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Multiplication on double coset space $L^1(K \setminus G/H)$

F. Fahimian[a], R. A. Kamyabi-Gol[b,*], F. Esmaeelzadeh[c]

[a] *Department of Pure Mathematics, Ferdowsi University of Mashhad, P.O.Box 1159-91775, Mashhad, Islamic Republic of Iran.*
[b] *Department of Pure Mathematics, Ferdowsi University of Mashhad and Center of Excellence in Analysis on Algebric Structures (CEAAS) Islamic Republic of Iran.*
[c] *Department of Mathematics, Bojnourd Branch, Islamic Azad university, Bojnourd, Islamic Republic of Iran.*

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Abstract

Consider a locally compact group G with two compact subgroups H and K. Equip the double coset space $K \setminus G/H$ with the quotient topology. Suppose that μ is an N-relatively invariant measure, on $K \setminus G/H$. We define a multiplication on $L^1(K \setminus G/H, \mu)$ such that this space becomes a Banach algebra that possesses a left (right) approximate identity.

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1. Introduction and Preliminaries

Suppose that G is a locally compact group and that H is a closed subgroup of G and K a compact subgroup of G. It is a fundamental fact that any locally compact group possesses a left Haar measure (a positive Radon measure which is left invariant) that is unique up to a multiplication by constants ([3, Theorems 2.10, 2.20]) and we consider the Lebesgue spaces $L^1(G)$ with respect to this measure. It is also well known that any locally compact group G has a modular function Δ_G. Liu [7] introduced the *double coset space of G by H and K* as

$$K \setminus G/H = \{KxH : \ x \in G\}.$$

In fact, a double coset space such as $K \setminus G/H$ is a natural generalization of the coset spaces arising from each of those subgroups, simultaneously. The canonical mapping $q : G \to K \setminus G/H$ defined by $q(x) = KxH$, denoted by $\ddot{x}$, is surjective. If the double coset space $K \setminus G/H$ is equipped with the quotient topology, the largest topology that makes q continuous, then q is an open mapping. Therefore, $K \setminus G/H$ is a locally compact and Hausdorff space.
Note that when K is the trivial group, it becomes the homogeneous space G/H, and when $H = K$, the double coset space is a hypergroup. Homogeneous spaces and hypergroups play important roles in physics; see [8].
For a locally compact group G, it is very well known that $L^1(G)$ is Banach algebra with the convolution as the product which strongly depends on group operations (see [3]). For the homogeneous space G/H (that is not necessarily a group), a multiplication on $L^1(G/H)$ was defined in [5] that makes $L^1(G/H)$ a Banach algebra. In this note, we aim to extend this multiplication on double coset spaces.

Let N be the normalizer of K in G, that is,

$$N = \{g \in G : \ gK = Kg\}.$$

Then the natural mapping $\varphi : N \times K \setminus G/H \to K \setminus G/H$ defined by $\varphi(n, q(x)) = KnxH$ induces a well-defined continuous action of N to $K \setminus G/H$. Consider $K \setminus G/H$ with this action, we denote $\varphi(n, q(x))$ by $n \cdot q(x)$.
It is known that the mapping $Q : C_c(G) \to C_c(K \setminus G/H)$ defined by $Q(f)(\ddot{x}) = \int_{H\times K} f(k^{-1}xh)d(\nu_1 \times \nu_2)(h, k)$, is a well-defined continuous onto linear map, as well as $(Q(f)) \subseteq q((f))$, where ν_1 and ν_2 are left Haar measures for H and K, respectively, (see [1]).

In [7], it is shown that for $n \in N$,

$$Q(L_n f) = L_n(Q(f)) \qquad (f \in C_c(G)),$$

in which L_n is the left translation operator via n i.e $L_nQ(f)(KxH) = Q(f)(KnxH)$.

