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Using tensor product dual frames for phase retrieval problems

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Abstract
In this paper we consider conditions under which the property of being phase retrievable is invariant among a frame and its dual frames. We also investigate perturbation of phase retrievable frames. Finally, we obtain an explicit expression of tensor product dual frames which leads to reconstruction of a signal from magnitudes of frame coefficients.

Keywords Phase retrievable frame · Dual frame · Tensor product frame

Mathematics Subject Classification 42C15 · 94A12 · 47A80

1 Introduction and preliminaries

The topic of reconstructing a signal without using phase is known as *phase retrieval* and has become a very active area of research [3,4,8]. It has a wide range of applications in Physics and engineering such as X-ray tomography [9], electron microscopy [10], coherence theory and a number of other areas. Phase retrieval using frames was first introduced in 2006 by Balan, Casazza and Edidin [4], they gave a unique mathematical approach for phase retrieval, which quickly turned into an industry. Phase retrievable frames has been developed very fast over the last years [2,8,13].

Phase retrievable frames consider recovering a signal from the magnitudes of frame coefficients. Often times in engineering problems, the phase of a signal is lost and only magnitudes of frame coefficients are known. Therefore, it is necessary to provide

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methods for signal reconstruction with this lack of information, i.e. without using phase. Let $f = \{f_i\}_{i\in I}$ be a frame for a Hilbert space $\mathcal{H}$. Any signal $x \in \mathcal{H}$ is uniquely defined by magnitudes of its frame coefficients up to a unimodular scalar, if and only if f is phase retrievable [2]. Phase retrieval aims to recover signal x from magnitudes of frame coefficients $\{|\langle x, f_i\rangle|\}_{i\in I}$. In this paper, we study the property of phase retrievability for dual frames of a phase retrievable frame. Sometimes it is easier to check phase retrievability of a dual frame than a frame. We investigate conditions under which a frame is phase retrievable if and only if its dual frames are phase retrievable. We also consider stability of phase retrievable frames under small perturbations of the frame in a Hilbert space $\mathcal{H}$. For $x, y \in \mathcal{H}$, $x \otimes y$ on $\mathcal{H}$ is defined by $(x \otimes y)(z) = \langle z, y\rangle x$. Then $x \otimes x = y \otimes y$ if and only if $x = \alpha y$, where $|\alpha| = 1$. Thus recovering x up to a unimodular constant is the same as recovering the rank one operator $x \otimes x$ from the given magnitudes of frame coefficients $\{|\langle x, f_i\rangle|\}_{i\in I}$ by using dual frames. An explicit expression for one of the choices of a dual is obtained in [3,5]. Using dual frames, one can recover a signal x from the given magnitudes of frame coefficients.

The main result of this paper gives an explicit expression of all dual frames of tensor product frames. This leads to reconstruction of a signal from magnitudes of frame coefficients.

The organization of the paper is as follows. Sect. 1 contains basic notation and definitions related to frame theory and tensor product frames. In Sect. 2, we give conditions under which the property of being phase retrievable is invariant among a frame and its dual frames. We also propose a way of constructing a phase retrievable dual frame from a frame. Sect. 3 investigates methods to obtain tensor product dual frames which yield recovering a signal after loss of phase.

Throughout this paper, $\mathcal{H}$ is a Hilbert space, I and J are countable index sets and $I_{\mathcal{H}}$ is the identity operator on $\mathcal{H}$. We denote the set of all bounded linear operators between Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ by $B(\mathcal{H}_1, \mathcal{H}_2)$ and for $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$, it is represented by $B(\mathcal{H})$. Before discussing phase retrievable results, we briefly recall the definitions and basic properties of frame theory and tensor product of frames.

A sequence $f = \{f_i\}_{i\in I}$ in a Hilbert space $\mathcal{H}$ is a *frame* if there are constants $0 < A \le B < \infty$ so that for all $x \in \mathcal{H}$,

$$A\|x\|^2 \le \sum_{i\in I} |\langle x, f_i\rangle|^2 \le B\|x\|^2, \tag{1.1}$$

where A and B are the lower and upper frame bounds, respectively. If we can choose $A = B = 1$ then the frame is called a Parseval frame. If $f = \{f_i\}_{i\in I}$ is only assumed to satisfy the right hand of (1.1), then it is called a Bessel sequence. For a Bessel sequence $f = \{f_i\}_{i\in I}$, the *synthesis* operator $T_f : \ell^2(I) \to \mathcal{H}$ is defined by

$$T_f(\{c_i\}_{i\in I}) = \sum_{i\in I} c_i f_i.$$

Its adjoint operator T_f^* is called *the analysis operator* and the operator $S_f : \mathcal{H} \to \mathcal{H}$, which is defined by $S_f x = T_f T_f^* x = \sum_{i\in I} \langle x, f_i\rangle f_i$, for all $x \in \mathcal{H}$, is called *the* *frame operator*. The frame operator is positive, self-adjoint and invertible [6, Lemma 5.1.6]. A dual for a Bessel sequence $f = \{f_i\}_{i\in I}$ in $\mathcal{H}$ is a Bessel sequence $g = \{g_i\}_{i\in I}$ such that $T_g T_f^* = I_{\mathcal{H}}$. Let f be a frame for $\mathcal{H}$ with frame operator S_f. Then $\{S_f^{-1} f_i\}_{i\in I}$ is called the canonical dual frame for f. A Riesz basis for $\mathcal{H}$ is a family of the form $\{Ue_i\}_{i\in I}$, where $\{e_i\}_{i\in I}$ is an orthonormal basis for $\mathcal{H}$ and $U : \mathcal{H} \to \mathcal{H}$ is a bounded bijective operator.

