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The matrix Jacobson graph of fields

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Abstract. The notion of Jacobson graph and n-array Jacobson graph of a commutative ring were introduced in 2012 and 2018, respectively, by Azimi et al and Ghayour et al. In this article we generalize them to matrix Jacobson graph. Let R be a commutative ring. The matrix Jacobson graph of a ring R, denoted $\mathfrak{J}(R)^{m\times n}$, is defined as a graph with vertex set is the set of matrix of ring without the matrix of its Jacobson such that two distinct vertices A, B are adjacent if and only if $1 - \det(A^t B)$ is not a unit of ring. In this article we study the matrix Jacobson graph where the underlying ring R is a finite field. Since any matrix of size $m \times n$ over a field F can be considered as a linear mapping from linear space F^m to F^n, we employ the structure of linear mappings on finite dimensional vector spaces to derive some properties of square and non square matrix Jacobson graph of fields, including their diameters.

1. Introduction

Let R be a commutative ring and $U(R)$ denote the group of units of R. Let $\mathfrak{m}$ denote a maximal ideal of R, $J(R)$ denote the Jacobson radical of the ring R, that is $J(R) = \cap_{\mathfrak{m}}\mathfrak{m}$, where $\mathfrak{m}$ ranges over all maximal ideals of R (see [8] for more information about finite rings). A Jacobson graph of R, denoted $\mathfrak{J}_R = (V, E)$, is defined as a graph with vertex set is $V = R\backslash J(R)$ and for every two distinct vertices $x, y \in V$, we have $x \sim y$ (adjacent) if and only if $1 - xy \notin U(R)$, (see [2]). Many graph theoretical properties such as connectivity, planarity and perfectness were obtained for Jacobson graphs of various commutative rings (for more details, see [1],[3],[4],[5],[7]).

Let R^n be the set of n-arrays of elements of R. An n-array Jacobson graph of R, denoted $\mathfrak{J}_R^n = (V, E)$, is defined as a graph with vertex set is $V = R^n\backslash J(R)^n$ (n-array column of size n) and for every two vertices $\mathbf{x}, \mathbf{y} \in V, \mathbf{x} \neq \mathbf{y}$, we have $\mathbf{x} \sim \mathbf{y}$ if and only if $1 - \mathbf{x}^t\mathbf{y} \notin U(R)$. The n-array Jacobson graph was introduced by H. Ghayour et al (2018) in [6] as an extension of the notion of Jacobson graph from ring elements to n-array with entries are elements of the underlying ring.

In this article, we generalize the notion of Jacobson graphs and n-array Jacobson graph into matrix Jacobson graph. Let R be a commutative ring, **a matrix Jacobson graph of** R, denoted by $\mathfrak{J}_R^{m\times n} = (V, E)$, is defined as a graph with vertex set $V = R^{m\times n}\backslash J(R)^{m\times n}$ and for every two vertices $A, B \in V, A \neq B$ we have $A \sim B$ if and only if $1 - \det(A^t B) \notin U(R)$. Thus, according to this definition, the n-array Jacobson graph of a ring R by H. Ghayour et al (2018) is the $n \times 1$ matrix Jacobson graph of the ring R (for more details, see [6]).

Let F be a field, $U(F)$ be the group of unit of F. We have $U(F) = F\backslash 0$ dan $J(F) = 0$. Let F^n be the row vector space over the field F of dimension n. For the rest of this note, we assume the elements of F^n are in the form of row vectors. Let $\{\dot{e}_1, ..., \dot{e}_m\}$ be the standard basis of F^m and $\{\mathbf{e}_1, ..., \mathbf{e}_n\}$ be the standard basis of F^n. Let $A \in F^{m\times n}$. We may write a matrix $A \in F^{m\times n}$ as

$$A = (a_{ij}) = I_m \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_m \end{pmatrix} = (\dot{e}_1^t \cdots \dot{e}_m^t) \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_m \end{pmatrix} = \sum_{i=1}^{m} \dot{e}_i^t \mathbf{a}_i.$$

where $\mathbf{a}_i = (a_{i1}, a_{i2}, ..., a_{in}) \in F^n$.

The aim of this paper is to determine the diameter of some square and non square matrix Jacobson graphs especially of the field F. Since any matrix of size $m \times n$ over a field F can be considered as a linear mapping from linear space F^m to F^n, we employ the structure of linear mappings on finite dimensional vector spaces to derive our result.