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*Corresponding author
Email addresses: fatemeFahimian@gmail.com (F. Fahimian), kamyabi@um.ac.ir (R. A. Kamyabi-Gol), esmaeelzadeh@bojnourdiau.ac.ir (F. Esmaeelzadeh)

Also, we recall that a positive Radon measure μ on $K \setminus G/H$ is called an *N-relatively invariant* if there exists a positive character χ on N such that

$$\int_{K\setminus G/H} Q(f)(n\ddot{x})d\mu(\ddot{x}) = \chi(n) \int_{K\setminus G/H} Q(f)(\ddot{x})d\mu(\ddot{x}),$$

for all $n \in N$ and $f \in C_c(G)$. The character χ is called a *modular function* of μ. An N-relatively invariant measure is said to be an *N-invariant measure* if its modular function is identically 1.
For a positive Radon measure μ and $n \in N$, let μ_n denote its translate by n, that is, $\mu_n(E) = \mu(nE)$ for all Borel sets E in $K \setminus G/H$. A positive Radon measure μ is called an *N-strongly quasi-invariant measure*, if there exists a positive continuous function λ on $N \times K \setminus G/H$ such that $d\mu_n(\ddot{y}) = \lambda(n, \ddot{y})d\mu(\ddot{y})$.

For the triple (K, G, H), a *rho-function* ρ is a positive locally integrable function on G such that

$$\rho(kxh) = \frac{\Delta_H(h)\Delta_K(k)}{\Delta_G(h)}\rho(x),$$

for all $x \in G$, $h \in H$, and $k \in K$. In [1], it is explained that for each triple (K, G, H) there exists a strictly positive continuous rho-function ρ which constructs a N-strongly quasi invariant measure μ satisfying

$$\int_{K\setminus G/H} Q(f)(\ddot{x})d\mu(\ddot{x}) = \int_G f(x)\rho(x)dm(x), \tag{1.1}$$

for all $f \in C_c(G)$, where m is a left Haar measure on G. Also in [1, Theorem3.4], it is proven that $\rho : G \to (0, \infty)$ is a homomorphism if and only if there exists a N-relatively invariant measure on $K \setminus G/H$. Moreover, in this case we have

$$\chi(n) = \frac{\rho(n)}{\rho(e)},$$

and

$$\rho(nm) = \frac{\rho(n)\rho(m)}{\rho(e)}, \tag{1.2}$$

for all $m, n \in N$.
From now on, we consider the double coset space $K \setminus G/H$ with N-relatively invariant measure μ that arises from the rho-function ρ.

When G/H equips with a relatively invariant measure μ, the authors of [6] defined a convolution on $L^1(G/H, \mu)$ and proved that $L^1(G/H, \mu)$ is a Banach algebra with this convolution. The main result of this paper is devoted to characterize the structure of $L^1(K \setminus G/H, \mu)$ as a Banach algebra.
More precisely, we define and generalize a convolution on the double coset space $K \setminus G/H$. To do this, let

$$C_c(K : G : H) = \{f \in C_c(G) : \ f(k^{-1}xh) = f(x), \quad \forall x \in G, \ \forall h \in H, \ \forall k \in K\},$$

and define $f \underset{N}{*} g(x) = \int_N f(n)g(n^{-1}x)d\omega(n)$ for each $f, g \in C_c(G)$, in which ω is a left Haar measure on N. Now for $f \in C_c(G)$ and $g, h \in C_c(K : G : H)$, it can be verified that $f \underset{N}{*} g \in C_c(K : G : H)$ and $h \underset{N}{*} f \in C_c(K : G : H)$. This implies that $C_c(K : G : H)$ is a left and right ideal and therefore is a subalgebra of $C_c(G)$. We consider $L^1(K : G : H)$ as the $\| \cdot \|_{L^1(G)}$-closure of $C_c(K : G : H)$.

2. Main results

Suppose that G is a locally compact group, that H and K are compact subgroups of G, and that N is the normalizer of K in G. Throughout this paper, we denote the left Haar measure on G, H, K, and N by dm, $d\nu_1$, $d\nu_2$, and $d\omega$ and their modular functions by Δ_G, Δ_H, Δ_K, and Δ_N, respectively and μ is a N-relatively invariant measure on $K \setminus G/H$ arising from a homomorphism rho-function ρ.
In the next proposition, we investigate some properties of the linear mapping Q_ρ between $C_c(G)$ and $C_c(K \setminus G/H)$. Note that compactness of K and H implies that Q_ρ in the following proposition is injective. A property that is needed in the following proposition.