Now we briefly recall some basic facts about tensor products of Hilbert spaces which are required in the sequel. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces. The tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ of $\mathcal{H}_1$ and $\mathcal{H}_2$ is defined as the space of all antilinear maps $T : \mathcal{H}_2 \to \mathcal{H}_1$ such that $\sum_{j\in J} \|Tu_j\|^2 < \infty$, for some orthonormal basis $\{u_j\}_{j\in J}$ for $\mathcal{H}_2$. For $T \in \mathcal{H}_1 \otimes \mathcal{H}_2$, we set $|||T|||^2 = \sum_{j\in J} \|Tu_j\|^2$. In [11, Theorem 7.12] it is shown that $\mathcal{H}_1 \otimes \mathcal{H}_2$ is a Hilbert space with the norm $|||.|||$ and the associated inner product

$$\langle Q, T\rangle = \sum_{j\in J} \left\langle Qu_j, Tu_j\right\rangle$$

for $Q, T \in \mathcal{H}_1 \otimes \mathcal{H}_2$. Let $x \in \mathcal{H}_1$, $y \in \mathcal{H}_2$. We define their *tensor product* $x \otimes y$ by

$$(x \otimes y)(z) = \langle z, y\rangle x,$$

for $z \in \mathcal{H}_2$. If $Q \in \mathcal{B}(\mathcal{H}_1)$ and $T \in \mathcal{B}(\mathcal{H}_2)$ the tensor product $Q \otimes T \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ is defined to be the bounded linear operator on $\mathcal{H}_1 \otimes \mathcal{H}_2$ such that $(Q \otimes T)(A) = QAT^*$ for every $A \in \mathcal{H}_1 \otimes \mathcal{H}_2$. For further information on tensor products of Hilbert spaces see [11]. It is easy to see that for any $x_1, x_2 \in \mathcal{H}_1$ and $y_1, y_2 \in \mathcal{H}_2$,

$$\langle x_1 \otimes x_2, y_1 \otimes y_2\rangle = \langle x_1, y_1\rangle \langle y_2, x_2\rangle,$$

and $(x_1 \otimes y_1)^* = y_1 \otimes x_1$. Moreover, if T and Q are invertible operators, then $T \otimes Q$ is an invertible operator and $(T \otimes Q)^{-1} = T^{-1} \otimes Q^{-1}$.

Let $\mathcal{H}$ be a Hilbert space, $f = \{f_i\}_{i\in I}$ and $g = \{g_j\}_{j\in J}$ be frames for $\mathcal{H}$. We write

$$f \otimes g = \left\{f_i \otimes g_j\right\}_{(i,j)\in I\times J}.$$

The synthesis (analysis) operator associated with the tensor product of two Bessel sequences is exactly the tensor product of their respective synthesis (analysis) operators [14, Lemma 2.1]. Furthermore, $S_{f\otimes g} = T_{f\otimes g} T_{f\otimes g}^*$ and $S_{f\otimes g}^{-1} = S_f^{-1} \otimes S_g^{-1}$.

In particular, the canonical dual frame of $f \otimes g$ is the tensor product of the canonical duals of f and g respectively [14, Corollary 2.1].

2 Phase retrievable frames

In [2–4], the authors highlighted some properties in phase retrievable frames. While in [2] it is shown that the canonical dual frame of a phase retrievable frame is also phase retrievable, but the property of phase retrievability has not been investigated for alternate duals. In this section, we consider the property of phase retrievability for alternate dual frames. we give several conditions under which a frame is phase retrievable if and only if its dual frames are phase retrievable. Moreover, we consider perturbations of phase retrievable frames.

Definition 2.1 Let $\mathcal{H}$ be a Hilbert space. A frame $\{f_i\}_{i\in I}$ for $\mathcal{H}$ is called a *phase retrievable frame* if for all $x, y \in \mathcal{H}$, satisfying $|\langle x, f_i\rangle| = |\langle y, f_i\rangle|$ for every i, then $x = \alpha y$, for some scaler α with $|\alpha| = 1$.

Let $f = \{f_i\}_{i\in I}$ be a frame for $\mathcal{H}$. In [2, Lemma 3.4] it is shown that if f is phase retrievable, then $\{Uf_i\}_{i\in I}$ is phase retrievable for any invertible operator U on $\mathcal{H}$. In the following proposition we prove $\{Uf_i\}_{i\in I}$ is phase retrievable, for an invertible operator U if and only if $\{Uf_i\}_{i\in I}$ is phase retrievable, for every invertible operator U.

