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Adjointness of Suspension and Shape Path Functors

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Abstract

In this paper, we introduce a subcategory $\widetilde{Sh}_*$ of Sh_* and obtain some results in this subcategory. First we show that there is a natural bijection $Sh(\Sigma(X,x),(Y,y)) \cong Sh((X,x),Sh((I,\dot{I}),(Y,y)))$, for every $(Y,y) \in \widetilde{Sh}_*$ and $(X,x) \in Sh_*$. By this fact, we prove that for any pointed topological space (X,x) in $\widetilde{Sh}_*$, $\check{\pi}_n^{top}(X,x) \cong \check{\pi}_{n-k}^{top}(Sh((S^k,*),(X,x)),e_x)$, for all $1 \leq k \leq n-1$.

Keywords: shape category, topological shape homotopy group, shape group, suspensions.

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1. Introduction and Motivation

Morón et al. [11] gave a complete, non-Archimedean metric (or ultrametric) on the set of shape morphisms between two unpointed compacta (compact metric spaces) X and Y, $Sh(X,Y)$. They mentioned that this construction can be translated to the pointed case. Consequently, as a particular case, they obtained a complete ultrametric induces a norm on the shape groups of a compactum Y and then presented some results on these topological groups [12]. Also, Cuchillo-Ibanez et al. [5] constructed several generalized ultrametrics in the set of shape morphisms

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between topological spaces and obtained semivaluations and valuations on the groups of shape equivalences and kth shape groups. On the other hand, Cuchillo-Ibanez et al. [6] introduced a topology on the set $Sh(X,Y)$, where X and Y are arbitrary topological spaces, in such a way that it extended topologically the construction given in [11]. Also, Moszyńska [10] showed that the kth shape group $\check{\pi}_k(X,x)$, $k \in \mathbb{N}$, is isomorphic to the set $Sh((S^k,*),(X,x))$ consists of all shape morphisms $(S^k,*) \to (X,x)$ with a group operation, for all compact Hausdorff space (X,x). Note that, Bilan [1] mentioned that this fact is true for all topological spaces.

The authors [13] applied this topology on the set of shape morphisms between pointed spaces and proved that the kth shape group $\check{\pi}_k(X,x)$, $k \in \mathbb{N}$, with the above topology is a Hausdorff topological group, denoted by $\check{\pi}_k^{top}(X,x)$. In this paper, we introduce a subcategory $\widetilde{Sh}_*$ of Sh_* and obtain some results in this subcategory. It is well-known that the pair (Σ,Ω) is an adjoint pair of functors on hTop_* and therefore, there is a natural bijection $Hom(\Sigma(X,x),(Y,y)) \cong Hom((X,x),\Omega(Y,y))$, for every pointed topological spaces (X,x) and (Y,y). In this paper, we show that there is a natural bijection

$$Sh(\Sigma(X,x),(Y,y)) \cong Sh((X,x),(Sh((I,\dot{I}),(Y,y)),e_y)),$$

for every $(Y,y) \in \widetilde{Sh}_*$ and $(X,x) \in Sh_*$. By this fact we conclude that the functor $Sh((I,\dot{I}),-)$ preserves inverse limits such as products, pullbacks, kernels, nested intersections and completions, provided inverse limit exists in the subcategory $\widetilde{Sh}_*$. Also, the functor Σ preserves direct limits of connected spaces in this subcategory. As a consequence, if $(X\times Y,(x,y))$ is a product of pointed spaces (X,x) and (Y,y) in the subcategory $\widetilde{Sh}_*$, then

$$\check{\pi}_1(X\times Y,(x,y)) \cong \check{\pi}_1(X,x)\times\check{\pi}_1(Y,y).$$

It is well-known that for any pointed space (X,x) and for all $1 \le k \le n-1$, $\pi_n(X,x) \cong \pi_{n-k}(\Omega(X,x),e_x)$. In this paper, we show that for any pointed topological space (X,x) in $\widetilde{Sh}_*$, $\check{\pi}_n(X,x) \cong \check{\pi}_{n-k}(Sh((S^k,*),(X,x)),e_x)$, for all $1 \le k \le n-1$. We then exhibit an example in which this result dose not hold in the category Sh_*.

Endowed with the quotient topology induced by the natural surjective map $q:\Omega^n(X,x)\to\pi_n(X,x)$, where $\Omega^n(X,x)$ is the nth loop space of (X,x) with the compact-open topology, the familiar homotopy group $\pi_n(X,x)$ becomes a quasitopological group which is called the quasitopological nth homotopy group of the pointed space (X,x), denoted by $\pi_n^{qtop}(X,x)$ (See [2, 3, 4, 8]). Nasri et al. [14], showed that for any pointed topological space (X,x), $\pi_n^{qtop}(X,x) \cong \pi_{n-k}^{qtop}(\Omega^k(X,x),e_x)$, for all $1 \le k \le n-1$. In this paper, we prove that for any pointed topological space (X,x) in $\widetilde{Sh}_*$, $\check{\pi}_n^{top}(X,x) \cong \check{\pi}_{n-k}^{top}(Sh((S^k,*),(X,x)),e_x)$, for all $1 \le k \le n-1$.