2. Peliminary Results

Proposition 1 *[2] The graph $\mathfrak{J}_R$ is an empty graph if and only if $R \cong \mathbb{Z}_2$ or $\mathbb{Z}_3$.*

Theorem 2 *[2] Let R be a finite ring. The graph $\mathfrak{J}_R$ is a complete graph if and only if R is a local ring with associated field of order 2.*

Theorem 3 *[2] Let R be a finite ring. Then the graph $\mathfrak{J}_R$ is bipartite graph if and only if either R is a field or it is isomorphic to one of the rings $\mathbb{Z}_4$, $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ or $\mathbb{Z}_2[x]/(x^2)$.*

3. Main Result

In this section, we identify the diameter by divided into two cases. There are square and non square matrix Jacobson graph.

3.1. Square Matrix Jacobson Graph of Fields

In general, the square matrix Jacobson graph $\mathfrak{J}_F^{n\times n}$ is described in the following two theorems. Both theorems are obtained by employing the fact that zero is the only non-unit element in F and so any two distinct elements A, B in $V(\mathfrak{J}_F^{n\times n})$ are adjacent if and only if $\det(AB) = 1$. Then, using the property $\det(AB) = \det(A)\det(B)$, it is happened if and if $\det(A)$ and $\det(B)$ are mutually inversed.

We classify the vertex set of $\mathfrak{J}_F^{n\times n}$ be

$$V(\mathfrak{J}_F^{n\times n}) = V_0 \cup V_{\mathfrak{u}(F)} \cup V_{\mathfrak{u}'(F)}$$

with $V_0 = \{A \in V(\mathfrak{J}_F^{n\times n}) | det(A) = 0\}$, $V_{\mathfrak{u}(F)} = \{A \in V(\mathfrak{J}_F^{n\times n}) | det(A) \in \mathfrak{u}(F)\}$, $V_{\mathfrak{u}'(F)} = \{A \in V(\mathfrak{J}_F^{n\times n}) | det(A) \in \mathfrak{u}'(F)\}$.

Theorem 4 *Let F be a field. The matrix Jacobson graph $\mathfrak{J}_F^{n\times n}$ consists of*

(i) $|V_0|$ be empty subgraph,

(ii) $|\mathfrak{u}(F)|$ components complete subgraphs on $V_{\mathfrak{u}(F)}$, and

(iii) $|\mathfrak{u}'(F)|/2$ components complete bipartite subgraphs on $V_{\mathfrak{u}'(F)}$. $\square$

The example of this theorem is $\mathfrak{J}_{\mathbb{Z}_2}^{2\times 2}$ which consists of 15 elements and is as shown in Figure 1. Let p be prime number, $\mathbb{Z}_p$ be the field. Note that we have the following detail of the above theorem. Let G_0 be the empty graph with the vertex set is V_0. Let V_i as defined in previous Theorem, for $i = 0, 1, \ldots, p-1$.

- If $p = 2$ then $\mathfrak{J}_{\mathbb{Z}_3}^{n\times n} = G_0 \cup K_{|V_1|}$.

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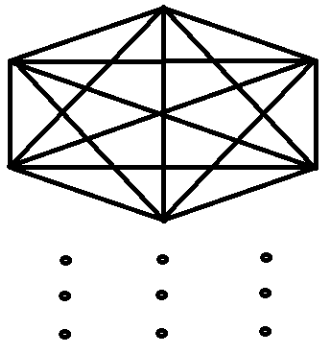

Figure 1. Graph $\mathfrak{J}_{\mathbb{Z}_2}^{2\times 2}$

- If $p = 3$ then $\mathfrak{J}_{\mathbb{Z}_3}^{n\times n} = G_0 \cup K_{|V_1|} \cup K_{|V_2|}$.
- If $p = 5$ then $\mathfrak{J}_{\mathbb{Z}_5}^{n\times n} = G_0 \cup K_{|V_1|} \cup K_{|V_2|,|V_3|} \cup K_{|V_4|}$.
- If $p = 7$ then $\mathfrak{J}_{\mathbb{Z}_7}^{n\times n} = G_0 \cup K_{|V_1|} \cup K_{|V_2|,|V_4|} \cup K_{|V_3|,|V_5|} \cup K_{|V_6|}$.