Proposition 2.1. *Suppose that H and K are compact subgroups of the locally compact group G and that μ is a relatively invariant measure on $K \setminus G/H$ that arises from the rho-function ρ. Then, for the linear mapping $Q_\rho : C_c(G) \to C_c(K \setminus G/H)$ defined by $Q_\rho(f)(\ddot{x}) = \int_{H\times K} \frac{f(k^{-1}xh)}{\rho(k^{-1}xh)} d(\nu_1 \times \nu_2)(h,k)$, we have*

(i) Q_ρ maps $C_c(K : G : H)$ onto $C_c(K \setminus G/H)$;

(ii) $C_c(K : G : H) = \{\varphi_\rho = \rho \cdot \varphi \circ q : \ \varphi \in C_c(K \setminus G/H)\}$;

(iii) $Q_\rho\big|_{C_c(K:G:H)}$ is injective.

Proof. For (i), suppose that $\varphi \in C_c(K \setminus G/H)$. Since $Q : C_c(G) \to C_c(K \setminus G/H)$ defined by $Q(f) = \int_{H\times K} f(k^{-1}xh)d(h,k)$ is surjective, there is $g \in C_c(G)$ such that $Q(g) = \varphi$. Now if we put $h = \rho \cdot g$, then $Q_\rho(h) = Q(g) = \varphi$.
To prove (ii), for $\varphi \in C_c(K \setminus G/H)$, since H and K are compact, so $\Delta_G\big|_K = \Delta_K = 1$ and $\Delta_G\big|_H = \Delta_H = 1$; hence $\varphi_\rho(kxh) = \rho \cdot \varphi \circ q(kxh) = (\frac{\Delta_H(h)\Delta_K(k)}{\Delta_G(h)}\rho(x))\varphi \circ q(kxh) = \rho(x)\varphi \circ q(x)$. Now these facts that φ_ρ is continuous and $(\varphi_\rho) \subseteq (\varphi \circ q)$ and $(\varphi \circ q)$ are compact, imply that $\varphi_\rho \in C_c(K : G : H)$. So if $f \in C_c(K : G : H)$, then $Q_\rho(f)$ is a member of $C_c(K \setminus G/H)$.
The proof of (iii) is immediate. □

Now by using the linear map Q_ρ, we are able to define a multiplication on $C_c(K \setminus G/H)$ as follows. For $\varphi, \psi \in C_c(K \setminus G/H)$ and the rho-function ρ, put $\varphi_\rho = \rho \cdot (\varphi \circ q)$ and $\psi_\rho = \rho \cdot (\psi \circ q)$, and consider

$$\begin{aligned} \sharp : C_c(K \setminus G/H) \times C_c(K \setminus G/H) &\to C_c(K \setminus G/H) \\ (\varphi, \psi) &\mapsto \varphi \sharp \psi := Q_\rho(\varphi_\rho *_N \psi_\rho). \end{aligned} \tag{2.1}$$

This linear map has the following properties. For $\varphi, \psi_1, \psi_2 \in K \setminus G/H$ we have,

(i) $\varphi\sharp(\psi_1 + \psi_2) = \varphi\sharp\psi_1 + \varphi\sharp\psi_2$.

(ii) $(\varphi + \psi_1)\sharp\psi_2 = \varphi\sharp\psi_2 + \psi_1\sharp\psi_2$.

(iii) $c(\varphi\sharp\psi) = (c\varphi)\sharp\psi = \varphi\sharp(c\psi)$.

(iv) $\varphi\sharp(\psi_1\sharp\psi_2) = (\varphi\sharp\psi_1)\sharp\psi_2$.

The properties (i), (ii), and (iii) are easy to check. For (iv), first note that by injectively Q_ρ we have $(\varphi\sharp\psi)_\rho = \varphi_\rho \underset{N}{*} \psi_\rho$ for all $\varphi, \psi \in C_c(K \setminus G/H)$. Therefore we may write