2. Preliminaries

In this section, we recall some of the main notions concerning the shape category and the pro-HTop (See [9]). Let $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ and $\mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M)$ be two inverse systems in HTop. A *pro-morphism* of inverse systems, $(f, f_\mu) : \mathbf{X} \to \mathbf{Y}$, consists of an index function $f : M \to \Lambda$ and of mappings $f_\mu : X_{f(\mu)} \to Y_\mu$, $\mu \in M$, such that for every related pair $\mu \leq \mu'$ in M, there exists a $\lambda \in \Lambda$, $\lambda \geq f(\mu), f(\mu')$ so that

$$q_{\mu\mu'} f_{\mu'} p_{f(\mu')\lambda} \simeq f_\mu p_{f(\mu)\lambda}.$$

The *composition* of two pro-morphisms $(f, f_\mu) : \mathbf{X} \to \mathbf{Y}$ and $(g, g_\nu) : \mathbf{Y} \to \mathbf{Z} = (Z_\nu, r_{\nu\nu'}, N)$ is also a pro-morphism $(h, h_\nu) = (g, g_\nu)(f, f_\mu) : \mathbf{X} \to \mathbf{Z}$, where $h = fg$ and $h_\nu = g_\nu f_{g(\nu)}$. The *identity pro-morphism* on $\mathbf{X}$ is pro-morphism $(1_\Lambda, 1_{X_\lambda}) : \mathbf{X} \to \mathbf{X}$, where 1_Λ is the identity function. A pro-morphism $(f, f_\mu) : \mathbf{X} \to \mathbf{Y}$ is said to be *equivalent* to a pro-morphism $(f', f'_\mu) : \mathbf{X} \to \mathbf{Y}$, denoted by $(f, f_\mu) \sim (f', f'_\mu)$, provided every $\mu \in M$ admits a $\lambda \in \Lambda$ such that $\lambda \geq f(\mu), f'(\mu)$ and

$$f_\mu p_{f(\mu)\lambda} \simeq f'_\mu p_{f'(\mu)\lambda}.$$

The relation $\sim$ is an equivalence relation. The *category* pro-HTop has as objects, all inverse systems $\mathbf{X}$ in HTop and as morphisms, all equivalence classes $\mathbf{f} = [(f, f_\mu)]$. The composition of $\mathbf{f} = [(f, f_\mu)]$ and $\mathbf{g} = [(g, g_\nu)]$ in pro-HTop is well defined by putting

$$\mathbf{gf} = \mathbf{h} = [(h, h_\nu)].$$

An HPol-expansion of a topological space X is a morphism $\mathbf{p} : X \to \mathbf{X}$ in pro-HTop, where $\mathbf{X}$ belongs to pro-HPol characterised by the following two properties:
(E1) For every $P \in HPol$ and every map $h : X \to P$ in HTop, there is a $\lambda \in \Lambda$ and a map $f : X_\lambda \to P$ in HPol such that $fp_\lambda \simeq h$.
(E2) If $f_0, f_1 : X_\lambda \to P$ satisfy $f_0 p_\lambda \simeq f_1 p_\lambda$, then there exists a $\lambda' \geq \lambda$ such that $f_0 p_{\lambda\lambda'} \simeq f_1 p_{\lambda\lambda'}$.

Let $\mathbf{p} : X \to \mathbf{X}$ and $\mathbf{p}' : X \to \mathbf{X}'$ be two HPol-expansions of an space X in HTop, and let $\mathbf{q} : Y \to \mathbf{Y}$ and $\mathbf{q}' : Y \to \mathbf{Y}'$ be two HPol-expansions of an space Y in HTop. Then there exist two natural isomorphisms $\mathbf{i} : \mathbf{X} \to \mathbf{X}'$ and $\mathbf{j} : \mathbf{Y} \to \mathbf{Y}'$ in pro-HTop. A morphism $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ is said to be *equivalent* to a morphism $\mathbf{f}' : \mathbf{X}' \to \mathbf{Y}'$, denoted by $\mathbf{f} \sim \mathbf{f}'$, provided the following diagram in pro-HTop commutes:

$$\begin{array}{ccc} \mathbf{X} & \xrightarrow{\mathbf{i}} & \mathbf{X}' \\ \downarrow{\scriptstyle \mathbf{f}} & & {\scriptstyle \mathbf{f}'}\downarrow \\ \mathbf{Y} & \xrightarrow{\mathbf{j}} & \mathbf{Y}'. \end{array}$$

Now, the *shape category* Sh is defined as follows: The objects of Sh are topological spaces. A morphism $F : X \to Y$ is the equivalence class $< \mathbf{f} >$ of a mapping $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ in pro-HTop. The *composition* of $F =< \mathbf{f} >: X \to Y$ and $G =< \mathbf{g} >: Y \to Z$ is defined by the representatives, i.e., $GF =< \mathbf{gf} >: X \to Z$.