The greater the prime integer p is, the more bipartite subgraphs are obtained.

We denote the connected subgraph of the square matrix Jacobson graph for field F by $\left(\mathfrak{J}_F^{n\times n}\right)^*$.

Corollary 5 *Diam* $\left(\mathfrak{J}_F^{n\times n}\right)^* = 1$ *or* 2. □

3.2. Non-square Matrix Jacobson Graph of fields

In this section we identify the matrix Jacobson graph $\mathfrak{J}_F^{m\times n}$ of F, for $m \neq n$. Since F is a field, hence any $m \times n$ matrix with entries in F can be consider as a linear transformation. In this study we will be utilizing some known properties and facts concerning linear transformation to derive our results.

Theorem 6 *Let $\mathfrak{J}_F^{m\times n}$ be the matrix Jacobson graph for F. If $m < n$ then $\mathfrak{J}_F^{m\times n}$ is an empty graph.*

Proof. Let $A, B \in V(\mathfrak{J}_F^{m\times n})$, $A \neq B$. Consider A, B as linear transformations,

$$\begin{array}{cccc} B^t: & F^n & \longrightarrow & F^m \\ & \mathbf{x} & \mapsto & \mathbf{x}B^t \end{array}$$

$$\begin{array}{cccc} B^tA: & F^n & \longrightarrow & F^n \\ & \mathbf{y} & \mapsto & \mathbf{y}B^tA \end{array}$$

We obtain

$$\text{Rank}(B^tA) \leq min\{\text{Rank}(B^t), \text{Rank}(A)\} \leq m < n.$$

Since B^tA is of size $n \times n$, we have B^tA is singular and hence $1 - det(B^tA) = 1$. Thus A does not adjacent to B. So, $\mathfrak{J}_F^{m\times n}$ with $m < n$ is an empty graph. □

Theorem 7 *If $m > n$ then $\mathfrak{J}_F^{m\times n}$ consists of an empty subgraph and a connected subgraph with $diam(\mathfrak{J}_F^{m\times n}) \leq 4$.*

Proof. Let

$$V_0 = \{A \in V(\mathfrak{J}_F^{m\times n}) \mid \mathrm{Rank}(A) < n\}$$
$$V_1 = \{A \in V(\mathfrak{J}_F^{m\times n}) \mid \mathrm{Rank}(A) = n\}.$$

Then $V(\mathfrak{J}_F^{m\times n}) = V_0 \cup V_1$. By using similar argument to be used to prove Theorem 6, we obtain the set V_0 forms an empty subgraph.

Now, we shall prove that the subgraph formed by V_1 is connected. For any $A \in V_1$, we may write

$$A = \sum_{i=1}^{m} \dot{e}_i^t \mathbf{a}_i \quad \text{for some} \quad \mathbf{a}_i \in F^n, \quad i = 1, \ldots, m$$

and denote a set of n-array row indeces of A that forms a basis of F^n

$$I(A) = \left\{ (k_1, k_2, \ldots, k_n) \mid 1 \le k_i < k_{i+1} \le m,\ \det\left(\sum_{i=1}^{n} e_i \mathbf{a}_{k_i}\right) \neq 0 \right\}.$$

Then for any $A \in V_1$, the set $I(A)$ is not empty since $\mathrm{Rank}(A) = n$.

Take any two distinct vertices $A, B \in V_1$ and suppose

$$A = \sum_{i=1}^{m} \dot{e}_i^t \mathbf{a}_i, \qquad B = \sum_{i=1}^{m} \dot{e}_i^t \mathbf{b}_i$$

for some $\mathbf{a}_i, \mathbf{b}_i \in \mathbb{Z}_p^n$ where $i = 1, \ldots, m$. In the following we show that A and B are connected which is divided into two cases, the case $I(A) \cap I(B) \neq \emptyset$ and the case $I(A) \cap I(B) = \emptyset$.