Proposition 2.2 *Let $f = \{f_i\}_{i\in I}$ be a frame for $\mathcal{H}$. Then the following are equivalent.*

(1) The frame f is phase retrievable.
(2) There exists an invertible operator $U \in B(\mathcal{H})$, such that $\{Uf_i\}_{i\in I}$ is phase retrievable.
(3) For every invertible operator $U \in B(\mathcal{H})$, $\{Uf_i\}_{i\in I}$ is phase retrievable.

Proof Put $U = S_f^{-1}$, in which S_f is the frame operator of f. Then (1)⇒(2) is obvious. The only implication which needs proof is (2)⇒(1). For this, let $|\langle x, f_i\rangle| = |\langle y, f_i\rangle|$, for $x, y \in \mathcal{H}$, and every i. Then

$$\begin{aligned}\left|\left\langle \left(U^{-1}\right)^* y, Uf_i\right\rangle\right| &= \left|\left\langle \left(U^{-1}U\right)^* y, f_i\right\rangle\right| \\ &= |\langle y, f_i\rangle| \\ &= |\langle x, f_i\rangle| \\ &= \left|\left\langle \left(U^{-1}\right)^* x, Uf_i\right\rangle\right|.\end{aligned}$$

Since $\{Uf_i\}_{i\in I}$ is phase retrievable, $\left(U^{-1}\right)^* x = \alpha \left(U^{-1}\right)^* y$, where $|\alpha| = 1$. Therefore, $x = \alpha y$ with $|\alpha| = 1$. □

As a consequence of Proposition 2.2 we have the following corollary .

Corollary 2.3 *Let $f = \{f_i\}_{i\in I}$ be a frame for $\mathcal{H}$ with the frame operator S_f. Then*

(1) $\{S_f f_i\}_{i\in I}$ is phase retrievable if and only if f is phase retrievable.
(2) $\{S_f^{-1} f_i\}_{i\in I}$ is phase retrievable if and only if f is phase retrievable.
(3) If f is phase retrievable then $\{U^* f_i\}_{i\in I}$ is phase retrievable, for every left invertible operator $U \in B(\mathcal{H})$.

The following corollary considers conditions under which the property of being phase retrievable is invariant among a frame and its dual frames.

Corollary 2.4 *Let $f = \{f_i\}_{i\in I}$ be a frame for $\mathcal{H}$ and $\widetilde{f} = \{\widetilde{f}_i\}_{i\in I}$ be a dual frame of f. Under any of the following assumptions f is phase retrievable if and only if $\widetilde{f}$ is phase retrievable.*

(1) f is a Riesz basis.
(2) For $i, j \in I$,

$$\langle \widetilde{f}_j, f_i \rangle = \langle f_j, \widetilde{f}_i \rangle . \tag{2.1}$$

(3) There is no Bessel sequence k in $\mathcal{H}$ such that $T_f T_k^* = 0$.

Proof Assume (1). Since the dual of a Riesz basis is its canonical dual frame [6, Theorem 5.2.1], by Corollary 2.3, f is phase retrievable if and only if the canonical dual $\widetilde{f}$ of f is phase retrievable.

Assume (2). By [12, Proposition 3.3.5], $\widetilde{f}$ is the canonical dual frame of f if and only if (2.1) holds. This gives the result when combined with Corollary 2.3. Assume (3). If $\widetilde{f}$ is not the canonical dual frame of f, then there exists Bessel sequence $k = \{\widetilde{f}_i - S_f^{-1} f_i\}_{i\in I}$ such that $T_f T_k^* = 0$. This contradicts our assumption, so $\widetilde{f}$ is the canonical dual frame of f. Using Corollary 2.3, the result follows immediately. □

Stability and perturbation of phase retrievable frames have been investigated in the case of $\mathcal{H} = \mathbb{C}^n$ in [2].

In the sequel, we investigate perturbation of phase retrievable frames for an arbitrary Hilbert space $\mathcal{H}$. Assume that f is a phase retrievable frame and $\acute{f}$ is a frame in some sense close to f. In the following proposition we show that $\acute{f}$ is also phase retrievable.

Proposition 2.5 *Let $f = \{f_i\}_{i\in I}$ be a phase retrievable frame for $\mathcal{H}$ and $\acute{f} = \{\acute{f}_i\}_{i\in I}$ be a frame for $\mathcal{H}$ such that for every $x \in \mathcal{H}$,*

$$\left|\left\langle x, \acute{f}_i - f_i \right\rangle\right| < \epsilon, \quad i \in I.$$

Then $\acute{f}$ is phase retrievable.

