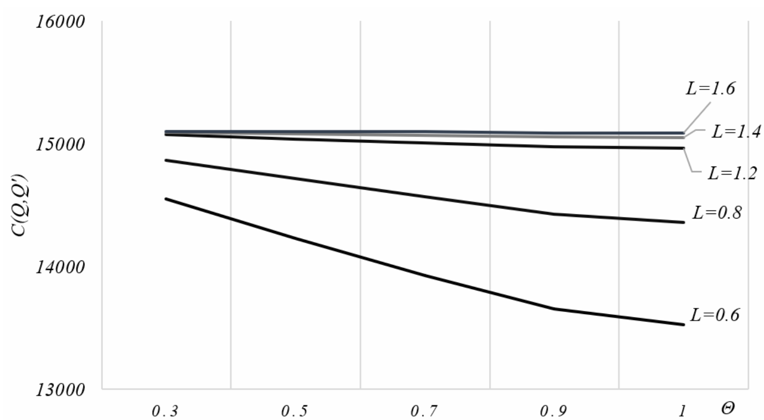


Figure 10 Variation of θ effects on C^* for different values of L



Higher shortage costs cause more Q and Q' to reduce the cost of lost sales. As it increases by 50%, the order quantity rises two fold. Sensitivity analysis of simultaneous variations of L and c_s (see *Table 1*) shows that by decreasing L and increasing shortage cost, better results are gained by increasing the quantities ordered from the supplier. When the supplier lead-time is longer, and values of c_s are low, the manufacturer does not order; however, when values of c_s are high, the manufacturer purchases part of the required inventory from the external supplier to avoid shortages from the lost sales. In *Table 1*, the condition with the lower total cost is shown in bold in each row.

Table 1 Sensitivity analysis with respect to shortage cost under the variation of L

c_s	$L = 0.5$					$L = 0.8$				
	Condition 1		*Condition 2*			*Condition 1*		*Condition 2*		
	Q	*C(Q,0)*	*Q*	*Q′*	*C(Q, Q′)*	*Q*	*C(Q,0)*	*Q*	*Q′*	*C(Q, Q′)*
32	209.9	13,456.8	4,476.6	2,790.26	*12,615.7*	209.9	*13,456.8*	5,714.9	2,539.05	13,487.2
34	479.4	13,742.3	5,118.8	3,017.64	*12,742.6*	479.4	13,742.3	6,058.7	2,656.79	*13,627.1*
36	891.4	14,213.9	5,683.8	3,232.7	*12,853.6*	891.34	14,213.9	6,711.6	2,883.52	*13,901*
38	1,251.2	14,639.1	6,189.7	3,440.29	*12,950.4*	1,251.2	14,639.1	7,323.1	3,102.96	*14,167.5*
40	1,500	15,043.7	6,646.9	3,642.38	*13,034.1*	1,599.9	15,043.2	7,897	3,317.61	*14,427*
42	1,500	15,444.4	7,063.1	3,839.98	*13,105.5*	1,958.2	15,435	8,436.3	3,528.65	*14,679.9*
44	1,500	15,845.1	7,443.9	4,033.72	*13,165.2*	2,341.3	15,818.8	8,944.3	3,736.77	*14,926.4*
46	1,500	16,245.8	7,794.2	4,224	*13,213.9*	2,400	16,198.8	9,423.9	3,942.39	*15,166.7*
48	1,500	16,646.5	8,117.4	4,411.2	*13,252*	2,400	16,578.7	9,877.9	4,145.82	*15,400.9*

c_s	$L = 1.1$					$L = 1.4$				
	Condition 1		*Condition 2*			*Condition 1*		*Condition 2*		
	Q	*C(Q,0)*	*Q*	*Q′*	*C(Q, Q′)*	*Q*	*C(Q,0)*	*Q*	*Q′*	*C(Q, Q′)*
32	209.9	*13,456.8*	3,300	925.45	13,569.6	209.9	*13,456.8*	4,200	661.12	13,616
34	479.4	*13,742.3*	6,167.1	2,230.29	13,905.5	479.4	*13,742.3*	4,200	793.73	13,982.3
36	891.4	*14,213.9*	6,858.7	2,459.91	14,242.7	891.4	*14,213.9*	4,200	921	14,348
38	1,251.25	14,639.1	7,516.4	2,676.08	*14,576.7*	1,251.2	*14,639.1*	7,413.4	2,263.34	14,698.5
40	1,599.99	15,043.2	8,145.7	2,885.6	*14,907.8*	1,599.9	*15,043.2*	8,062.6	2,466	15,054.1
42	1,958.28	15,435	8,748	3,091.14	*15,236.2*	1,958.2	15,435	8,688.6	2,660.94	*15,408.6*
44	2,341.34	15,818.8	9,324.4	3,293.95	*15,561.9*	2,341.3	15,818.8	9,293.6	2,851.73	*15,762.1*
46	2,764.57	16,197	9,876.5	3,494.69	*15,885*	2,764.5	16,197	9,878.1	3,039.96	*16,114.7*
48	3,242.78	16,571.2	10,406	3,693.77	*16,205.5*	3,242.7	16,571.2	10,442.7	3,226.44	*16,466.3*