$$\begin{aligned}
(\varphi\sharp(\psi_1\sharp\psi_2))(KxH) &= Q_\rho(\varphi_\rho \underset{N}{*}(\psi_1\sharp\psi_2)_\rho)(KxH)\\
&= \int_{H\times K} \frac{\varphi_\rho \underset{N}{*}(\psi_1\sharp\psi_2)_\rho(k^{-1}xh)}{\rho(k^{-1}xh)} d(\nu_1\times\nu_2)(h,k)\\
&= \int_{H\times K}\int_N \frac{\varphi_\rho(n)(\psi_1\sharp\psi_2)_\rho(n^{-1}k^{-1}xh)}{\rho(k^{-1}xh)} d\omega(n)d(\nu_1\times\nu_2)(h,k)\\
&= \int_{H\times K}\int_N\int_N \frac{\varphi_\rho(n)\psi_{1\rho}(m)\psi_{2\rho}(m^{-1}n^{-1}k^{-1}xh)}{\rho(k^{-1}xh)} d\omega(m)d\omega(n)d(\nu_1\times\nu_2)(h,k)\\
&= \int_{H\times K}\int_N\int_N \frac{\varphi_\rho(n)\psi_{1\rho}(m)\psi_{2\rho}((nm)^{-1}k^{-1}xh)}{\rho(k^{-1}xh)} d\omega(m)d\omega(n)d(\nu_1\times\nu_2)(h,k),
\end{aligned}$$

on the other hand,

$$\begin{aligned}
((\varphi\sharp\psi_1)\sharp\psi_2)(KxH) &= Q_\rho((\varphi\sharp\psi_1)_\rho \underset{N}{*} \psi_{2\rho})(KxH)\\
&= \int_{H\times K} \frac{(\varphi\sharp\psi_1)_\rho \underset{N}{*} \psi_{2\rho}(k^{-1}xh)}{\rho(k^{-1}xh)} d(\nu_1\times\nu_2)(h,k)\\
&= \int_{H\times K}\int_N \frac{(\varphi\sharp\psi_1)_\rho(n)\psi_{2\rho}(n^{-1}k^{-1}xh)}{\rho(k^{-1}xh)} d\omega(n)d(\nu_1\times\nu_2)(h,k)\\
&= \int_{H\times K}\int_N\int_N \frac{\varphi_\rho(m)\psi_{1\rho}(m^{-1}n)\psi_{2\rho}(n^{-1}k^{-1}x)}{\rho(k^{-1}xh)} d\omega(m)d\omega(n)d(\nu_1\times\nu_2)(h,k)\\
&= \int_{H\times K}\int_N\int_N \frac{\varphi_\rho(m)\psi_{1\rho}(n)\psi_{2\rho}((mn)^{-1}k^{-1}xh)}{\rho(k^{-1}xh)} d\omega(n)d\omega(m)d(\nu_1\times\nu_2)(h,k).
\end{aligned}$$

Proposition 2.2. *Suppose that H and K are compact subgroups of the locally compact group G and that μ is an N-relatively invariant measure on $K \setminus G/H$ that arises from the rho-function ρ. Then, for all $\varphi, \psi \in C_c(K \setminus G/H)$, the multiplication defined above satisfies*

$$\varphi\sharp\psi = Q_\rho(\varphi_\rho \underset{N}{*} g),$$

for all $g \in C_c(G)$ with $Q_\rho(g) = \psi$.

Proof. Suppose that $\varphi, \psi \in C_c(K\setminus G/H)$ and $g \in C_c(G)$ with $Q_\rho(g) = \psi$. Note that in [7] it has been shown that the measure on K is invariant under inner automorphism N, that is $\nu_2(n^{-1}En) = \nu_2(E)$, for all $n \in N$ and each Borel set $E \subseteq K$. Then by this we get,