Proof For every $x \in \mathcal{H}$, we have

$$\begin{aligned}\left|\left\langle x, \acute{f}_i \right\rangle\right| &= \left|\left\langle x, \acute{f}_i - f_i + f_i \right\rangle\right| \\ &\leq \left|\left\langle x, \acute{f}_i - f_i \right\rangle\right| + |\langle x, f_i \rangle| \\ &\leq \epsilon + |\langle x, f_i \rangle| .\end{aligned}$$

By a similar argument,

$$|\langle x, f_i \rangle| \leq \epsilon + \left|\left\langle x, \acute{f}_i \right\rangle\right| .$$

Now assume that, for every $x, y \in \mathcal{H}$, $\left|\left\langle x, \acute{f}_i \right\rangle\right| = \left|\left\langle y, \acute{f}_i \right\rangle\right|$. Then

$$|\langle x, f_i \rangle| = \left|\left\langle x, \acute{f}_i \right\rangle\right| = \left|\left\langle y, \acute{f}_i \right\rangle\right| = |\langle y, f_i \rangle| .$$

Since f is phase retrievable, it follows that $\acute{f}$ is also phase retrievable. □

Let $f=\{f_i\}_{i\in I}$ be a frame for $\mathcal{H}$ and $\{\delta_i\}_{i\in I}$ be the canonical basis for $\ell^2(I)$. By [1, Theorem 2.1], dual frames are precisely the families of the form $\left\{S_f^{-1}f_i+W^*\delta_i\right\}_{i\in I}$ where $W\in B\left(\mathcal{H},\ell^2\right)$ such that $T_fW=0$. In the following proposition we establish conditions under which the property of being phase retrievable is invariant among a frame and its dual frames.

Proposition 2.6 *Let $f=\{f_i\}_{i\in I}$ be a frame for $\mathcal{H}$. Suppose that $\widetilde{f}=\left\{S_f^{-1}f_i+W^*\delta_i\right\}_{i\in I}$ is a dual frame of f where $W\in B\left(\mathcal{H},\ell^2\right)$ such that $T_fW=0$. We denote the i-th component of Wx by $(Wx)_i$ for every $x\in\mathcal{H}$.*

(1) If $\left|(Wx)_i\right|<\epsilon$, for every $x\in\mathcal{H}$ and $i\in I$, then f is phase retrievable if and only if $\widetilde{f}$ is phase retrievable.
(2) If $(Wf_i)_j=\left(Wf_j\right)_i$, for every $x\in\mathcal{H}$ and $i,j\in I$, then f is phase retrievable if and only if $\widetilde{f}$ is phase retrievable.

Proof Let f be phase retrievable. By Corollary 2.3, $\{S_f^{-1}f_i\}_{i\in I}$ is also phase retrievable. For every $x\in\mathcal{H}$ we have

$$\left|\left\langle x,\widetilde{f_i}-S_f^{-1}f_i\right\rangle\right|=\left|\langle x,W^*\delta_i\rangle\right|=\left|(Wx)_i\right|<\epsilon.$$

By Proposition 2.5, $\widetilde{f}$ is phase retrievable. Analysis similar to that in the above shows that f is phase retrievable if $\widetilde{f}$ is phase retrievable. For (2) by [12, Proposition 3.3.5], $\widetilde{f}$ is the canonical dual frame of f if and only if

$$\langle\widetilde{f_j},f_i\rangle=\langle f_j,\widetilde{f_i}\rangle,\quad i,j\in I.$$

Now observe that

$$\begin{aligned}\langle\widetilde{f_j},f_i\rangle&=\left\langle S_f^{-1}f_j+W^*\delta_j,f_i\right\rangle\\&=\left\langle S_f^{-1}f_j,f_i\right\rangle+(Wf_i)_j\\&=\left\langle f_j,S_f^{-1}f_i\right\rangle+\left(Wf_j\right)_i\\&=\langle f_j,\widetilde{f_i}\rangle.\end{aligned}$$

Therefor $\widetilde{f}$ is the canonical dual frame of f. Corollary 2.3 gives the desired conclusion. □

The next proposition, gives a perturbation of phase retrievable dual frames.

Proposition 2.7 *Assume that $f=\{f_i\}_{i\in I}$ is a frame for $\mathcal{H}$ and $\widetilde{f}=\{\widetilde{f_i}\}_{i\in I}$ is a dual frame of f. Let $\theta=\{\theta_i\}_{i\in I}$ be a phase retrievable frame such that*

$$\|T_{\widetilde{f}}-T_\theta\|\le\|T_f\|^{-1}.$$

Then $T_\theta T_f^*$ *is invertible and* $\left(T_\theta T_f^*\right)^{-1}\theta$ *is a phase retrievable dual frame of* f.