The *identity shape morphism* on a space X, $1_X : X \to X$, is the equivalence class $< 1_{\mathbf{X}} >$ of the identity morphism $1_{\mathbf{X}}$ in pro-HTop.

Let $\mathbf{p} : X \to \mathbf{X}$ and $\mathbf{q} : Y \to \mathbf{Y}$ be HPol-expansions of X and Y, respectively. Then for every morphism $f : X \to Y$ in HTop, there is a unique morphism $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ in pro-HTop such that the following diagram commutes in pro-HTop.

$$\begin{array}{ccc} \mathbf{X} & \xleftarrow[\mathbf{p}]{} & X \\ \Big\downarrow \mathbf{f} & & f \Big\downarrow \\ \mathbf{Y} & \xleftarrow[\mathbf{q}]{} & Y. \end{array}$$

If we take other HPol-expansions $\mathbf{p}' : X \to \mathbf{X}'$ and $\mathbf{q}' : Y \to \mathbf{Y}'$, we obtain another morphism $\mathbf{f}' : \mathbf{X}' \to \mathbf{Y}'$ in pro-HTop such that $\mathbf{f}'\mathbf{p}'^{*} = \mathbf{q}'f$ and so we have $\mathbf{f} \sim \mathbf{f}'$. Hence every morphism $f \in HTop(X, Y)$ yields an equivalence class $< [\mathbf{f}] >$, i.e., a shape morphism $F : X \to Y$ which is denoted by $\mathcal{S}(f)$. If we put $\mathcal{S}(X) = X$ for every topological space X, then we obtain a functor $\mathcal{S} : HTop \to Sh$, called the *shape functor*. Also if $Y \in$HPol, then every shape morphism $F : X \to Y$ admits a unique morphism $f : X \to Y$ in HTop such that $F = \mathcal{S}(f)$ [9, Theorem 1.2.4].

Similarly, we can define the categories pro-HTop$_*$ and Sh$_*$ on pointed topological spaces (See [9]).

3. Main Results

In this section, we introduce a subcategory $\widetilde{Sh}_*$ of Sh$_*$ consists of all pointed topological spaces having bi-expansions. Then we consider the well-known suspension functor $\Sigma : Sh_* \to Sh_*$ (See [9]) and $Sh((I, \dot{I}), -) : Sh_* \to Sh_*$ and show that there is a natural bijection $Sh(\Sigma(X, x), (Y, y)) \cong Sh((X, x), (Sh((I, \dot{I}), (Y, y)), e_y))$, for every $(Y, y) \in \widetilde{Sh}_*$ and $(X, x) \in Sh_*$. Then using this bijection we conclude some results in subcategory $\widetilde{Sh}_*$.

Definition 3.1. We say that a pointed topological space (X, x) has a bi-expansion $\mathbf{p} : (X, x) \to (\mathbf{X}, \mathbf{x})$ whenever $\mathbf{p}$ is an HPol$_*$-expansion of (X, x) such that $\mathbf{p}_* : Sh((I, \dot{I}), (X, x)) \to \mathbf{Sh}((I, \dot{I}), (X, x))$ is an HPol$_*$-expansion of $Sh((I, \dot{I}), (X, x))$.

In follow, we recall some conditions on topological space X under which X has a bi-expansion.

Remark 1. [13, Remark 4.11]. If $\mathbf{p} : (X, x) \to (\mathbf{X}, \mathbf{x})$ is an HPol$_*$-expansion of X, then $\mathbf{p}_* : Sh((S^k, *), (X, x)) \to \mathbf{Sh}((S^k, *), (X, x))$ is an inverse limit of $\mathbf{Sh}((S^k, *), (X, x)) = (Sh((S^k, *), (X_\lambda, x_\lambda)), (p_{\lambda\lambda'})_*, \Lambda)$ (See [6, Theorem 2]). Moreover, if $Sh((S^k, *), (X, x))$ is compact and $Sh((S^k, *), (X_\lambda, x_\lambda))$ is compact polyhedron for all $\lambda \in \Lambda$, then by [7, Remark 1], $\mathbf{p}_*$ is an HPol$_*$-expansion of $Sh((S^k, *), (X, x))$.

Lemma 3.2. [13, Lemma 4.12] *Let* (X, x) *have an* $HPol_*$*-expansion* $\mathbf{p} : (X, x) \to ((X_\lambda, x_\lambda), p_{\lambda\lambda'}, \Lambda)$ *such that* $\pi_k(X_\lambda, x_\lambda)$ *is finite, for every* $\lambda \in \Lambda$. *Then* $\mathbf{p}_* : Sh((S^k, *), (X, x)) \to \mathbf{Sh}((S^k, *), (X, x))$ *is an* $HPol_*$*-expansion of* $Sh((S^k, *), (X, x))$*, for all* $k \in \mathbb{N}$.