$$
\begin{aligned}
Q_\rho(\varphi_\rho \underset{N}{*} g)(KxH) &= \int_{H\times K} \frac{(\varphi_\rho \underset{N}{*} g)(k^{-1}xh)}{\rho(k^{-1}xh)} d(\nu_1\times\nu_2)(h,k) \\
&= \int_{H\times K}\int_N \varphi_\rho(n) g(n^{-1}k^{-1}xh)\frac{\rho(e)}{\rho(n)\rho(n^{-1}k^{-1}xh)} d\omega(n) d(\nu_1\times\nu_2)(h,k) \\
&= \rho(e)\int_N \frac{\varphi_\rho(n)}{\rho(n)}\int_{H\times K}\frac{g(n^{-1}k^{-1}xh)}{\rho(n^{-1}k^{-1}xh)} d(\nu_1\times\nu_2)(h,k) d\omega(n) \\
&= \rho(e)\int_N \frac{\varphi_\rho(n)}{\rho(n)}\int_{H\times K}\frac{g(k^{-1}n^{-1}xh)}{\rho(k^{-1}n^{-1}xh)} d(\nu_1\times\nu_2)(h,k) d\omega(n) \\
&= \rho(e)\int_N \frac{\varphi_\rho(n)}{\rho(n)} Q_\rho g(Kn^{-1}xH) d\omega(n) \\
&= \frac{\rho(e)}{\rho(e)}\rho(x^{-1})\int_N \varphi_\rho(n)\rho(n^{-1}x)\psi(Kn^{-1}xH) d\omega(n) \\
&= \rho(x^{-1})\int_N \varphi_\rho(n)\psi_\rho(n^{-1}x) d\omega(n) \\
&= \rho(x^{-1})\varphi_\rho \underset{N}{*}{}_{,} \psi_\rho(x),
\end{aligned}
$$

for all $x \in G$. Furthermore, the equality $(\varphi\sharp\psi)_\rho = \varphi_\rho \underset{N}{*} \psi_\rho$, implies that $\rho.(\varphi\sharp\psi)\circ q(x) = \rho(x) Q_\rho(\varphi_\rho \underset{N}{*} g)(\ddot{x})$. So, $(\varphi\sharp\psi)(\ddot{x}) = Q_\rho(\varphi_\rho \underset{N}{*} g)(\ddot{x})$.

□

At this point, we recall that if X and Y are dense subspaces of Banach spaces $\tilde{X}$ and $\tilde{Y}$, respectively, then every bounded linear map $T : X \to Y$ has a unique extension $\tilde{T} : \tilde{X} \to \tilde{Y}$.
In the following theorem, we show that the convolution defined in Proposition 2.1 can be extended to a convolution on $L^1(K\setminus G/H, \mu)$.

Theorem 2.3. *With the assumptions as in Proposition 2.2, the convolution defined in Proposition 2.2 can be uniquely extended to a convolution*

$$\sharp : L^1(K\setminus G/H, \mu)\times L^1(K\setminus G/H, \mu) \to L^1(K\setminus G/H, \mu),$$

which makes $L^1(K\setminus G/H, \mu)$ into a Banach algebra.

Proof. Suppose that $\varphi \in C_c(K\setminus G/H)$. Equation (1.1) implies that

$$
\begin{aligned}
\|\varphi\|_1 &= \int_{K\setminus G/H} |\varphi|(\ddot{x}) d\mu(\ddot{x}) \\
&= \int_{K\setminus G/H} Q_\rho(|\varphi_\rho|)(\ddot{x}) d\mu(\ddot{x}) \\
&= \int_G |\varphi|_\rho(x) dm(x) = \|\varphi_\rho\|_1.
\end{aligned}
$$

Now let $\varphi, \psi \in C_c(K \setminus G/H)$; then

$$\begin{aligned}
\|\varphi \sharp \psi\|_1 &= \|Q_\rho(\varphi_\rho \underset{N}{*} \psi_\rho)\|_1 \\
&= \|\varphi_\rho \underset{N}{*} \psi_\rho\|_{L^1(G)} \\
&\leq \|\varphi_\rho\|_1 \|\psi_\rho\|_1 \\
&= \|Q_\rho(\varphi_\rho)\|_1 \|Q_\rho(\psi_\rho)\|_1 \\
&= \|\varphi\|_1 \|\psi\|_1.
\end{aligned}$$

Hence, $\sharp$ can be extended to $L^1(K \setminus G/H, \mu)$. □

The following corollary shows that $L^1(K : G : H)$ and $L^1(K \setminus G/H, \mu)$ are isometrically isomorphic.