1. **For the case** $I(A) \cap I(B) \neq \emptyset$, let $(k_1, \ldots, k_n) \in I(A) \cap I(B)$. Then $\alpha = \det(\sum_{i=1}^n \mathbf{e}_i^t \mathbf{a}_{k_i}) \neq 0$, $\beta = \det(\sum_{i=1}^n \mathbf{e}_i^t \mathbf{b}_{k_i}) \neq 0$ which imply the square matrices $\sum_{i=1}^n \mathbf{e}_i^t \mathbf{a}_{k_i}$ and $\sum_{i=1}^n \mathbf{e}_i^t \mathbf{b}_{k_i}$ are invertible. Let us denote their inverse matrices are respectively $A' = (\sum_{i=1}^n \mathbf{e}_i^t \mathbf{a}'_{k_i})^t$ and $B' = (\sum_{i=1}^n \mathbf{e}_i^t \mathbf{b}'_{k_i})^t$. In this case we have

$$A' \sum_{i=1}^{n} \mathbf{e}_i^t \mathbf{a}_{k_i} = \sum_{i=1}^{n} \mathbf{a}'^t_{k_i} \mathbf{a}_{k_i} = I_n \quad \text{and} \quad B' \sum_{i=1}^{n} \mathbf{e}_i^t \mathbf{b}_{k_i} = \sum_{i=1}^{n} \mathbf{b}'^t_{k_i} \mathbf{b}_{k_i} = I_n.$$

Moreover A and $\sum_{i=1}^n \dot{e}_{k_i}^t \mathbf{a}'_{k_i}$ are adjacent; and so are $\sum_{i=1}^n \dot{e}_{k_i}^t \mathbf{b}'_{k_i}$ and B. To generate a path from A to B, we consider two cases.

- The case $A = \sum_{i=1}^n \dot{e}_{k_i}^t \mathbf{a}_{k_i}$ and $B = \sum_{i=1}^n \dot{e}_{k_i}^t \mathbf{b}_{k_i}$ which is equivalent to $\mathbf{a}_i = 0, \mathbf{b}_i = 0$ for all $i \notin \{k_1, \ldots, k_n\}$. Let $k_{n+1} \in \{1, \ldots, m\} \backslash \{k_1, \ldots, k_n\}$. Then we obtain a path form A to B as follows:

$$A \sim \sum_{i=1}^{n} \dot{e}_{k_i}^t \mathbf{a}'_{k_i} + \dot{e}_{k_{n+1}}^t (\mathbf{a}'_{k_1}/\alpha\beta) \sim \sum_{i=2}^{n} \dot{e}_{k_i}^t \mathbf{b}_{k_i} + \dot{e}_{k_{n+1}}^t \mathbf{b}_{k_1} \sim \sum_{i=1}^{n} \dot{e}_{k_i}^t \mathbf{b}'_{k_i} + \dot{e}_{k_{n+1}}^t \mathbf{b}'_{k_1} \sim B.$$

  On the other notions, we may write :

$$A = \begin{pmatrix} \mathbf{a}_{k_1} \\ \mathbf{a}_{k_2} \\ \vdots \\ \mathbf{a}_{k_n} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \sim \begin{pmatrix} \mathbf{a}'_{k_1} \\ \mathbf{a}'_{k_2} \\ \vdots \\ \mathbf{a}'_{k_n} \\ \frac{\mathbf{a}'_{k_1}}{\alpha\beta} \\ \vdots \\ 0 \end{pmatrix} \sim \begin{pmatrix} 0 \\ \mathbf{b}_{k_2} \\ \vdots \\ \mathbf{b}_{k_n} \\ \mathbf{b}_{k_1} \\ \vdots \\ 0 \end{pmatrix} \sim \begin{pmatrix} \mathbf{b}'_{k_1} \\ \mathbf{b}'_{k_2} \\ \vdots \\ \mathbf{b}'_{k_n} \\ \mathbf{b}'_{k_1} \\ \vdots \\ 0 \end{pmatrix} \sim \begin{pmatrix} \mathbf{b}_{k_1} \\ \mathbf{b}_{k_2} \\ \vdots \\ \mathbf{b}_{k_n} \\ 0 \\ \vdots \\ 0 \end{pmatrix} = B.$$