Example 3.3. [13, Example 4.13] (See also [9]). Let $\mathbb{R}P^2$ be the real projective plane. Consider the map $\bar{f} : \mathbb{R}P^2 \to \mathbb{R}P^2$ induced by the following commutative diagram:

$$\begin{array}{ccc} D^2 & \xleftarrow[f]{} & D^2 \\ \Big\downarrow{\scriptstyle \phi} & & {\scriptstyle \phi}\Big\downarrow \\ \mathbb{R}P^2 & \xleftarrow[\bar{f}]{} & \mathbb{R}P^2, \end{array}$$

where $D^2 = \{z \in \mathbb{C} \mid |z| \leq 1\}$ is the unit 2-cell, $f(z) = z^3$ and $\phi : D^2 \to \mathbb{R}P^2$ is the quotient map identifies pairs of points $\{z, -z\}$ of S^1. We consider X as the inverse sequence

$$\mathbb{R}P^2 \xleftarrow{\bar{f}} \mathbb{R}P^2 \xleftarrow{\bar{f}} \cdots .$$

Since $\mathbb{R}P^2$ is compact polyhedron, by [7, Remark 1] X is compact and $\mathbf{p} : X \to (\mathbb{R}P^2, \bar{f}, \mathbb{N})$ is an HPol-expansion of X. Since $\bar{f}$ is onto and $\pi_k(\mathbb{R}P^2) \cong \mathbb{Z}_2$ is finite, $\mathbf{p}_* : Sh((S^k, *), (X, x)) \to \mathbf{Sh}((S^k, *), (X, x))$ is an HPol$_*$-expansion of $Sh((S^k, *), (X, x))$, for all $k \in \mathbb{N}$.

The well-known suspension functor $\Sigma : HTop_* \to HTop_*$ is extended to a suspension functor $\Sigma : Sh_* \to Sh_*$ (See [9]). Note that, if (X, x) is a pointed topological space, then $\Sigma(X, x) = (\Sigma X, \Sigma x)$ is also a pointed topological space. Therefore, whenever $\mathbf{p} : (X, x) \to (\mathbf{X}, \mathbf{x})$ is an HPol$_*$-expansion of (X, x), then $\mathbf{\Sigma p} : \Sigma(X, x) \to \Sigma(\mathbf{X}, \mathbf{x}) = (\Sigma(X_\lambda, x_\lambda), \Sigma p_{\lambda\lambda'}, \Lambda)$ is an HPol$_*$-expansion of $\Sigma(X, x)$.

Remark 2. Let (X, x) be a connected topological space and $\mathbf{p} : (X, x) \to (\mathbf{X}, \mathbf{x}) = ((X_\lambda, x_\lambda), p_{\lambda\lambda'}, \Lambda)$ be an HPol$_*$-expansion of (X, x). Since X is connected, one can assume that all X_λ are connected, by [9, Remark 4.1.1] and so $\pi_1(\Sigma(X_\lambda, x_\lambda)) = 0$, for all $\lambda \in \Lambda$ (by Van Kampen Theorem). Therefore, the HPol$_*$-expansion $\mathbf{\Sigma p} : \Sigma(X, x) \to \Sigma(\mathbf{X}, \mathbf{x})$ satisfies in the conditions of Lemma 3.2 and so $\Sigma(X, x) \in \widetilde{Sh}_*$.

Let $F : \Sigma(X, x) \to (Y, y)$ be a shape morphism represented by $\mathbf{f} : \Sigma(\mathbf{X}, \mathbf{x}) \to (\mathbf{Y}, \mathbf{y})$ consists of $f : M \to \Lambda$ and $f_\mu : \Sigma(X_{f(\mu)}, x_{f(\mu)}) \to (Y_\mu, y_\mu)$. If (Y, y) has a bi-expansion $\mathbf{q} : (Y, y) \to (\mathbf{Y}, \mathbf{y})$, then F determines a map $F^\sharp : (X, x) \to (Sh((I, \dot{I}), (Y, y)), e_y)$ represented by $\mathbf{f}^\sharp : (\mathbf{X}, \mathbf{x}) \to (\mathbf{Sh}((I, \dot{I}), (Y, y)), \mathbf{e_y})$ consists of $f : M \to \Lambda$ and $f^\sharp_\mu : (X_{f(\mu)}, x_{f(\mu)}) \to (Sh((I, \dot{I}), (Y_\mu, y_\mu)), e_{y_\mu})$ which is defined as $f^\sharp_\mu(x) = \mathcal{S}(l_{x\mu})$, where $l_{x\mu} : (I, \dot{I}) \to (Y_\mu, y_\mu)$ is a map in HTop$_*$ such that $l_{x\mu}(t) = f_\mu([x, t])$.

In the following lemma we show that $F^\sharp$ is a shape morphism.

