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Coordinate space representation for renormalization of quantum electrodynamics

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Abstract We present a systematic treatment, up to order α, for the fundamental renormalization of quantum electrodynamics in real space. Although the standard renormalization is an old school problem for this case, it has not been completely done in position space yet. The most difference with well-known differential renormalization is that we do the whole procedure in coordinate space without needing to transform to momentum space. Specially, we directly derive the counterterms in coordinate space. This problem becomes crucial when the translational symmetry of the system breaks somehow explicitly (by nontrivial boundary condition (BC) on the fields). In this case, one is not able to move to momentum space by a simple Fourier transformation. In the context of the renormalized perturbation theory, counterterms in coordinate space will depend directly on the fields BCs (or background topology). Trivial BC or trivial background leads to the usual standard counterterms. If the counterterms are modified, then the quantum corrections of any physical quantity are different from those in free space where we have the translational invariance. We also show that, up to order α, our counterterms reduce to the usual standard one.

1 Introduction

From its early stages, quantum field theory (QFT) encountered some infinities leading to meaningless results and required to eliminate. These ultraviolet (UV) infinities are related to the quantum corrections of some physical quantities, such as electron mass and charge [1]. Very many attempts, starting with Kramers in the 1940s [2], have been done to control and remove these ultraviolet divergences. To calculate a physical quantity (for instance, electron mass) in an interacting field theory, in addition to its 'bare' value, we must take into account quantum corrections, Δm:

$$m_{\text{physical}} = m_{\text{bare}} + \Delta m, \tag{1}$$

where Δm is almost infinite due to undetermined momenta in loop quantum corrections.

The renormalization technique is a recipe that consistently removes and, also, controls all infinities that appeared in theory. The importance of the renormalization procedure is not

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only to absorb divergences but also to complete the definition of the quantized field theory, i.e., the finite parts of the renormalization constants -fixed by the renormalization conditions-influence the results of the calculation of radiative corrections and physically observable quantities. In QFT, there are two completely equivalent methods for the systematic of renormalization; First, Bare perturbation theory: working with the Bare parameters and relate them to their physical values at the end of calculations. Roughly speaking, the divergences absorb by *redefinition* of unmeasurable bare quantities. Second, renormalized perturbation theory: splitting the parameters appeared in the Lagrangian into two parts: physical part and *counterterm* part that absorbs the unphysical part. The unobservable shifts between the bare and the physical parameters absorb by counterterms. Both methods are required to give us precise definitions of the physical mass and coupling constants by applying renormalization conditions. The differences between two renormalization procedures are purely a matter of bookkeeping.

There are many investigations related to renormalization programs concerned with quantum electrodynamics (QED) [3], Quantum chromodynamics (QCD) [4], and scalar field with various self interactions [5–8]. All of these theories are renormalizable in 4-D since their coupling constants are dimensionless (Weinberg theorem) [9]. On the other hand, the renormalization group (RG) methods have been vastly considered (see [10–13]).

We should do, in principle, the renormalization in position space. However, for ease of calculation, we do it in momentum space. There is a duality transformation from p- to x-space renormalization, especially when we have translational symmetry. One moves from position to momentum space by a simple Fourier transformation. It is easy to do if our wave functions are plane waves. But, if the translational symmetry breaks somehow explicitly, then the momentum is not a good quantum number. In this case, the wave functions are not plane waves that the transformation to momentum space is no longer so simple and trivial. In this case, field propagators will depend on nontrivial properties that break translational symmetry (nontrivial boundary conditions (BC) or nontrivial background), all n-point functions and consequently all counterterms will depend on those nontrivial properties. (Please note that it is not possible to remedy the renormalization in momentum space by any perturbation since a nontrivial BC or a nonzero background is not a perturbative phenomenon [14]). For example, in the calculations of the radiative corrections to the Casimir effect, one usually encounters counterterms. In this case, the non-trivial boundary conditions on the walls break the translational invariance. For ϕ^4 theory, such problems have been investigated in Refs. [15–19]. Another example for the nontrivial background is the radiative correction to the mass of the kink [20], where the existence of a constant background breaks the translational symmetry. An important real example for QED is the Lamb shift which is a α^5 order effect. In this case, the responsible for symmetry breaking is the Coulomb potential.