6.2 *Sensitivity analysis based on failure and repair rates*

It is obvious that the manufacturer decision, to make or make not order quantities, is highly dependent on the machine failure rate as well as repair times. In this section, we examine the effect of these parameters on the optimal values of the decision variables and the objective function.

The behaviour of expected total cost per unit time in both conditions based on the variation of μ_1 and μ_2 illustrated in *Table 2* and *Table 3* respectively. According to Table 2, as μ_1 decreases (the mean time for the correcting repair increases), the optimal value of cost function increases in both conditions. In return, by increasing of μ_1, the model decreases the amount of cost function. It can be seen that for the chosen data set, the purchasing policy is always preferable. The similar situation would happen when the preventive repair rate changes. With the increase of mean time for preventive repair (decrease of μ_2), the expected total cost per unit time increases in both conditions and with the increase of μ_2, it decreases (see *Table 3*). As the table shows, $C(Q, 0)$ and $C(Q, Q')$ are less sensitive to changes in μ_2 rather than changes in μ_1. The reason is that usually, the unit preventive repair cost is much less, than the unit corrective repair cost.

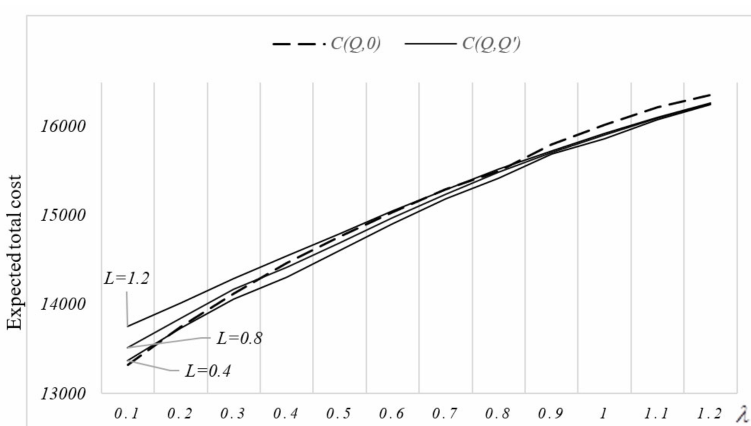
Table 2 Behaviour of expected total cost per unit time by changing corrective repair rate μ_1

	μ_1					
	0.3	*0.5*	*0.7*	*0.9*	*1.1*	*1.3*
$C(Q, 0)$	16,800	15,770.3	15,043.2	14,516.8	14,122.2	13,817.5
$C(Q, Q')$	16,165.9	15,035.8	14,427	14,052.2	13,790.1	13,532.1

Table 3 Behaviour of expected total cost per unit time by changing preventive repair rate μ_2

	μ_1					
	1	*1.3*	*1.6*	*1.9*	*2.2*	*2.5*
$C(Q, 0)$	15,101.1	15,073.3	15,043.2	15,001.8	14,943.8	14,857.1
$C(Q, Q')$	14,428.9	14,427.8	14,427	14,427	14,426.9	14,426.8

Figure 11 A comparison of *ETC* in both conditions for different values of *L* with respect to λ variation