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- The case $A \neq \sum_{i=1}^{n} \dot{e}_{k_i}^t \mathbf{a}_{k_i}$ or $B \neq \sum_{i=1}^{n} \dot{e}_{k_i}^t \mathbf{b}_{k_i}$, without loss of generality, let $k_{n+1} \in \{1, \ldots, m\} \backslash \{k_1, \ldots, k_n\}$ such that

$$\mathbf{a}_{k_{n+1}} \neq 0 \quad \text{and} \quad \mathbf{a}_{k_{n+1}} = \sum_{i=1}^{n} \lambda_i \mathbf{a}_{k_i}, \quad \text{with} \quad \lambda_1 \neq 0.$$

Denote

$$C = \sum_{i=2}^{n+1} \dot{e}_{k_i}^t \mathbf{c}_{k_i}, \quad \text{where} \quad \mathbf{c}_{k_i} = \mathbf{a}'_{k_i} - \frac{\lambda_i}{\lambda_1} \mathbf{a}'_{k_1}, \quad \text{for} \quad i = 2, \ldots, n \quad \text{and} \quad \mathbf{c}_{k_{n+1}} = \frac{\mathbf{a}'_{k_1}}{\lambda_1}.$$

It is a routine to obtain $C^t A = \sum_{i=2}^{n+1} \mathbf{c}_{k_i}^t \mathbf{a}_{k_i} = I_n$. Then we have

$$A \sim C \sim \sum_{i=2}^{n+1} \dot{e}_{k_i}^t \mathbf{a}_{k_i} + (\dot{e}_{k_1}^T \mathbf{a}_{k_1} \beta / \alpha) \sim \sum_{i=1}^{n} \dot{e}_{k_i}^t \mathbf{b}_{k_i} \sim B.$$

On the other notions, we may write :

$$A = \begin{pmatrix} \mathbf{a}_{k_1} \\ \mathbf{a}_{k_2} \\ \vdots \\ \mathbf{a}_{k_n} \\ \mathbf{a}_{k_{n+1}} \\ \vdots \\ \mathbf{a}_{k_m} \end{pmatrix} \sim \begin{pmatrix} 0 \\ \mathbf{a}'_{k_2} - \frac{\lambda_2}{\lambda_1} \mathbf{a}'_{k_1} \\ \vdots \\ \mathbf{a}'_{k_n} - \frac{\lambda_n}{\lambda_1} \mathbf{a}'_{k_1} \\ \frac{\mathbf{a}'_{k_1}}{\lambda_1} \\ \vdots \\ 0 \end{pmatrix} \sim \begin{pmatrix} \frac{\mathbf{a}_{k_1} \beta}{\alpha} \\ \mathbf{a}_{k_2} \\ \vdots \\ \mathbf{a}_{k_n} \\ \mathbf{a}_{k_{n+1}} \\ \vdots \\ 0 \end{pmatrix} \sim \begin{pmatrix} \mathbf{b}'_{k_1} \\ \mathbf{b}'_{k_2} \\ \vdots \\ \mathbf{b}'_{k_n} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \sim \begin{pmatrix} \mathbf{b}_{k_1} \\ \mathbf{b}_{k_2} \\ \vdots \\ \mathbf{b}_{k_n} \\ \mathbf{b}_{k_{n+1}} \\ \vdots \\ \mathbf{b}_{k_m} \end{pmatrix} = B$$

for $\mathbf{a}_{k_{n+1}} = \sum_{i=1}^{n} \lambda_i \mathbf{a}_{k_i}$ with $\lambda_1 \neq 0$.

2. **For the case** $I(A) \cap I(B) = \emptyset$, we divide it into two subcases; the subcase there exists $(k_1, \ldots, k_n) \in I(A)$ and $(l_1, \ldots, l_n) \in I(B)$ such that $\{k_1, \ldots, k_n\} \cap \{l_1, \ldots, l_n\} \neq \emptyset$ and its complement, the subcase for any $(k_1, \ldots, k_n) \in I(A)$ and $(l_1, \ldots, l_n) \in I(B)$, $\{k_1, \ldots, k_n\} \cap \{l_1, \ldots, l_n\} = \emptyset$.