Proof Since $\widetilde{f}$ is a dual frame of f, so $T_{\widetilde{f}}T_f^* = I_{\mathcal{H}}$ and

$$\begin{aligned}\|I_{\mathcal{H}} - T_\theta T_f^*\| &= \|T_{\widetilde{f}}T_f^* - T_\theta T_f^*\| \\ &\leq \|T_{\widetilde{f}} - T_\theta\|\|T_f^*\| < 1.\end{aligned}$$

Hence, $T_\theta T_f^*$ is invertible and $\left(T_\theta T_f^*\right)^{-1}\theta$ is a dual frame of f [7]. Since θ is a phase retrievable frame and $\left(T_\theta T_f^*\right)^{-1}$ is invertible, by Proposition 2.2, $\left(T_\theta T_f^*\right)^{-1}\theta$ is phase retrievable. □

3 Duality of tensor product frames

Let $f = \{f_i\}_{i\in I}$ be a frame for $\mathcal{H}$ and $x \in \mathcal{H}$. Clearly the magnitudes of frame coefficients $|\langle x, f_i\rangle|$ for $i \in I$ are the same for both x and αx for every unimodular scalar α. The phase retrievable problem asks to recover a signal x up to a unimodular scalar from its magnitudes of frame coefficients. While some papers [3,4,10] present conditions for reconstruction of a signal, the general problem of finding efficient algorithms is still open. Notice that if $\{R_{i,i}\}_{(i,i)\in I\times I}$ is a dual frame of $f \otimes f$, then for every $x \in \mathcal{H}$ we have

$$x \otimes x = \sum_{(i,i)} \langle x \otimes x, f_i \otimes f_i\rangle\, R_{i,i} = \sum_{(i,i)} |\,\langle x, f_i\rangle\,|^2 R_{i,i}. \tag{3.1}$$

So a signal x can be recovered up to a unimodular scalar, by factorizing dual frames of $f \otimes f$.

Using Grammian matrix, an explicit formula for one of the choices of a dual $\{R_{i,i}\}_{(i,i)\in I\times I}$ is obtained in [3,5]. In this section we discuss another algorithms that characterize alternate dual frames of $f \otimes f$ which lead to reconstruction of a signal up to a unimodular scalar, by (3.1). A trivial dual frame of $f \otimes f$ that makes (3.1) allowable, is the canonical dual frame $\{S_f^{-1} f_i \otimes S_f^{-1} f_i\}_{(i,i)\in I\times I}$.

In the sequel we mention methods to obtain alternate dual frames of $f \otimes f$ such that (3.1) is satisfied.

Proposition 3.1 *Let* $f = \{f_i\}_{i\in I}$ *be a frame for* $\mathcal{H}$ *and* $K = \{k_{i,i}\}_{(i,i)\in I\times I}$ *be a Bessel sequence in* $\mathcal{H} \otimes \mathcal{H}$. *Then* $R = \{R_{i,i}\}_{(i,i)\in I\times I}$ *is a dual frame of* $f \otimes f = \{f_i \otimes f_i\}_{(i,i)\in I\times I}$ *if and only if* $R_{i,i} = S_{f\otimes f}^{-1}(f_i \otimes f_i) + k_{i,i}$, *where* $T_K T_{f\otimes f}^* = 0$.

Proof It is easy to see that $T_R T_{f\otimes f}^* = I_{\mathcal{H}\otimes\mathcal{H}}$. This completes the proof. □

The following proposition which is [14, Theorem 2.1] derives a necessary and sufficient condition for two tensor product Bessel sequences to be a pair of dual frames.

Proposition 3.2 *Assume that $f = \{f_i\}_{i\in I}$ and $\widetilde{f} = \{\widetilde{f}_i\}_{i\in I}$ are Bessel sequences in Hilbert space $\mathcal{H}_1$, $g = \{g_i\}_{i\in I}$ and $\widetilde{g} = \{\widetilde{g}_i\}_{i\in I}$ are Bessel sequences in Hilbert space $\mathcal{H}_2$. Then $f \otimes g$ and $\widetilde{f} \otimes \widetilde{g}$ form a pair of dual frames in $\mathcal{H}_1 \otimes \mathcal{H}_2$ if and only if there exist constants a, b with $ab = 1$ such that $T_f T^*_{\widetilde{f}} = aI_{\mathcal{H}_1}$ and $T_g T^*_{\widetilde{g}} = bI_{\mathcal{H}_2}$.*

Let $f = \{f_i\}_{i\in I}$ be a frame for $\mathcal{H}$. In the following theorem, using an idea of the preceding proposition, we characterize dual frames of $f\otimes f$ that makes (3.1) allowable.

Theorem 3.3 *Assume that $f = \{f_i\}_{i\in I}$ and $g = \{g_i\}_{i\in I}$ are frames for $\mathcal{H}$ and $\{\delta_i\}_{i\in I}$ is the canonical orthonormal basis for $\ell^2(I)$.*

(1) The dual frames of $f \otimes g$ are precisely $\widetilde{f} \otimes \widetilde{g} = \{V\delta_i \otimes U\delta_i\}_{(i,i)\in I\times I}$, where $U, V \in B\left(\ell^2, \mathcal{H}\right)$ and there exist constants a and b with $ab = 1$ such that $VT^*_f = aI_{\mathcal{H}}$ and $UT^*_g = bI_{\mathcal{H}}$.