Lemma 3.4. *The map* $F^\sharp$ *defined in the above is a shape morphism.*

Proof. With the above notation, first we show that $f^\sharp_\mu : X_{f(\mu)} \to Sh((I,\dot{I}),(Y_\mu,y_\mu))$ is continuous. Since Y_μ is a polyhedron, the space $Sh((I,\dot{I}),(Y_\mu,y_\mu))$ is discrete by [6, Corollary 1]. Therefore, it is sufficient to show that $f^\sharp_\mu$ is locally constant. Let $x \in X_{f(\mu)}$. Since $X_{f(\mu)}$ is polyhedron, there is an open neighborhood V_x of x that is contractible to x in $X_{f(\mu)}$. We will show that $f^\sharp_\mu$ is constant on V_x. Let $x' \in V_x$, then by path connectedness of V_x, there exists a path $\alpha : I \to X_{f(\mu)}$ such that $\alpha(0) = x$ and $\alpha(1) = x'$. We define the map $H : I \times I \to Y_\mu$ by $H(t,s) = f_\mu([\alpha(s),t])$. Since f_μ and α are continuous and V_x is contractible to x in $X_{f(\mu)}$, the map H is well-defined and continuous. Moreover, H is a relative homotopy between $f_\mu([x,-])$ and $f_\mu([x',-])$. Hence $l_{x\mu} \simeq l_{x'\mu}$ $(rel\{\dot{I}\})$ and so $\mathcal{S}(l_{x\mu}) = \mathcal{S}(l_{x'\mu})$. Therefore $f^\sharp_\mu(x) = f^\sharp_\mu(x')$ and so $f^\sharp_\mu$ is constant on V_x. Finally, we conclude that $f^\sharp_\mu$ is continuous.

Now, let $\mathbf{p} : (X,x) \to (\mathbf{X},\mathbf{x})$ be an HPol$_*$-expansion of (X,x) and $\mathbf{q} : (Y,y) \to (\mathbf{Y},\mathbf{y})$ be a bi-expansion of (Y,y). The map $\mathbf{f}^\sharp$ is a morphism in pro-HTop$_*$. Indeed, for any pair $\mu' \geq \mu$, there is a $\lambda \geq f(\mu), f(\mu')$ such that

$$f_\mu \circ \Sigma p_{f(\mu)\lambda} \simeq q_{\mu\mu'} \circ f_{\mu'} \circ \Sigma p_{f(\mu')\lambda} \quad (rel\{\Sigma x_\lambda\}). \tag{1}$$

Also, for every $x \in X_\lambda$,

$$f^\sharp_\mu(p_{f(\mu)\lambda}(x)) = \mathcal{S}(l_{p_{f(\mu)\lambda}(x)\mu}),$$

and for every $t \in I$,

$$l_{p_{f(\mu)\lambda}(x)\mu}(t) = f_\mu([p_{f(\mu)\lambda}(x),t]) = f_\mu \circ \Sigma p_{f(\mu)\lambda}([x,t])$$

$$(q_{\mu\mu'})_* \circ l_{p_{f(\mu')\lambda}(x)\mu'}(t) = q_{\mu\mu'} \circ f_{\mu'}([p_{f(\mu')\lambda}(x),t]) = q_{\mu\mu'} \circ f_{\mu'} \circ \Sigma p_{f(\mu')\lambda}([x,t]).$$

By Equation (1), $l_{p_{f(\mu)\lambda}(x)\mu} \simeq (q_{\mu\mu'})_* \circ l_{p_{f(\mu')\lambda}(x)\mu'}$ $(rel\{\dot{I}\})$. Therefore

$$f^\sharp_\mu \circ p_{f(\mu)\lambda}(x) = \mathcal{S}(l_{p_{f(\mu)\lambda}(x)\mu}) = \mathcal{S}((q_{\mu\mu'})_* \circ l_{p_{f(\mu')\lambda}(x)\mu'}) = (q_{\mu\mu'})_* \circ f^\sharp_{\mu'}(p_{f(\mu')\lambda}(x)).$$

□

On the other hand, let $G : (X,x) \to (Sh((I,\dot{I}),(Y,y)),e_y)$ be a shape morphism represented by $\mathbf{g} : (\mathbf{X},\mathbf{x}) \to (\mathbf{Sh}((I,\dot{I}),(Y,y)),\mathbf{e_y})$ consists of $g : M \to \Lambda$ and $g_\mu : (X_{g(\mu)},x_{g(\mu)}) \to (Sh((I,\dot{I}),(Y_\mu,y_\mu)),e_{y_\mu})$. Then we define $G^\flat : \Sigma(X,x) \to (Y,y)$ represented by $\mathbf{g}^\flat : \Sigma(\mathbf{X},\mathbf{x}) \to (\mathbf{Y},\mathbf{y})$ in pro-HTop$_*$ consists of $g : M \to \Lambda$ and $g^\flat_\mu : \Sigma(X_{g(\mu)},x_{g(\mu)}) \to (Y_\mu,y_\mu)$ given by $g^\flat_\mu([x,t]) = g'_{\mu x}(t)$, where $g'_{\mu x}$ is a unique morphism in HTop$_*$ with $\mathcal{S}(g'_{\mu x}) = g_\mu(x)$ (See [9, Theorem 1.2.4]).