We should here note that Differential Renormalization (DR) procedure [21,22], which has been investigated in the literature vastly, is done in coordinate space, though the traditional method of renormalization in momentum space (for review see [23,24]). DR is equivalent to traditional renormalization [25–27]. It is based on the observation that the UV divergence reflects in the fact that the higher-order amplitude cannot have a Fourier transform into momentum space due to the short-distance singularity. Thus one can, first, regulate such an amplitude by writing its singular parts as the derivatives of the normal functions, which have well-defined Fourier transformation. Second, by performing the Fourier transformation in partial integration and discarding the surface term, directly, get the renormalized result. In this procedure, the surface terms which drop during the renormalization have just corresponded to the counterterms. Therefore, to get the hidden counterterms, we have to move to momentum space again.

The derivation of standard counterterms from scattering amplitudes has been investigated from many years ago. In the context of DR, there also exist some works in massive and massless QED [28,29]. However, its program in position space has not been surveyed yet. Besides, the large order behavior of ϕ^4 theory for the nonzero background field considers in [30]. Also, this theory in $1 + 1$ dimensions, renormalization in real space, has been done in Ref. [14]. Applications of this theory, where we have nontrivial BCs such as Dirichlet BC or nonzero background such as a kink, have been used in Refs. [15–17], respectively. In $3 + 1$ dimensions, it has partially done in Ref. [20]. In [31,32] perturbative QFT in configuration, space has developed on curved space. Also, one can follow several recent works in this area. Amplitudes in a massless QFT [33], and relativistic causality and position space renormalization [34], is considered.

In this paper, we shall derive the counterterms by imposing reasonable renormalization conditions in configuration space. The resultant counterterms should be equivalent to ones derived by the standard renormalization in momentum space with the translational invariance. We will also present and check this equivalence.

We have organized the paper as the following. We review the systematics of the renormalization for QED theory in momentum space in Sect. 2. The renormalized perturbation theory of QED as a program in position space considers in Sect. 3. In Sect. 4, we compare our results with those in momentum space. Section 5 summarizes our results and conclusions.

2 Renormalization of QED in momentum space: a brief review

We review the systematics of renormalization for QED theory in momentum space. In general, any renormalizable QFT involves only a few superficially divergent amplitudes. In QED, there

are three amplitudes involving four infinite constants; vertex correction , vacuum

polarization and electron self energy . The renormalized perturbation theory of QED aims to absorb these constants into the four unobservable parameters: the bare mass, the bare coupling constant, the electron field strength, and the photon field strength. The original QED Lagrangian is

$$\mathcal{L}_{\text{QED}} = -\frac{1}{4}(F^{\mu\nu})^2 + \bar{\Psi}(i\slashed{\partial} - m_0)\Psi - e_0\bar{\Psi}\gamma_\mu\Psi A^\mu. \tag{2}$$

where m_0 and e_0 are the bare mass and the bare electric charge, respectively. The $\Psi(x)$ and $A^\mu(x)$ are fermion and photon fields, respectively, and can be written as

$$\Psi(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \sum_{s=1,2} \frac{1}{\sqrt{2E_\mathbf{p}}} \left[c_\mathbf{p}^s \psi^s(x) + d_\mathbf{p}^{s\,\dagger} \phi^s(x) \right] \tag{3}$$

$$A_\mu(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \sum_{s=0}^{3} \frac{1}{\sqrt{2\omega_\mathbf{p}}} \left[a_\mathbf{p}^s \widetilde{A}_\mu^s(x) + a_\mathbf{p}^{s\,\dagger} \widetilde{A}_\mu^{s*}(x) \right], \tag{4}$$

where, in the first line, $c_\mathbf{p}^{s\,\dagger}$ ($c_\mathbf{p}^s$) and $d_\mathbf{p}^{s\,\dagger}$ ($d_\mathbf{p}^s$) create (annihilate) a fermion and anti-fermion with momentum $\mathbf{p}$ and spin direction s, respectively. Here, $\psi^s(x)$ and $\phi^s(x)$ are the particle

and anti-particle solutions of the Dirac equation, respectively. In the second line, $a_{\mathbf{p}}^{s\,\dagger}$ ($a_{\mathbf{p}}^{s}$) creates (annihilates) a photon with momentum $\mathbf{p}$ and polarization $\varepsilon_{\mu}^{s}(\mathbf{p})$, and $\widetilde{A}_{\mu}^{r}(x)$ are the momentum-space solution of the equation $\partial^{\mu} A_{\mu} = 0$.