2.a The first case, let $(k_1, \ldots, k_n) \in I(A)$ and $(l_1, \ldots, l_n) \in I(B)$ such that $\{k_1, \ldots, k_n\} \cap \{l_1, \ldots, l_n\} \neq \emptyset$. Similar to the previous discussion, let $\alpha = \det(\sum_{i=1}^{n} \mathbf{e}_i^t \mathbf{a}_{k_i}) \neq 0, \beta = \det(\sum_{i=1}^{n} \mathbf{e}_i^t \mathbf{b}_{k_i}) \neq 0$, and $\mathbf{a}'_{k_1}, \ldots, \mathbf{a}'_{k_n}, \mathbf{b}'_{l_1}, \ldots, \mathbf{b}'_{l_n} \in \mathbb{Z}_p^n$ such that

$$(\sum_{i=1}^{n} \mathbf{e}_i^t \mathbf{a}'_{k_i})^t \sum_{i=1}^{n} \mathbf{e}_i^t \mathbf{a}_{k_i} = \sum_{i=1}^{n} \mathbf{a}'^t_{k_i} \mathbf{a}_{k_i} = I_n \quad \text{and} \quad (\sum_{i=1}^{n} \mathbf{e}_i^t \mathbf{b}'_{l_i})^t \sum_{i=1}^{n} \mathbf{e}_i^t \mathbf{b}_{l_i} = \sum_{i=1}^{n} \mathbf{b}'^t_{l_i} \mathbf{b}_{l_i} = I_n.$$

Denote $\{k_1, \ldots, k_n\} \cap \{l_1, \ldots, l_n\} = \{k_1, \ldots, k_q\}$ where $q < n$ and reindex

$$\{l_1, \ldots, l_n\} = \{k_1, \ldots, k_q, l_{q+1}, \ldots, l_n\}.$$

We obtain

$$A \sim \sum_{i=1}^{n} \dot{e}_{k_i}^t \mathbf{a}'_{k_i} \sim \sum_{i=1}^{n} \dot{e}_{k_i}^t \mathbf{a}_{k_i} + \sum_{i=q+2}^{n} \dot{e}_{l_i}^t \mathbf{a}_{k_i} + \dot{e}_{l_{q+1}}^t \mathbf{a}_{k_{q+1} \beta / \alpha} \sim \sum_{i=1}^{n} \dot{e}_{l_i}^t \mathbf{b}'_{l_i} \sim B.$$

On the other notions, we may write :

$$\begin{pmatrix} \mathbf{a}_{k_1} \\ \mathbf{a}_{k_2} \\ \vdots \\ \mathbf{a}_{k_q} \\ \mathbf{a}_{k_{q+1}} \\ \vdots \\ \mathbf{a}_{k_n} \\ \mathbf{X} \\ \mathbf{X} \\ \vdots \\ \mathbf{X} \\ \mathbf{X} \\ \vdots \\ \mathbf{X} \end{pmatrix} \sim \begin{pmatrix} \mathbf{a}'_{k_1} \\ \mathbf{a}'_{k_2} \\ \vdots \\ \mathbf{a}'_{k_q} \\ \mathbf{a}'_{k_{q+1}} \\ \vdots \\ \mathbf{a}'_{k_n} \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \sim \begin{pmatrix} \mathbf{a}_{k_1} \\ \mathbf{a}_{k_2} \\ \vdots \\ \mathbf{a}_{k_q} \\ \mathbf{a}_{k_{q+1}} \\ \vdots \\ \mathbf{a}_{k_n} \\ \frac{\mathbf{a}_{k_{q+1}}\beta}{\alpha} \\ \mathbf{a}_{k_{q+2}} \\ \vdots \\ \mathbf{a}_{k_n} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \sim \begin{pmatrix} \mathbf{b}'_{k_1} \\ \mathbf{b}'_{k_2} \\ \vdots \\ \mathbf{b}'_{k_q} \\ 0 \\ \vdots \\ 0 \\ \mathbf{b}'_{l_{q+1}} \\ \mathbf{b}'_{l_{q+2}} \\ \vdots \\ \mathbf{b}'_{l_n} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \sim \begin{pmatrix} \mathbf{b}_{k_1} \\ \mathbf{b}_{k_2} \\ \vdots \\ \mathbf{b}_{k_q} \\ \mathbf{X} \\ \vdots \\ \mathbf{X} \\ \mathbf{b}_{l_{q+1}} \\ \mathbf{b}_{l_{q+2}} \\ \vdots \\ \mathbf{b}_{l_n} \\ \mathbf{X} \\ \vdots \\ \mathbf{X} \end{pmatrix}.$$