(2) The frame $\widetilde{f} \otimes \widetilde{g}$ is a dual frame of $f \otimes g$ if and only if

$$\widetilde{f} \otimes \widetilde{g} = \left\{S_f^{-1} f_i \otimes S_g^{-1} g_i + W_1^*\delta_i \otimes W_2^*\delta_i\right\}_{(i,i)\in I\times I} \quad (3.2)$$

for some operators $W_1, W_2 \in B\left(\mathcal{H}, \ell^2\right)$ such that $T_f W_1 \otimes T_g W_2 = 0$.

Proof (1) By [6, Lemma 5.7.2], the dual frames of $f \otimes g$ are precisely the families of the form $\{(V \otimes U)(\delta_i \otimes \delta_i)\}_{(i,i)\in I\times I}$, where $U, V \in B\left(\ell^2, \mathcal{H}\right)$ and the tensor product $V \otimes U$ is the left inverse of $T^*_{f\otimes g}$. It is obvious that $\{(V \otimes U)(\delta_i \otimes \delta_i)\}_{(i,i)\in I\times I} = \{V\delta_i \otimes U\delta_i\}_{(i,i)\in I\times I}$. On the other hand by [14, Proposition 1.1] $(V \otimes U)\, T^*_{f\otimes g} = I_{\mathcal{H}\otimes\mathcal{H}}$ if and only if there exist constants a and b with $ab = 1$ such that $VT^*_f = aI_{\mathcal{H}}$ and $UT^*_g = bI_{\mathcal{H}}$. This completes the proof.

(2) By [1, Theorem 2.1], the dual frames of $f \otimes g$ are precisely the families of the form

$$\widetilde{f} \otimes \widetilde{g} = \left\{S^{-1}_{f\otimes g}(f_i \otimes g_i) + (W_1 \otimes W_2)^*(\delta_i \otimes \delta_i)\right\}_{(i,i)\in I\times I}$$

for some operators $W_1, W_2 \in B\left(\mathcal{H}, \ell^2\right)$ such that $T_{f\otimes g}(W_1 \otimes W_2) = 0$. As $S^{-1}_{f\otimes g}(f_i \otimes g_i) = S_f^{-1} f_i \otimes S_g^{-1} g_i$ and

$$\begin{aligned} T_{f\otimes g}(W_1 \otimes W_2) &= \left(T_f \otimes T_g\right)(W_1 \otimes W_2) \\ &= T_f W_1 \otimes T_g W_2, \end{aligned}$$

we have (3.2). □

As tensor product of dual frames $f = \{f_i\}_{i\in I}$ and $g = \{g_i\}_{i\in I}$ is a dual frame of $f \otimes g$ [14], we immediately have the following proposition.

Proposition 3.4 *Assume that $f = \{f_i\}_{i\in I}$ and $g = \{g_i\}_{i\in I}$ are frames for $\mathcal{H}$ and $\{\delta_i\}_{i\in I}$ is the canonical orthonormal basis for $\ell^2(I)$.*

(1) The frame $\left\{S_f^{-1} f_i \otimes V\delta_i\right\}_{(i,i)\in I\times I}$ *is a dual frame of* $f \otimes g$*, where* $V \in B\left(\ell^2, \mathcal{H}\right)$ *is a left inverse of* T_g^**.*

(2) The frame $\left\{V\delta_i \otimes S_g^{-1} g_i\right\}_{(i,i)\in I\times I}$ *is a dual frame of* $f \otimes g$*, where* $V \in B\left(\ell^2, \mathcal{H}\right)$ *is a left inverse of* T_f^**.*

(3) The frame $\left\{V\delta_k \otimes \left(S_g^{-1} g_i + W^*\delta_i\right)\right\}_{(i,i)\in I\times I}$ *is a dual frame of* $f \otimes g$*, in which* $V \in B\left(\ell^2, \mathcal{H}\right)$ *is a left inverse of* T_f^* *and* $W \in B\left(\mathcal{H}, \ell^2\right)$ *such that* $T_g W = 0$*.*

(4) The frame $\left\{S_f^{-1} f_i \otimes \left(S_g^{-1} g_i + W^*\delta_i\right)\right\}_{(i,i)\in I\times I}$ *is a dual frame of* $f \otimes g$*, in which* $W \in B\left(\mathcal{H}, \ell^2\right)$ *such that* $T_g W = 0$*.*

In the next proposition, we give dual frames for tensor product of two frames. Its proof is straightforward and so is omitted.

Proposition 3.5 *Assume that* $f = \{f_i\}_{i\in I}$ *and* $g = \{g_i\}_{i\in I}$ *are frames for* $\mathcal{H}$ *and* $\{\delta_i\}_{i\in I}$ *is the canonical orthonormal basis for* $\ell^2(I)$*.*

(1) If $\|I_{\mathcal{H}} - T_g T_f^*\| < 1$ *and* $S_{g,f} := \left(T_g T_f^*\right)^{-1}$ *then* $\{S_{g,f} g_i \otimes S_{g,f} g_i\}_{i\in I}$ *is a dual frame of* $f \otimes f$*.*

(2) If $\|I_{\mathcal{H}} - T_g T_f^*\| < 1$ *and* $S_{g,f} := \left(T_g T_f^*\right)^{-1}$ *then* $\{S_{g,f} g_i \otimes S_g^{-1} g_i\}_{i\in I}$ *is a dual frame of* $f \otimes g$*.*

The next proposition, gives a perturbation of tensor product dual frames.