Lemma 3.5. *The map $G^\flat$ defined in the above is a shape morphism.*

Proof. First we show that $g^\flat_\mu$ is continuous. It is sufficient to show that $\overline{g^\flat_\mu}$: $(X_{g(\mu)} \times I, \{x_{g(\mu)}\} \times \dot{I}) \to (Y_\mu,y_\mu)$ is continuous. We claim that the map e_μ :

$Sh((I,\dot{I}),(Y_\mu,y_\mu))\times I\to Y_\mu$ given by $e_\mu(F,t)=F'(t)$ is continuous, where F' is a unique morphism in HTop$_*$ with $\mathcal{S}(F')=F$ (See [9, Theorem 1.2.4]). To prove the continuity of e_μ, let U be an open set containing an arbitrary point $e_\mu(F,t)=F'(t)$. Since F' is continuous, there is an open neighbourhood V of t in I such that $F'(V)\subseteq U$. Hence the set $\{F\}\times V$ is an open neighbourhood of (F,t) in $Sh((I,\dot{I}),(Y_\mu,y_\mu))\times I$ such that $e_\mu(\{F\}\times V)\subseteq U$. Now, the map $\overline{g_\mu^\flat}$ is equal to the composition $e_\mu\circ(g_\mu\times id)$ and so it is continuous.

Let $\mathbf{p}:(X,x)\to(\mathbf{X},\mathbf{x})$ and $\mathbf{q}:(Y,y)\to(\mathbf{Y},\mathbf{y})$ be HPol$_*$-expansions of (X,x) and (Y,y), respectively. The map $\mathbf{g}^\flat:\Sigma(\mathbf{X},\mathbf{x})\to(\mathbf{Y},\mathbf{y})$ is a morphism in pro-HTop$_*$. To prove this, let $\mu'\geq\mu$, then there is a $\lambda\geq g(\mu),g(\mu')$ such that

$$(g_{\mu\mu'})_*\circ g_{\mu'}\circ p_{g(\mu')\lambda}\simeq g_\mu\circ p_{g(\mu)\lambda}\ \ (rel\{x_\lambda\}).$$

Since Y_μ is a polyhedron, the space $Sh((I,\dot{I}),(Y_\mu,y_\mu))$ is discrete by [6, Corollary 1]. But homotopic maps in a discrete space are equal, so

$$(g_{\mu\mu'})_*\circ g_{\mu'}\circ p_{g(\mu')\lambda}=g_\mu\circ p_{g(\mu)\lambda}. \tag{2}$$

Also, for every $x\in X_\lambda$ and $t\in I$,

$$g_\mu^\flat\circ\Sigma p_{g(\mu)\lambda}([x,t])=g_\mu^\flat([p_{g(\mu)\lambda}(x),t])=g'_{\mu p_{g(\mu)\lambda}(x)}(t)$$

and

$$q_{\mu\mu'}\circ g_{\mu'}^\flat\circ\Sigma p_{g(\mu')\lambda}([x,t])=q_{\mu\mu'}\circ g_{\mu'}^\flat([p_{g(\mu')\lambda}(x),t])=q_{\mu\mu'}\circ g'_{\mu' p_{g(\mu')\lambda}(x)}(t).$$

Also,

$$\mathcal{S}(g'_{\mu p_{g(\mu)\lambda}(x)})=g_\mu(p_{g(\mu)\lambda}(x))$$

and

$$\mathcal{S}(q_{\mu\mu'}\circ g'_{\mu' p_{g(\mu)\lambda}(x)})=q_{\mu\mu'}\circ g_{\mu'}(p_{g(\mu')\lambda}(x)).$$

Hence, using Equation (2) and [6, Theorem 1.2.4],

$$g'_{\mu p_{g(\mu)\lambda}(x)}\simeq q_{\mu\mu'}\circ g'_{\mu' p_{g(\mu)\lambda}(x)}\ \ (rel\{\dot{I}\})$$

and so $g_\mu^\flat\circ\Sigma p_{g(\mu)\lambda}\simeq q_{\mu\mu'}\circ g_{\mu'}^\flat\circ\Sigma p_{g(\mu')\lambda}\ \ (rel\{\Sigma x_\lambda\})$. □

Let $\widetilde{Sh}_*$ be a subcategory of Sh$_*$ consists of all pointed topological spaces having bi-expansions. In follow, we conclude some results in the subcategory $\widetilde{Sh}_*$. It is well-known that the pair (Σ,Ω) is an adjoint pair of functors on hTop$_*$. In the following theorem we prove similar result on subcategory $\widetilde{Sh}_*$.