2.b The second case, let for any $(k_1, \ldots, k_n) \in I(A)$ and $(l_1, \ldots, l_n) \in I(B)$, $\{k_1, \ldots, k_n\} \cap \{l_1, \ldots, l_n\} = \emptyset$. In this case, we have $2n \le m$. Let $(k_1, \ldots, k_n) \in I(A)$ and $(l_1, \ldots, l_n) \in I(B)$. Then $|\{k_1, \ldots, k_n, l_1, \ldots, l_n\}| = 2n$ and we obtain the following path from A to B.

$$A \sim \sum_{i=1}^{n} \dot{e}_{k_i}^t \mathbf{a}'_{k_i} \sim \sum_{i=1}^{n} \dot{e}_{k_i}^t \mathbf{a}_{k_i} + \sum_{i=1}^{n} \dot{e}_{l_i}^t \mathbf{b}_{l_i} \sim \sum_{i=1}^{n} \dot{e}_{l_i}^t \mathbf{b}'_{l_i} \sim B$$

$$A = \begin{pmatrix} \mathbf{a}_{k_1} \\ \mathbf{a}_{k_2} \\ \vdots \\ \mathbf{a}_{k_n} \\ \mathbf{X} \\ \vdots \\ \mathbf{X} \\ \mathbf{X} \\ \vdots \\ \mathbf{X} \end{pmatrix} \sim \begin{pmatrix} \mathbf{a}'_{k_1} \\ \mathbf{a}'_{k_2} \\ \vdots \\ \mathbf{a}'_{k_n} \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \sim \begin{pmatrix} \mathbf{a}_{k_1} \\ \mathbf{a}_{k_2} \\ \vdots \\ \mathbf{a}_{k_n} \\ \mathbf{b}_{l_1} \\ \vdots \\ \mathbf{b}_{l_n} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \sim \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \mathbf{b}'_{l_1} \\ \vdots \\ \mathbf{b}'_{l_n} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \sim \begin{pmatrix} \mathbf{X} \\ \mathbf{X} \\ \vdots \\ \mathbf{X} \\ \mathbf{b}_{l_1} \\ \vdots \\ \mathbf{b}_{l_n} \\ \mathbf{X} \\ \vdots \\ \mathbf{X} \end{pmatrix} = B,$$

where $(\sum_{i=1}^{n} \mathbf{e}_i \mathbf{a}'_{k_i})^t$ and $(\sum_{i=1}^{n} \mathbf{e}_i \mathbf{b}'_{l_i})^t$ are respectively the inverse of $\sum_{i=1}^{n} \mathbf{e}_i \mathbf{a}_{k_i}$ and $\sum_{i=1}^{n} \mathbf{e}_i \mathbf{b}_{l_i}$.

square

We denote the connected subgraph of a non-square matrix Jacobson graph for F by $\left(\mathfrak{J}_F^{m\times n}\right)^*$.

Theorem 8 *Let $\mathfrak{J}_{\mathbb{Z}_2}^{m\times 2}$ be a matrix Jacobson graph of $\mathbb{Z}_2$. Then $diam(\mathfrak{J}_{\mathbb{Z}_2}^{m\times 2})^* = 2$ for $m \ge 3$.*

Proof.

Let $\{\mathbf{e}_1, \mathbf{e}_2\}$ be the standard basis of $\mathbb{Z}_2^2$ and $\{\dot{e}_1, \dot{e}_2, ..., \dot{e}_m\}$ be the standard basis of $\mathbb{Z}_2^m$. Let $A, B \in V(\mathfrak{J}_{\mathbb{Z}_2}^{m\times 2})^*$ non-adjacent vertices. We write $A = \sum_{i=1}^{m} \dot{e}_i^t \mathbf{a}_i$ and $B = \sum_{i=1}^{m} \dot{e}_i^t \mathbf{b}_i$ for some $\mathbf{a}_i, \mathbf{b}_i \in \mathbb{Z}_2^2$. We divide into two cases:

(i) The first case there exists $1 \leq i < j \leq m$ such that both sets $\{\mathbf{a}_i, \mathbf{a}_j\}$ and $\{\mathbf{b}_i, \mathbf{b}_j\}$ are linearly independent. Hence

$$\det(\mathbf{e}_1^t \mathbf{a}_i + \mathbf{e}_2^t \mathbf{a}_j) = 1 \quad \text{and} \quad \det(\mathbf{e}_1^t \mathbf{b}_i + \mathbf{e}_2^t \mathbf{b}_j) = 1.$$