By replacing $\Psi(x) = \sqrt{z_2}\Psi_r(x)$ and $A^{\mu}(x) = \sqrt{z_3}A_r^{\mu}(x)$, we have

$$\mathcal{L}_{\text{QED}} = -\frac{1}{4}z_3(F_r^{\mu\nu})^2 + z_2\bar{\Psi}_r(i\slashed{\partial} - m_0)\Psi_r - e_0 z_2\sqrt{z_3}\,\bar{\Psi}_r\gamma_{\mu}\Psi_r A_r^{\mu}, \tag{5}$$

where z_2 and z_3 are the field-strength renormalizations for Ψ and A^{μ}, respectively. We define a scaling factor z_1 as $ez_1 = e_0 z_2\sqrt{z_3}$ and split each term of the Lagrangian into two pieces

$$\begin{aligned}
\mathcal{L}_{\text{QED}} = {} & -\frac{1}{4}\left(F_r^{\mu\nu}\right)^2 + \bar{\Psi}_r\left(i\slashed{\partial} - m\right)\Psi_r - e\bar{\Psi}_r\gamma^{\mu}\Psi_r A_r^{\mu} \\
& -\frac{1}{4}\delta_3\left(F_r^{\mu\nu}\right)^2 + i\delta_2\bar{\Psi}_r\slashed{\partial}\Psi_r - (\delta_m + m\delta_2)\,\bar{\Psi}_r\Psi_r - e\delta_1\bar{\Psi}_r\gamma_{\mu}\Psi_r A_r^{\mu},
\end{aligned} \tag{6}$$

with $z_3 = 1 + \delta_3$, $z_2 = 1 + \delta_2$, $m_0 = m + \delta_m$ and $z_1 = 1 + \delta_1$, where δ_1, δ_2, δ_3 and δ_m are counterterms. Here, m and e are the physical mass and physical charge of the electron measured at large distances. Now, the Feynman rules for the above Lagrangian are:

$$= -ie\gamma^{\mu} \tag{7}$$

$$= -ie\delta_1\gamma^{\mu} \tag{8}$$

$$\mu\,{\sim\sim\sim}\,\nu = \frac{-ig^{\mu\nu}}{q^2 + i\epsilon} \quad \text{(Feynman gauge)} \tag{9}$$

$$\mu\,{\sim\!\otimes\!\sim}\,\nu = -i(g^{\mu\nu}q^2 - q^{\mu}q^{\nu})\delta_3 \tag{10}$$

$$\xrightarrow{\;\;p\;\;} = \frac{i}{\slashed{p} - m + i\epsilon} \tag{11}$$

$$-\!\!\otimes\!\!\!\rightarrow\!- = i(\slashed{p}\delta_2 - \delta_m - m\delta_2). \tag{12}$$

We use the following notations:

$$-\!\!\!\!(\text{1PI})\!\!\!-\!\!= -i\,\Sigma(\not{p}) \tag{13}$$

$$_{\mu}\!\!\sim\!\!(\text{1PI})\!\!\sim\!\!_{\nu} = i\,\Pi^{\mu\nu}(q) = i(g^{\mu\nu}q^2 - q^{\mu}q^{\nu})\Pi(q^2), \tag{14}$$

$$= -ie\Gamma^{\mu}(p', p). \tag{15}$$

Here '1PI' denotes a *one-particle irreducible* diagram which is the sum of any diagram that cannot split in two by removing a single line. To fix the pole of the fermion propagator at the physical mass m we need two renormalization conditions:

$$\Sigma(\not{p} = m) = 0 \tag{16}$$

$$\left.\frac{\mathrm{d}\Sigma(\not{p})}{\mathrm{d}\not{p}}\right|_{\not{p}=m} = 0. \tag{17}$$