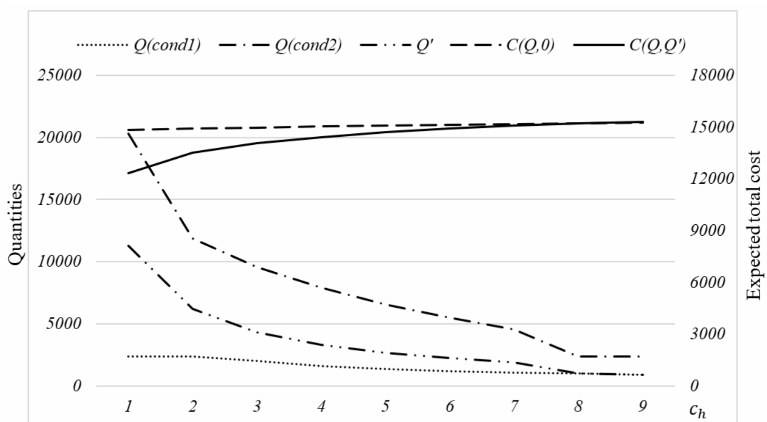


The dependence of the optimal production policy on the parameter of λ based on different values of L is shown in *Figure 11*. Studying this figure gives remarkable results. When λ takes larger values, the expected total cost in both conditions increases. According to this figure, for the higher failure rate, the purchasing strategy is more economical even for the large values of L. as λ decreases, the *ETC* decreases in both conditions, so that $C^*(Q, 0)$ is less than $C^*(Q, Q')$. For large values of L, this superiority occurs earlier. It can be observed from *Figure 11* that the $C^*(Q, 0)$ and $C^*(Q, Q')$ will be equal when $L \to 0$ and $\lambda \to 0$.

Table 4 The amount of shortage cost in *ETC* by variation of c_h

c_h	*Condition 1*		*Condition 2*		*Manufacturer decision*
	$C^*(Q, 0)$	*Shortage cost*	$C^*(Q, Q')$	*Shortage cost*	
1	14,823.3	7,597.6	*12,348.2*	1,330.5	*Condition 2*
2	14,901.9	7,597.6	*13,515.3*	3,503.2	*Condition 2*
3	14,978.4	7,722.3	*14,071.5*	4,521.2	*Condition 2*
4	15,043.2	7,940.7	*14,427*	5,123.4	*Condition 2*
5	15,099.2	8,132.3	*14,689.8*	5,533	*Condition 2*
6	15,149	8,305.9	*14,900.6*	5,847.5	*Condition 2*
7	15,194	8,466.4	*15,077.2*	6,128.1	*Condition 2*
8	15,235.2	8,616.4	*15,209.7*	6,925.4	*Condition 2*
9	15,273.2	8,757.9	15,308.9	6,997.6	*Condition 1*

Figure 12 The effects of c_h variation on optimal values of Q^*, Q'^* and C^* in both conditions



6.3 *Effect of unit holding cost*

Although purchasing strategy raising manufacturer service level through reducing lost sales, but by increasing the unit holding cost, the model prefers lost sales (*Table 4*). As unit holding cost rises, the manufacturer prefers to produce a smaller lot size for each production run. Furthermore, for $c_h \geq 9$ given that expected total cost (ETC) of *condition 1* is lower than ETC of *condition 2*, no purchasing order is given by manufacturer. While for low values of c_h, the manufacturer increases production and purchasing order quantities leading to a reduction in expected total cost (*Figure 12*). This shows that purchasing from an external supplier is only applicable for the products with reasonable unit holding costs.

Table 5 Sensitivity analysis on other parameters of the numerical example

Parameters variation (in %)		*Variations in condition 1*		*Variations in condition 2*			*Manufacturer decision*
		Q^*	$C^*(Q, 0)$	Q^*	Q'^*	$C^*(Q, Q')$	
c_P	–25%	+800.01	–1,534.4	+98.64	–364.99	–1,420.3	Condition 2
	–10%	+358.19	–608.2	+31.33	–144.94	–565.8	Condition 2
	+10%	–348.84	+595.9	–22.37	+143.66	+562.8	Condition 2
	+25%	–903.33	+1,444	–41.93	+356.95	+1,401.5	Condition 2
c'	–50%	_	_	+1,723.2	+1,352.4	–909.9	Condition 2
	–25%	_	_	+949.82	+688.5	–422	Condition 2
	+25%	_	_	–1,155.6	–732.54	+349.3	Condition 2
	+50%	_	_	–5,696.9	–3,317.6	+635.5	Condition 1
c_r	–50%	+10.02	–17.2	+0.76	–4.12	–16.1	Condition 2
	–25%	+5.01	–8.6	+0.39	–2.06	–8	Condition 2
	+25%	–5.01	+8.6	–0.38	+2.06	+8.1	Condition 2
	+50%	–10.02	+17.2	–0.75	+4.12	+16.2	Condition 2
c_m	–50%	–5.87	–0.5	–2.17	–0.0009	–0.0390	Condition 2
	–25%	–2.93	–0.2	–1.08	–0.0001	–0.0393	Condition 2
	+25%	+2.93	+0.3	+1.1	+0.0015	+0.0399	Condition 2
	+50%	+5.86	+0.5	+2.18	+0.0024	+0.0410	Condition 2
c_0	–50%	–38.12	–18.9	–16.75	–3.62	–14.1	Condition 2
	–25%	–19.07	–9.4	–8.37	–1.81	–7	Condition 2
	+25%	+19.08	+9.4	+8.38	+1.81	+7.1	Condition 2
	+50%	+38.17	+18.8	+16.76	+3.62	+14.2	Condition 2
c'_0	–50%	_	_	2.35	–0.28	–1.1	Condition 2
	–25%	_	_	1.18	–0.14	–0.5	Condition 2
	+25%	_	_	–1.17	+0.14	+0.6	Condition 2
	+50%	_	_	–2.35	+0.28	+1.2	Condition 2
c_w	–50%	+800.1	–422.1	+389.01	–109.21	–427.1	Condition 2
	–25%	+610.22	–203.7	+188.23	–54.5	–213.3	Condition 2
	+25%	–594.5	+166.5	–176.77	+54.29	+213	Condition 2
	+50%	–1254.41	+221.4	–343.11	+108.37	+425.4	Condition 2
α	–50%	–101.06	–50.6	–44.48	–9.61	–37.4	Condition 2
	–25%	–49.41	–24.6	–21.73	–4.69	–18.3	Condition 2
	+25%	+47.27	+23.3	+20.75	+4.48	+17.5	Condition 2
	+50%	+92.51	+45.3	+40.55	+8.76	+34.2	Condition 2