As a result we obtain a path

$$A \sim \dot{e}_i^t \mathbf{a}_i + \dot{e}_j^t \mathbf{a}_j \sim B.$$

(ii) The second case is that for any $1 \leq i < j \leq m$ with $\{\mathbf{a}_i, \mathbf{a}_j\}$ is linearly independent implies $\{\mathbf{b}_i, \mathbf{b}_j\}$ is linearly dependent. So, let $1 \leq i < j \leq m$ with $\{\mathbf{a}_i, \mathbf{a}_j\}$ is linearly independent. Then $\{\mathbf{b}_i, \mathbf{b}_j\}$ is dependent. we have three subcases:

a. the case $\mathbf{b}_i = \mathbf{b}_j = 0$. Let $1 \leq k, l \leq m$ such that $\{\mathbf{b}_k, \mathbf{b}_l\}$ is independent. In this case, i, j, k, l are all different integers and the set $\{\mathbf{a}_k, \mathbf{a}_l\}$ is dependent. Then we have three cases.

- If $\mathbf{a}_k = \mathbf{a}_l = 0$ then we obtain the following path

$$A \sim \dot{e}_i^t \mathbf{a}_i + \dot{e}_j^t \mathbf{a}_j + \dot{e}_k^t \mathbf{a}_i + \dot{e}_l^t \mathbf{a}_j \sim B.$$

- If $\mathbf{a}_k = \mathbf{a}_l \neq 0$ then $\{\mathbf{a}_i, \mathbf{a}_k\}$ or $\{\mathbf{a}_j, \mathbf{a}_k\}$ are independent and we have the following path

$$A \sim \dot{e}_i^t \mathbf{a}_j + \dot{e}_k^t \mathbf{a}_i + \dot{e}_l^t \mathbf{a}_j \sim B, \quad \text{or}$$
$$A \sim \dot{e}_j^t \mathbf{a}_j + \dot{e}_k^t \mathbf{a}_i + \dot{e}_l^t \mathbf{a}_j \sim B.$$

- Without loss of generality suppose $\mathbf{a}_k \neq 0$ and $\mathbf{a}_l = 0$, then we obtain the following path

$$A \sim \dot{e}_i^t \mathbf{b}_l + \dot{e}_k^t \mathbf{b}_k + \dot{e}_l^t \mathbf{b}_l \sim B, \quad \text{or}$$
$$A \sim \dot{e}_j^t \mathbf{a}_l + \dot{e}_k^t \mathbf{b}_k + \dot{e}_l^t \mathbf{b}_l \sim B.$$

b. If $\mathbf{b}_i = \mathbf{b}_j \neq 0$ then there is k such that $\{\mathbf{b}_i, \mathbf{b}_k\}$ is linear independent. In this case we have $\{\mathbf{a}_i, \mathbf{a}_k\}$ is dependent. Then we obtain

$$A \sim \dot{e}_i^t \mathbf{a}_i + \dot{e}_j^t \mathbf{a}_j + \dot{e}_k^t \mathbf{a}_j \sim B.$$

c. Without loss of generality suppose $\mathbf{b}_i = 0$ but $\mathbf{b}_j \neq 0$. Then there exists k so that $\{\mathbf{b}_j, \mathbf{b}_k\}$ is an independent set. In this case $\{\mathbf{a}_j, \mathbf{a}_k\}$ is dependent. Then we have

$$A \sim \dot{e}_i^t \mathbf{a}_i + \dot{e}_j^t \mathbf{a}_j + \dot{e}_k^t \mathbf{a}_j \sim B. \ \square$$

4. Conclusion

Let F be a field. The square matrix Jacobson graph consists of 3 kinds of subgraphs which are empty subgraph, complete subgraph and bipartite subgraph. The non square matrix Jacobson graph consist of empty subgraph and connected subgraph. For connected subgraph, it diameter ≤ 4.

4.1. Concluding Remarked:

In this article some properties concerning matrix Jacobson graph of a field F. For further research we will extend these result to matrix Jacobson graphs of any finite local commutative ring.

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