Theorem 3.6. *For every $(Y,y)\in\widetilde{Sh}_*$ and $(X,x)\in Sh_*$, there is a natural bijection*

$$Sh(\Sigma(X,x),(Y,y))\cong Sh((X,x),(Sh((I,\dot{I}),(Y,y)),e_y)). \tag{3}$$

Proof. Let $\mathbf{p} : (X, x) \to (\mathbf{X}, \mathbf{x})$ be an HPol$_*$-expansion of (X, x) and $\mathbf{q} : (Y, y) \to (\mathbf{Y}, \mathbf{y})$ be a bi-expansion of (Y, y). We define

$$\tau_{XY} : Sh(\Sigma(X, x), (Y, y)) \to Sh((X, x), (Sh((I, \dot{I}), (Y, y)), e_y)),$$

by $\tau_{XY}(F) = F^\sharp$ and

$$\theta_{XY} : Sh((X, x), (Sh((I, \dot{I}), (Y, y)), e_y)) \to Sh(\Sigma(X, x), (Y, y)),$$

by $\theta_{XY}(G) = G^\flat$. By Lemmas 3.4 and 3.5, the maps τ_{XY} and θ_{XY} are well-defined. It is easy to see that $\theta_{XY} \circ \tau_{XY} = id$, $\tau_{XY} \circ \theta_{XY} = id$ and τ_{XY} is natural in each variable. Hence the result holds. □

Using natural bijection Equation (3), one can see that the functor $Sh((I, \dot{I}), -)$ preserves inverse limits such as products, pullbacks, kernels, nested intersections and completions, provided inverse limit exists in the subcategory $\widetilde{Sh}_*$. Also, the functor Σ preserves direct limits of connected spaces in this subcategory. Hence if $(X \times Y, (x, y))$ is a product of pointed spaces (X, x) and (Y, y) in the subcategory $\widetilde{Sh}_*$, then

$$Sh((I, \dot{I}), (X \times Y, (x, y))) = Sh((I, \dot{I}), (X, x)) \times Sh((I, \dot{I}), (Y, y)),$$

and so

$$\check{\pi}_1(X \times Y, (x, y)) = \check{\pi}_1(X, x) \times \check{\pi}_1(Y, y).$$

Lemma 3.7. *The mappings τ_{XY} and θ_{XY} are continuous.*

Proof. First, we show that τ_{XY} is continuous. Let V_μ^F be a basis element of $Sh((X, x), (Sh((I, \dot{I}), (Y, y)), e_y))$ containing F. We will show that $\tau_{XY}(V_\mu^{F^\flat}) \subseteq V_\mu^F$. Let $G \in V_\mu^{F^\flat}$. By definition, $q_\mu \circ F^\flat = q_\mu \circ G$ as homotopy classes to Y_μ, or equivalently $f_\mu^\flat \circ \Sigma p_{f(\mu)} \simeq g_\mu \circ \Sigma p_{g(\mu)} \ \ (rel\{\Sigma x\})$. It is sufficient to show that $(q_\mu)_* \circ F = (q_\mu)_* \circ G^\sharp$ as homotopy classes to $Sh(I, Y_\mu)$ or equivalently $f_\mu \circ p_{f(\mu)} \simeq g_\mu^\sharp \circ p_{g(\mu)} \ \ (rel\{x\})$. For every $x \in X$,

$$g_\mu^\sharp \circ p_{g(\mu)}(x) = \mathcal{S}(l_{p_{g(\mu)}(x)\mu}),$$

and for every $t \in I$,

$$l_{p_{g(\mu)}(x)\mu}(t) = g_\mu([p_{g(\mu)}(x), t]) = g_\mu \circ \Sigma p_{g(\mu)}([x, t]).$$

Also

$$\begin{aligned} f_\mu^\flat \circ \Sigma p_{f(\mu)}([x, t]) &= f_\mu^\flat([p_{f(\mu)}(x), t]) \\ &= f'_{\mu p_{f(\mu)}(x)}(t), \end{aligned}$$

where $\mathcal{S}(f'_{\mu p_{f(\mu)}(x)}) = f_\mu(p_{f(\mu)}(x))$. Since $f_\mu^\flat \circ \Sigma p_{f(\mu)} \simeq g_\mu \circ \Sigma p_{g(\mu)} \ \ (rel\{\Sigma x\})$, by the above equalities, $l_{p_{g(\mu)}(x)\mu} \simeq f'_{\mu p_{f(\mu)}(x)} \ \ (rel\{\dot{I}\})$. Thus

$$g_\mu^\sharp \circ p_{g(\mu)}(x) = \mathcal{S}(l_{p_{g(\mu)}(x)\mu}) = \mathcal{S}(f'_{\mu p_{f(\mu)}(x)}) = f_\mu(p_{f(\mu)}(x)).$$

So $\tau_{XY}(G) = G^\sharp \in V_\mu^F$, and therefore τ_{XY} is continuous. Similarly, θ_{XY} is continuous. □

In particular, we can conclude that for any pointed topological space (X,x), $Sh((I,\dot{I}),(Sh((I,\dot{I}),(X,x)),e_x)) \cong Sh((I^2,\dot{I^2}),(X,x))$. We know that for any pointed space (X,x) and for all $1 \le k \le n-1$, $\pi_n(X,x) \cong \pi_{n-k}(\Omega(X,x),e_x)$. As a result of Theorem 3.6, we have the following corollary:

Corollary 3.8. *Let (X,x) be a pointed topological space in $\widetilde{Sh}_*$. Then for all $1 \le k \le n-1$*

$$\check{\pi}_n(X,x) \cong \check{\pi}_{n-k}(Sh((S^k,*),(X,x)),e_x).$$

Proof. By the definition of the shape homotopy group and using Theorem 3.6 and Lemma 3.7, we have

$$\begin{aligned}\check{\pi}_n(X,x) &= Sh((S^n,*),(X,x)) \cong Sh((\Sigma^n S^0,*),(X,x))\\ &\cong Sh((\Sigma^{n-k}S^0,*),(Sh((S^k,*),(X,x)),e_x))\\ &\cong Sh((S^{n-k},*),(Sh((S^k,*),(X,x)),e_x))\\ &= \check{\pi}_{n-k}(Sh((S^k,*),(X,x)),e_x),\end{aligned}$$

as desired. □

In follow, we exhibit an example in which the above corollary and therefore Theorem 3.6 do not hold in the category Sh_*.

Remark 3. The pair $(\Sigma, Sh((I,\dot{I}),-))$ is not an adjoint pair of functors on the category Sh_*. By contrary, if the pair $(\Sigma, Sh((I,\dot{I}),-))$ is an adjoint pair on Sh_*, with the same argument we obtain $\check{\pi}_n(X,x) \cong \check{\pi}_{n-k}(Sh((S^k,*),(X,x)),e_x)$, for all $1 \le k \le n-1$ and for all pointed topological space (X,x). But this isomorphism does not hold in general. Put $X = S^2$ and $n = 2$, we have $\check{\pi}_2(S^2) = \pi_2(S^2) = \mathbb{Z}$ while $\check{\pi}_1(Sh(S^1,S^2))$ is trivial. Note that, S^2 is a polyhedron and so $Sh(S^1,S^2)$ is discrete by [13, Theorem 4.4]. Hence $\check{\pi}_1(Sh(S^1,S^2))$ is trivial.

Nasri et al. in [14] showed that for any pointed topological space (X,x), $\pi_n^{qtop}(X,x) \cong \pi_{n-k}^{qtop}(\Omega^k(X,x),e_x)$, for all $1 \le k \le n-1$. In the following corollary we prove this result for $\check{\pi}_n^{top}$. The following result is an immediate consequence of Corollary 3.8 and Lemma 3.7.

Corollary 3.9. *Let (X,x) be a pointed topological space in $\widetilde{Sh}_*$. Then for all $1 \le k \le n-1$*

$$\check{\pi}_n^{top}(X,x) \cong \check{\pi}_{n-k}^{top}(Sh((S^k,*),(X,x)),e_x).$$

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References

[1] N. K. Bilan, The coarse shape groups, *Topol. Appl.* **157** (2010) 894 – 901.

[2] D. Biss, The topological fundamental group and generalized covering spaces, *Topol. Appl.* **124** (2002) 355 – 371.

[3] J. Brazas, The topological fundamental group and free topological groups, *Topol. Appl.* **158** (2011) 779 – 802.

[4] J. Brazas, The fundamental group as topological group, *Topol. Appl.* **160** (2013) 170 – 188.

[5] E. Cuchillo-Ibanez, M. A. Morón and F. R. Ruiz del Portal, Ultrametric spaces, valued and semivalued groups arising from the theory of shape, *Mathematical Contributions in Honor of Juan Tarrés (Spanish),* 81 – 92*, Univ. Complut, Madrid, Fac. Mat., Madrid*, 2012.

[6] E. Cuchillo-Ibanez, M. A. Morón, F. R. Ruiz del Portal and J. M. R. Sanjurjo, A topology for the sets of shape morphisms, *Topol. Appl.* **94** (1999) 51 – 60.

[7] H. Fischer and A. Zastrow, The fundamental groups of subsets of closed surfaces inject into their first shape groups, *Algebra. Geom. Topol.* **5** (2005) 1655 – 1676.

[8] H. Ghane, Z. Hamed, B. Mashayekhy and H. Mirebrahimi, Topological homotopy groups, *Bull. Belg. Math. Soc. Simon Stevin* **15** (3) (2008) 455 – 464.

[9] S. Mardesic and J. Segal, *Shape Theory*, North-Holland, Amsterdam, 1982.

[10] M . Moszyńska, Various approach es to fundamental groups, *Fund. M ath.* **78** (1973) 107 – 118.

[11] M. A. Morón and F. R. Ruiz del Portal, Shape as a Cantor completion process, *Math. Z.* **225** (1997) 67 – 86.

[12] M. A. Morón and F. R. Ruiz del Portal, Ultrametrics and infinite dimensional Whitehead theorems in shape theory, *Manuscr. Math.* **89** (1996) 325 – 333.

[13] T. Nasri, F. Ghanei, B. Mashayekhy and H. Mirebrahimi, On topological shape homotopy groups, *Topol. Appl.* **198** (2016) 22 – 33.

[14] T. Nasri, H. Mirebrahimi and H. Torabi, Some results in topological homotopy groups, *Ukrainian Math. J.*, to appear.

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