The renormalization condition which fixes the mass of the photon to zero is

$$\Pi(q^2 = 0) = 0. \tag{18}$$

Given the above conditions, finally, the physical electron charge is derived by the following renormalization condition:

$$-ie\Gamma^{\mu}(p' - p = 0) = -ie\gamma^{\mu}. \tag{19}$$

Now, using the dimensional regularization we are able to compute $-i\,\Sigma(\not{p})$, $i\,\Pi(q^2)$ and $-ie\Gamma^{\mu}(p', p)$. Applying the above renormalization conditions, up to leading order in α, the divergent parts of the counterterms are derived as

$$\delta_2 \sim -\frac{e^2}{8\pi^2\epsilon}, \tag{20}$$

$$\delta_m \sim -\frac{3me^2}{8\pi^2\epsilon}, \tag{21}$$

$$\delta_3 \sim -\frac{e^2}{6\pi^2\epsilon}, \tag{22}$$

$$\delta_1 = \delta_2 \sim -\frac{e^2}{8\pi^2\epsilon}, \tag{23}$$

where $d = 4 - \epsilon$ is the spacetime dimension so that we should take the limit $\epsilon \to 0$. These counterterms can remove all UV divergences of the QED theory in free space.

3 Renormalization in position space

We will survey the renormalization for QED in coordinate space within the renormalized perturbation theory. Naturally, when a systematic treatment of the renormalization program does, the counterterms automatically turn out to be dependent on the functional form of the fields. Besides, the RG may lead to position-dependent mass and charge, as a manifestation of the explicitly broken translational symmetry of the system. It is worth mentioning that our main scheme is following the standard renormalization approach in momentum space. In this case, we have the translational invariance. In the next three subsections, we separately consider electron self-energy, photon self-energy, and vertex correction and derive the counterterms by imposing proper renormalization conditions in the configuration space.

3.1 Electron self-energy

According to the Lagrangian (6), the perturbation expansion of the full electron propagator up to order α is

$$-i\Sigma = \quad \begin{array}{c}\text{(diagram)}\end{array} = \quad \begin{array}{c}\text{(diagram)}\end{array} + \begin{array}{c}\text{(diagram)}\\x \quad y\end{array} + \begin{array}{c}\text{(diagram)}\\x\end{array}. \tag{24}$$

We choose our renormalization condition in such a way that the pole of the first term of the right-hand side (RHS) gives the physical mass m at $x = x_0$. It requires that the sum of remaining diagrams, which we call it $-i\widetilde{\Sigma}(x)$ vanishes at this point, namely

$$-i\widetilde{\Sigma}(x)\Big|_{x=x_0} = \left(\begin{array}{c}\text{(diagram)}\\x \quad y\end{array} + \begin{array}{c}\text{(diagram)}\\x\end{array}\right)_{x=x_0} = 0, \quad \text{and} \quad \frac{d\left[-i\widetilde{\Sigma}(x)\right]}{dx}\Big|_{x=x_0} = 0. \tag{25}$$

We can write $-i\widetilde{\Sigma}$ to order α as

$$-i\widetilde{\Sigma}(x) = \int d^d y\,\overline{\psi}(y)\left[-i\Sigma_2(x,y)\right]\psi(x) + \overline{\psi}(x)\left[-\delta_2(x)\slashed{\partial} - im\delta_2(x) - i\delta_m(x)\right]\psi(x) \tag{26}$$

Thus, the first condition in Eq. (25) yields

$$-i\widetilde{\Sigma}(x_0) = \left\{\int d^d y\,\overline{\psi}(y)\left[-i\Sigma_2(x,y)\right]\psi(x) + \overline{\psi}(x)\left[-\delta_2(x)\slashed{\partial} - im\delta_2(x) - i\delta_m(x)\right]\psi(x)\right\}_{x=x_0}$$
$$= 0, \tag{27}$$

where $-i\Sigma_2$ is $O(\alpha)$ electron self-energy diagram. Now, using Dirac equation $(i\slashed{\partial} - m)\psi = 0$, up to order α we obtain

$$\delta_m = \frac{-1}{\overline{\psi}(x_0)\psi(x_0)}\int d^d y\,\overline{\psi}(y)\Sigma_2(x,y)\psi(x)\Big|_{x=x_0}. \tag{28}$$