6.4 Effect of other parameters

We have changed the values of the other parameters in the model by both positive and negative percentages. *Table 5* reports the positive and negative changes in the optimal values of decision variables and objective function in both conditions.

Based on *Table 5* the following features are observed:

- Except for c', changes in other parameters examined in *Table 5*, do not affect manufacturer's decision to order purchasing quantity and in all scenarios, $C^*(Q, Q')$ will be less than $C^*(Q, 0)$. As c' increases, $C^*(Q, Q')$ increases and with an increase of more than 50% in unit purchasing price, $C^*(Q, 0)$ will be less than $C^*(Q, Q')$. Moreover, Q'^* is highly sensitive to changes in purchasing cost. Lower c' causes more order quantity lot size to reduce cost of lost sale during stock-out situations.
- The optimal production lot size in *condition 2* is fairly sensitive to changes in c_P; additionally, Q^* in *condition 1*, Q'^*, $C^*(Q, 0)$ and $C^*(Q, Q')$ are highly sensitive to changes in c_P. Moreover, Q^* decreases and Q'^* increases with increases in c_P. Consequently $C^*(Q, 0)$ and $C^*(Q, Q')$ increase automatically.
- Both decision variables Q^*, Q'^* and also objective functions are less sensitive to changes in c_r and c_m; but changes in c_r result in more variation in cost function than changes in c_m. Similarly, sensitivity of Q^*, Q'^*, $C^*(Q, 0)$ and $C^*(Q, Q')$ in c_0 and c'_0 are negligible.
- Total lot size decreases to reduce higher rework cost (c_w) or higher rate of defective items produced (α). Variation of order quantity in these situations is negligible.

7 Conclusions

The classical economic production models assume that the production facilities always are failure-free. However, in practical situations, they usually are failure-prone. Since failures are unavoidable, the production manager should have practical solutions to deal with such disruptions. Proactive measures such as inspection, keeping safety stock have been carried out in this field to mitigate machine breakdowns consequences. In terms of expensive unit holding cost or low warehouse capacity, emergency replenishment could be a better option than keeping safety stock. In this study, we have developed a production system under random machine breakdown and two types of repair. By starting the production process, a facility may shift from an 'in-control' state to an 'out-of-control' state at any random time. If machine breakdown occurs during the production run time, then corrective repair is done; otherwise, preventive repair is performed at the end of the production run time. The model is developed under general machine breakdown and general repair time distributions. Assuming conditions like higher holding cost or lower warehouse capacity, we investigated how the manufacturer can benefit from using an external supplier as an alternative option to keeping safety stock during a stoppage in the production process caused by corrective or preventive repair. We also assumed that the external supplier is unreliable and has a lead-time. Our numerical study has been found that inventory holding costs and shortage costs play a critical role in determining the purchase situation from an external supplier. The study also shows that supplier lead-time is more important than supplier reliability; and for longer values of L, supplier reliability does not influence the manufacturer's decision. However, when L is decrease, reliability plays an essential role in determining order quantity. The analysis conducted on the other important parameters in Section 6, shows that the purchasing policy always imposes a less expected cost to the system. Our proposed model could be improved by investigating the influence of variable demand on the manufacturer decision in purchasing policy. Moreover, investigating the influence of contractual agreement between manufacturer and supplier on lead-time in purchasing strategy could be scope for a future study.

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