Proposition 3.6 *Assume that* $f = \{f_i\}_{i\in I}$ *is a frame for* $\mathcal{H}$ *such that* $\widetilde{f} \otimes \widetilde{f} = \{\widetilde{f}_i \otimes \widetilde{f}_i\}_{i\in I}$ *is a dual frame of* $f \otimes f$*. Let* $\Theta = \{\Theta_i \otimes \Theta_i\}_{i\in I}$ *be a Bessel sequence such that*

$$\|T_\Theta - T_{\widetilde{f}\otimes\widetilde{f}}\| < \|T_{f\otimes f}\|^{-1}.$$

Then $T_\Theta T_{f\otimes f}^*$ *is invertible and* $\left(T_\Theta T_{f\otimes f}^*\right)^{-1} \Theta$ *is a dual frame of* $f \otimes f$*.*

Proof It is easy to see that $\left\|I_{\mathcal{H}\otimes\mathcal{H}} - T_\Theta T_{f\otimes f}^*\right\| < 1$. This completes the proof. □

In the following proposition we give necessary and sufficient conditions under which a dual of a tensor product is canonical.

Proposition 3.7 *Let* $f = \{f_i\}_{i\in I}$ *and* $\widetilde{f} = \{\widetilde{f}_i\}_{i\in I}$ *be two frames for* $\mathcal{H}$ *such that* $\widetilde{f} \otimes \widetilde{f} = \left\{\widetilde{f}_i \otimes \widetilde{f}_i\right\}_{(i,i)\in I\times I}$ *is a dual frame of* $f \otimes f$*. Then* $\widetilde{f} \otimes \widetilde{f}$ *is the canonical dual frame of* $f \otimes f$ *if and only if*

$$\left|\langle f_i, \widetilde{f}_j\rangle\right| = \left|\langle \widetilde{f}_i, f_j\rangle\right|, \quad for\ i, j \in I. \tag{3.3}$$

Proof If $\widetilde{f} \otimes \widetilde{f}$ is the canonical dual, obviously (3.3) holds. Conversely, if (3.3) holds, then

$$\begin{aligned}\left\langle f_i \otimes f_i, \widetilde{f}_j \otimes \widetilde{f}_j \right\rangle &= \left|\left\langle f_i, \widetilde{f}_j \right\rangle\right|^2 \\ &= \left|\left\langle \widetilde{f}_i, f_j \right\rangle\right|^2 \\ &= \left\langle \widetilde{f}_i \otimes \widetilde{f}_i, f_j \otimes f_j \right\rangle.\end{aligned}$$

For $x \in \mathcal{H}$ we have

$$\begin{aligned}\langle x \otimes x, f_k \otimes f_k \rangle &= \sum_{(j,j)} \left\langle x \otimes x, f_j \otimes f_j \right\rangle \left\langle \widetilde{f}_j \otimes \widetilde{f}_j, f_k \otimes f_k \right\rangle \\ &= \sum_{(j,j)} \left\langle x \otimes x, f_j \otimes f_j \right\rangle \left\langle f_j \otimes f_j, \widetilde{f}_k \otimes \widetilde{f}_k \right\rangle \\ &= \left\langle S_{f \otimes f} x \otimes x, \widetilde{f}_k \otimes \widetilde{f}_k \right\rangle.\end{aligned}$$

Hence $\widetilde{f}_k \otimes \widetilde{f}_k = S^{-1}_{f \otimes f} \left(f_k \otimes f_k\right)$. □

One can easily generalize Proposition 3.7, for the tensor product dual of two different frames.

Proposition 3.8 *Let $f = \{f_i\}_{i \in I}$ be a frame for a Hilbert space $\mathcal{H}_1$ and $\widetilde{g} = \{\widetilde{g}_i\}_{i \in I}$ and $\widetilde{k} = \{\widetilde{k}_i\}_{i \in I}$ be frames for a Hilbert space $\mathcal{H}_2$ such that $\widetilde{g} \otimes \widetilde{k}$ is a dual frame of $f \otimes f$. Then $\widetilde{g} \otimes \widetilde{k}$ is the canonical dual frame of $f \otimes f$ if and only if*

$$\left\langle \widetilde{g}_j \otimes \widetilde{k}_j, f_i \otimes f_i \right\rangle = \left\langle f_j \otimes f_j, \widetilde{g}_i \otimes \widetilde{k}_i \right\rangle.$$

The next proposition gives another characterization of the canonical dual frames for tensor product of two frames.