To simplify the second condition in Eq. (25) we note that the $\widetilde{\Sigma}(x)$ is, in fact, a function of $\overline{\psi}(x), \psi(x), \slashed{\partial}\overline{\psi}(x)$ and $\slashed{\partial}\psi(x)$ so that

$$\frac{d\widetilde{\Sigma}(x)}{dx} = \frac{\partial\psi}{\partial x}\frac{\partial\widetilde{\Sigma}}{\partial\psi} + \frac{\partial\overline{\psi}}{\partial x}\frac{\partial\widetilde{\Sigma}}{\partial\overline{\psi}} + \frac{\partial(\slashed{\partial}\overline{\psi})}{\partial x}\frac{\partial\widetilde{\Sigma}}{\partial(\slashed{\partial}\overline{\psi})} + \frac{\partial(\slashed{\partial}\psi)}{\partial x}\frac{\partial\widetilde{\Sigma}}{\partial(\slashed{\partial}\psi)}. \tag{29}$$

Due to the opposite sign of the momentum for particles and anti-particles, the first two terms cancel each other. The third term is also zero because there is no derivative of $\overline{\psi}$ in $\widetilde{\Sigma}$. Thus, we obtain

$$\frac{\partial \left[-i\widetilde{\Sigma}(x)\right]}{\partial (\partial\!\!\!/\psi)}\bigg|_{x=x_0} = 0. \tag{30}$$

We can derive $\delta_2(x_0)$ by using the above equation and Eq. (26)

$$\frac{\partial \left[-i\widetilde{\Sigma}(x)\right]}{\partial (\partial\!\!\!/\psi)}\bigg|_{x=x_0} = \int d^d y \frac{\partial \left[\overline{\psi}(y)(-i\Sigma_2(x,y))\psi(x)\right]}{\partial (\partial\!\!\!/\psi(x))}\bigg|_{x=x_0} - \overline{\psi}(x_0)\delta_2(x_0) = 0 \tag{31}$$

$$\Rightarrow \delta_2 = \frac{1}{\overline{\psi}(x_0)}\int d^d y \frac{\partial \left[\overline{\psi}(y)(-i\Sigma_2(x,y))\psi(x)\right]}{\partial (\partial\!\!\!/\psi(x))}\bigg|_{x=x_0}. \tag{32}$$

3.2 Photon self-energy

For the photon propagator we again expand the full propagator as

$$i\Pi = \text{ }\langle\!\langle\!\bullet\!\rangle\!\rangle\text{ } = \text{ }\langle\!\langle\!\langle\!\langle\!\langle\!\langle\text{ } + \text{ }\langle\!\langle\bigcirc\!\rangle\text{ } + \text{ }\langle\!\langle\!\otimes\!\langle\!\langle\text{ } + \dots. \tag{33}$$

To have a massless photon, at $x = x_0$, we need only the first term on the RHS with a pole that is fixed definitely on zero. Therefore, the rest of the perturbation series must vanish so that up to order α we have

$$i\widetilde{\Pi}(x)\bigg|_{x=x_0} = \left(\begin{array}{c} \mu\!\cdot\!\langle\!\langle\!\bigcirc\!\langle\!\langle\!\cdot\!\nu + \mu\!\cdot\!\langle\!\langle\!\otimes\!\langle\!\langle\!\cdot\!\nu \end{array}\right)_{x=x_0} = 0 \tag{34}$$

or equivalently,

$$i\widetilde{\Pi}(x_0) = \left\{\int d^d y \widetilde{A}_\mu^*(y)\left[i\Pi_2^{\mu\nu}(x,y)\right]\widetilde{A}_\nu(x) + \widetilde{A}_\mu^*(x)\,\delta_3(x)\left[-i\left(g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu\right)\right]\widetilde{A}_\nu(x)\right\}_{x=x_0}$$
$$= 0, \tag{35}$$

where $i\Pi_2^{\mu\nu}(x,y)$ is $O(\alpha)$ photon self-energy diagram. Therefore,

$$\delta_3 = \int d^d y \frac{-\widetilde{A}_\mu^*(y)\Pi_2^{\mu\nu}(x,y)\widetilde{A}_\nu(x)}{\widetilde{A}_\mu^*(x)(g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu)\widetilde{A}_\nu(x)}\bigg|_{x=x_0}. \tag{36}$$

3.3 Vertex correction

Formally, the vertex corrections give us the physical charge of electron. Diagrammatically, we have

$$-ie\Gamma^\mu(x) = \overbrace{}^{\mu} = \overbrace{}^{\mu}{}_{x} + \overbrace{}^{\mu}{}_{y\ \ z} + \overbrace{}^{\mu}{}_{x} + \dots \tag{37}$$

Our renormalization condition for the electron charge is to fix it on physical e at $x = x_0$. We can do this by using the first term on RHS of Eq. (37) so that the remaining diagrams should cancel each other,

$$
-ie\widetilde{\Gamma}^\mu(x_0) = \left(\begin{array}{c} \text{(diagram)} \end{array} \right)_{x=x_0} . \tag{38}
$$

We can equivalently write the above equation as,

$$
-ie\widetilde{\Gamma}^\mu(x_0) = \left\{ \int d^d y\, d^d z\, \overline{\psi}(z)[-ie\delta\Gamma^\mu(x, y, z)]\psi(y) + \overline{\psi}(x)\left[-ie\delta_1(x)\gamma^\mu\right]\psi(x) \right\}_{x=x_0}
$$
$$
= 0, \tag{39}
$$

where $-ie\delta\Gamma^\rho$ is the vertex correction diagram to order α. Therefore, we find

$$
\delta_1\gamma^\mu = \int d^d y\, d^d z\, \frac{\overline{\psi}(z)\delta\Gamma^\mu(x, y, z)\psi(y)}{\overline{\psi}(x)\psi(x)}\bigg|_{x=x_0} . \tag{40}
$$

Accordingly, we may derive counterterms required for the renormalization of QED in coordinate space. These counterterms could apply for problems in which the translational invariance breaks explicitly. If we work in free space, with the translational symmetry, they should reduce to those in the standard prevalent derived in momentum space. We show this equivalence in the next section.

4 Comparison to momentum space (free space)

In this section, as a special case, we compare our results with the renormalization of QED in free space. In free space, the wave functions of fermions and photons consider as plane waves. We start with Eq. (28) by inserting $\psi(x) = u^s(p)e^{-ip.x}$ (from here on we drop the superscript s for simplicity). Then, the numerator of the integrand becomes

$$
\int d^d y\, \overline{\psi}(y)[-i\Sigma_2(x, y)]\psi(x)
$$
$$
= -e^2 \int d^d y\, \overline{u}(p)\, e^{ip.y}\gamma^\mu S(x-y)\, \gamma^\nu D_{\mu\nu}(y-x)u(p)\, e^{-ip.x}
$$
$$
= -e^2\overline{u}(p)\left[\int d^d y\, \frac{d^d k}{(2\pi)^d}\frac{d^d k'}{(2\pi)^d}\gamma^\mu \frac{\slashed{k}-m}{k^2-m^2}\frac{\gamma_\mu}{k'^2}e^{-i(k+k'-p).y}e^{-i(p-k'-k).x} \right]u(p), \tag{41}
$$

where $S(x-y)$ and $D^{\mu\nu}(y-x)$ are the propagators of fermion and photon in d spacetime dimensions, respectively,

$$
S(x-y) = \int \frac{d^d k}{(2\pi)^d}\frac{i}{\slashed{k}-m}e^{-ik.(x-y)}, \tag{42}
$$

and,

$$
D^{\mu\nu}(x-y) = \int \frac{d^d k}{(2\pi)^d}\frac{-ig^{\mu\nu}}{k^2}e^{-ik.(x-y)}. \tag{43}
$$

Integrating over position and then k' in Eq. (41) yields

$$\int d^d y \, \overline{\psi}(y)[-i\Sigma_2(x, y)]\psi(x) = -e^2 \overline{u}(p) \left[\int \frac{d^d k}{(2\pi)^d} \gamma^\mu \frac{1}{\not{k} - m} \gamma_\mu \frac{1}{(p-k)^2} \right] u(p). \quad (44)$$

In terms of $\epsilon = 4 - d$, the above equation becomes

$$\int d^d y \, \overline{\psi}(y)[-i\Sigma_2(x, y)]\psi(x) = \overline{u}(p) \frac{-ie^2}{8\pi^2 \epsilon} (-\not{p} + 4m) u(p) + O(\epsilon^0) \quad (45)$$

$$= \frac{-3ime^2}{8\pi^2 \epsilon} \overline{u}(p)u(p) + O(\epsilon^0). \quad (46)$$

Finally, using Eq. (28) we have

$$\delta_m = \frac{-1}{\overline{u}u} \frac{3me^2 \overline{u}u}{8\pi^2 \epsilon} + O(\epsilon^0)$$

$$= \frac{-3me^2}{8\pi^2 \epsilon} + O(\epsilon^0), \quad (47)$$

The above result is independent of x_0, manifesting the translational invariance of the system. It is also in agreement with Eq. (21), the standard common counterterm derived directly in free space.