Proposition 3.9 *Suppose $f = \{f_i\}_{i \in I}$ and $g = \{g_i\}_{i \in I}$ are frames for $\mathcal{H}$ and $g \otimes g$ is a dual frame of $f \otimes f$. The following are equivalent.*

(1) $g \otimes g$ is the canonical dual frame of $f \otimes f$.
(2) $S_{g \otimes g} = S^{-1}_{f \otimes f}$.
(3) $S_{g \otimes g} = T_{g \otimes g} T^*_{\Theta}$, for every dual frame $\Theta = \{\Theta_{i,i}\}_{(i,i) \in I}$ of $f \otimes f$.

Proof (1) $\Rightarrow$ (2). Let $g \otimes g$ be the canonical dual frame of $f \otimes f$. Then $g \otimes g = S^{-1}_{f \otimes f} f \otimes f$. So

$$\begin{aligned}S_{g \otimes g} &= T_{g \otimes g} T^*_{g \otimes g} \\ &= S^{-1}_{f \otimes f} T_{f \otimes f} T^*_{f \otimes f} S^{-1}_{f \otimes f} \\ &= S^{-1}_{f \otimes f}.\end{aligned}$$

(2) $\Rightarrow$ (1). Assume that $g \otimes g$ is a dual frame of $f \otimes f$. Using Theorem 3.3 there exist operators $W_1, W_2 \in B\left(\mathcal{H}, \ell^2\right)$ such that $T_{f\otimes f}\left(W_1 \otimes W_2\right) = 0$ and

$$g \otimes g = \left\{S^{-1}_{f\otimes f}\left(f_i \otimes f_i\right) + \left(W_1 \otimes W_2\right)^*\left(\delta_i \otimes \delta_i\right)\right\}_{(i,i)\in I\times I}. \tag{3.4}$$

Put $W = W_1 \otimes W_2$. Since $T_{f\otimes f}W = 0$, for any $x \in \mathcal{H}$

$$\begin{aligned}
&\left\langle S_{g\otimes g}\left(x \otimes x\right), x \otimes x\right\rangle \\
&\quad= \left\langle \sum_{(i,i)} \left\langle x \otimes x, S^{-1}_{f\otimes f}\left(f_i \otimes f_i\right) + W^*\left(\delta_i \otimes \delta_i\right)\right\rangle\right. \\
&\quad\left.\left\langle S^{-1}_{f\otimes f}\left(f_i \otimes f_i\right) + W^*\left(\delta_i \otimes \delta_i\right), x \otimes x\right\rangle\right\rangle \\
&\quad= \left\langle S^{-1}_{f\otimes f}\left(x \otimes x\right), x \otimes x\right\rangle + \left\| W\left(x \otimes x\right)\right\|^2.
\end{aligned}$$

Therefore, if $S_{g\otimes g} = S^{-1}_{f\otimes f}$, then $g \otimes g$ is the canonical dual frame of $f \otimes f$.

(2) $\Rightarrow$ (3). We observe that

$$\begin{aligned}
S_{g\otimes g} &= S^{-1}_{f\otimes f} \\
&= S^{-1}_{f\otimes f} T_{f\otimes f} T^*_{\Theta} \\
&= T_{S^{-1}_{f\otimes f} f\otimes f} T^*_{\Theta} \\
&= T_{g\otimes g} T^*_{\Theta}.
\end{aligned}$$

(3) $\Rightarrow$ (2). Put $\Theta = S^{-1}_{f\otimes f} f \otimes f$. Then

$$\begin{aligned}
S_{g\otimes g} &= T_{g\otimes g} T^*_{S^{-1}_{f\otimes f} f\otimes f} \\
&= T_{g\otimes g} T^*_{f\otimes f} S^{-1}_{f\otimes f} \\
&= S^{-1}_{f\otimes f}.
\end{aligned}$$

□

Let $f_j, g_i \in \mathbb{C}^n$ be such that

$$\sum_{j=1}^{m} \left|\left\langle x, f_j\right\rangle\right|^2 f_j \otimes f_j = \sum_{i=1}^{m} \left|\left\langle x, g_i\right\rangle\right|^2 g_i \otimes g_i$$

for every $x \in \mathbb{C}^n$. In [8, Corollary 4.6] it is shown that $\{f_j\}_{j=1}^m$ is phase retrievable if and only if $\{g_i\}_{i=1}^m$ is phase retrievable.

If there exist parseval frames $\{f_j\}_{j=1}^m$ and $\{g_i\}_{i=1}^m$ for $\mathbb{C}^n$ then

$$x \otimes x = \sum_{j=1}^{m} \left|\left\langle x, f_j \right\rangle\right|^2 f_j \otimes f_j = \sum_{i=1}^{m} |\langle x, g_i \rangle|^2 g_i \otimes g_i.$$

Hence by [8, Corollary 4.6], $\{f_j\}_{j=1}^n$ is phase retrievable if and only if $\{g_i\}_{i=1}^m$ is phase retrievable. So, the following proposition can be concluded.

Proposition 3.10 *Let $\mathcal{H} = \mathbb{C}^n$. The following are equivalent.*

(1) There exists a Parseval phase retrievable frame for $\mathcal{H}$.
(2) Every Parseval frame is phase retrievable for $\mathcal{H}$.

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