We similarly derive the second counterterm, δ_2. Now, using Eq. (45) and the fact that $\not{\partial}\psi = \not{\partial}[u(p)e^{-ip.x}] = -i\not{p}\psi$, we can rewrite Eq.(32) as follows:

$$\delta_2 = \frac{1}{\overline{u}(p)} \frac{\partial \left[\overline{u}(p) \frac{-ie^2}{8\pi^2 \epsilon} (-\not{p} + 4m) u(p) \right]}{\partial \left[-i\not{p}u(p) \right]} = \frac{-e^2}{8\pi^2 \epsilon} + O(\epsilon^0), \quad (48)$$

which is precisely in agreement with Eq. (20). Again we see that the position dependence cancels out as expected.

To compute δ_3 in free space, we use $\widetilde{A}_\mu(p, x) = \varepsilon_\mu^s(p)e^{-ip.x}$ in Eq. (36). The numerator becomes

$$\int d^d y \widetilde{A}_\mu^*(y) \left(i\Pi_2^{\mu\nu} \right) \widetilde{A}_\nu(x) = \varepsilon_\mu^* \left[-ie^2 \int d^d y \, \gamma^\mu S(x-y) \gamma^\nu S(y-x) e^{-iq.x} e^{iq.y} \right] \varepsilon_\nu$$

$$= \varepsilon_\mu^* \left[ie^2 \int d^d y \int \frac{d^d k}{(2\pi)^d} \frac{d^d k'}{(2\pi)^d} \gamma^\mu \frac{1}{\not{k} - m} \right.$$

$$\left. \gamma^\nu \frac{1}{\not{k'} - m} e^{-i(q+k-k').x} e^{-i(-k-q+k').y} \right] \varepsilon_\nu. \quad (49)$$

Integrating over y and k', the RHS gives,

$$\varepsilon_\mu^* \left[ie^2 \int \frac{d^d k}{(2\pi)^4} \gamma^\mu \frac{1}{\not{k} - m} \gamma^\nu \frac{1}{\not{q} + \not{k} - m} \right] \varepsilon_\nu. \quad (50)$$

By simple calculations we finally have,

$$\int d^d y \widetilde{A}_\mu^*(i\Pi_2^{\mu\nu})\widetilde{A}_\nu = \varepsilon_\mu^* \frac{-ie^2}{6\pi^2 \epsilon} (g^{\mu\nu}k^2 - k^\mu k^\nu)\varepsilon_\nu + O(\epsilon^0). \quad (51)$$

Inserting the above calculation in Eq. (36) and using $\widetilde{A}_\mu^*(x)\widetilde{A}_\nu(x) = \varepsilon_\mu^* \varepsilon_\nu$ we derive,

$$
\begin{aligned}
\delta_3 &= \frac{-ie^2}{6\pi^2\epsilon} \frac{\varepsilon_\mu^* \left(g^{\mu\nu}k^2 - k^\mu k^\nu\right)\varepsilon_\nu}{\widetilde{A}_\mu^*[-i\left(g^{\mu\nu}\left(-k^2\right) + k^\mu k^\nu\right)]\widetilde{A}_\nu} + O(\epsilon^0) \\
&= -\frac{e^2}{6\pi^2\epsilon} + O(\epsilon^0),
\end{aligned}
\tag{52}
$$

which is in accordance with Eq. (22).

For the last counterterm, δ_1, the numerator in Eq. (40) can be rewritten as

$$
\begin{aligned}
&\int d^d z d^d y\, \overline{\psi}(z)\delta\Gamma^\mu(x, y, z)\psi(y) \\
&= -e^2 \int d^d z d^d y\, \overline{\psi}\left(p', z\right)\gamma^\alpha S\left(z, x\right)\gamma^\mu S\left(x, y\right)\gamma^\beta \psi\left(p, y\right) D_{\alpha\beta}\left(y, z\right) \\
&= e^2 \int d^d z d^d y\, \overline{u}(p')\left[e^{ip'\cdot z} \int \frac{d^d k}{(2\pi)^d}\frac{d^d k'}{(2\pi)^d}\frac{d^d k''}{(2\pi)^d} \right. \\
&\quad \times \gamma^\alpha \frac{e^{-ik'\cdot(z-x)}}{\slashed{k}'-m}\gamma^\mu\frac{e^{-ik\cdot(x-y)}}{\slashed{k}-m}\gamma^\beta e^{-ip\cdot y}\frac{-ig_{\alpha\beta}}{k''^2}e^{-ik''\cdot(z-y)} \left.\right] u(p) \\
&= -ie^2\overline{u}(p')\int\frac{d^d k}{(2\pi)^d}\frac{d^d k'}{(2\pi)^d}\frac{d^d k''}{(2\pi)^d}\gamma^\alpha\frac{1}{\slashed{k}'-m}\gamma^\mu\frac{1}{\slashed{k}-m}\gamma^\beta\frac{g_{\alpha\beta}}{k''^2} \\
&\quad \times (2\pi)^{2d}\,\delta^{(d)}\left(k+k''-p\right)\delta^{(d)}\left(p'-k'-k''\right)e^{i(k'-k)\cdot x}u(p).
\end{aligned}
\tag{53}
$$

Taking integral of k' and k'' yields,

$$
\begin{aligned}
&\int d^d z d^d y\, \overline{\psi}(z)\delta\Gamma^\mu(x, y, z)\psi(y) \\
&= -ie^2\overline{u}(p')\left[\int\frac{d^d k}{(2\pi)^d}\gamma^\alpha\frac{1}{\slashed{k}'-m}\gamma^\mu\frac{1}{\slashed{k}-m}\gamma^\beta\frac{-g_{\alpha\beta}}{(p-k)^2}e^{i(p'-p)\cdot x} \right] u(p) \\
&= \overline{u}(p')\left[\frac{-e^2}{8\pi^2\epsilon}\gamma^\mu e^{i(p'-p)\cdot x}\right]u(p) + O(\epsilon^0).
\end{aligned}
\tag{54}
$$

Replacing this result in Eq. (40) we find,

$$
\begin{aligned}
\delta_1\gamma^\mu &= \int d^d z d^d y\, \frac{\overline{\psi}(y)\delta\Gamma^\mu(x, y, z)\psi(x)}{\overline{\psi}(x)\psi(x)}\bigg|_{x=x_0} \\
&= \frac{\overline{u}(p')\left[\frac{-e^2}{8\pi^2\epsilon}\gamma^\mu e^{i(p'-p)\cdot x_0}\right]u(p)}{\overline{u}(p')u(p)e^{i(p'-p)\cdot x_0}} + O(\epsilon^0) \\
&\Rightarrow \delta_1 = -\frac{e^2}{8\pi^2\epsilon} + O(\epsilon^0),
\end{aligned}
\tag{55}
$$

which is again in complete agreement to Eq. (23). This counterterm is equal to δ_2 as it should be, due to the Ward identity. Consequently, up to order α, we show that our counterterms in position space are equal to the usual terms derived in momentum space. The results, in this case, do not depend on the spatial point x_0 where our renormalization conditions impose. It manifests the translational invariance of this problem.

5 Conclusions

Ultraviolet infinities of QED theory are basically due to three divergent Feynman diagrams: vertex correction, vacuum polarization, and electron self-energy. These infinities are controlled by four counterterms that derive by using the renormalization program in free space with translational symmetry. However, if the translational invariance of the system breaks strongly, then the momentum is no longer a good quantum number. Renormalization procedure in configuration space can be applied for such a situation, e.g. in problems with a nontrivial BC or a nonzero background that cannot treat as small perturbations. For example, the kink as a constant background in 1+1 dimensions breaks the translational invariance, or in the Casimir effect, we have nontrivial BC on the walls. Another real example for QED is the Lamb shift in which the Coulomb potential in hydrogen atom breaks the translational symmetry. In this paper, we have done the renormalization, up to order α, for the theory of QED in coordinate space and derived the general form of counterterms. The systematical treatment of the renormalized perturbation theory leads to x-independent counterterms. It indicates the dependency on the BCs of the fermion and photon fields directly. Finally, as a particular case, our results have been compared with those obtained in free space: we have shown the equivalence in the two cases is guaranteed, up to order α.

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