Corollary 2.4. *Suppose that H and K are compact subgroups of G, and let μ be a relatively invariant measure that arises from the rho-function ρ. Then $Q_\rho : L^1(K : G : H) \to L^1(K \setminus G/H, \mu)$ is an isometrical isomorphism.*

Proof. The first part of the proof of Theorem 2.3 shows that Q_ρ from $L^1(K : G : H)$ to $L^1(K \setminus G/H)$ is an isometry. Also since $\overline{C_c(K : G : H)}^{\|\cdot\|_1} = L^1(K : G : H)$ and $L^1(K \setminus G/H, \mu) = \overline{C_c(K \setminus G/H)}^{\|\cdot\|_1}$, then by Proposition 2.1 and by the statements preceding of Theorem 2.3, the result is achieved. □

Note that by Theorem 2.3 and Corollary 2.4, $L^1(K \setminus G/H, \mu)$ is a Banach algebra.
If $K \lhd G$ and μ is an N-strongly quasi-invariant measure that arises from the rho-function ρ, then $L^p(K \setminus G/H, \mu)$ is a Banach left $L^1(G)$-module for all $1 \leq p \leq +\infty$ and the left action is defined as

$$\begin{aligned}
L^1(G) \times L^p(K \setminus G/H, \mu) &\to L^p(K \setminus G/H, \mu) \\
(f, \psi) &\mapsto Q_p(f * g),
\end{aligned}$$

in which $g \in L^p(G)$ and $\psi = Q_p(g)$.
Generally, we can redefine the modular action as follows:

$$\begin{aligned}
L^1(G) \times_N L^p(K \setminus G/H, \mu) &\to L^p(K \setminus G/H, \mu) \\
(f, \psi) &\mapsto Q_p(f \underset{N}{*} g),
\end{aligned}$$

in which $g \in L^p(G)$, $\psi = Q_p(g)$ and

$$Q_p(f \underset{N}{*} g)(\ddot{x}) = \int_{H \times K} \frac{(f \underset{N}{*} g)(k^{-1}xh)}{\rho^{\frac{1}{p}}(k^{-1}xh)} d(\nu_1 \times \nu_2)(h, k). \tag{2.2}$$

This modular action is also well-defined. This is because, $\ker Q_p$ is an invariant subspace of $L^p(G)$ under the modular action and also if $f \in L^1(G)$ and $g \in \ker Q_p$, then $\rho^{\frac{1}{p}}(Q_p g \circ q) = 0$ in

$L^p(G)$. Hence for almost all $x \in G$ and almost all $\ddot{x} \in K \setminus G/H$, we have

$$\begin{aligned}
Q_p(f \underset{N}{*} g)(\ddot{x}) &= \int_{H\times K} \frac{(f \underset{N}{*} g)(k^{-1}xh)}{\rho^{\frac{1}{p}}(k^{-1}xh)} d(\nu_1 \times \nu_2)(h,k) \\
&= \int_{H\times K} \int_N \frac{f(n)g(n^{-1}k^{-1}xh)}{\rho^{\frac{1}{p}}(k^{-1}xh)} d\omega(n) d(\nu_1 \times \nu_2)(h,k) \\
&= \int_N \Big(\int_{H\times K} \frac{f(n)g(n^{-1}k^{-1}xh)}{\rho^{\frac{1}{p}}(k^{-1}xh)} d(\nu_1 \times \nu_2)(h,k) \Big) d\omega(n) \\
&= \frac{1}{\rho^{\frac{1}{p}}(x)} \int_{H\times K} \Big(\int_N f(n) g(kn^{-1}xh) d\omega(n) \Big) d(\nu_1 \times \nu_2)(h,k) \\
&= \frac{1}{\rho^{\frac{1}{p}}(x)} \int_N f(n) \Big(\int_{H\times K} \frac{g(k^{-1}n^{-1}xh)}{\rho^{\frac{1}{p}}(k^{-1}n^{-1}xh)} \rho^{\frac{1}{p}}(k^{-1}n^{-1}xh) d(\nu_1 \times \nu_2)(h,k) d\omega(n) \Big) \\
&= \frac{1}{\rho^{\frac{1}{p}}(x)} \int_N f(n) \rho^{\frac{1}{p}} (Q_p g \circ q)(n^{-1}x) d\omega(n) \\
&= \frac{1}{\rho^{\frac{1}{p}}(x)} f \underset{N}{*} \rho^{\frac{1}{p}} (Q_p g \circ q)(x) = 0.
\end{aligned}$$

In the following proposition, we show that the Banach algebra $L^1(K \setminus G/H, \mu)$ always possesses a right approximation identity.

Proposition 2.5. *Suppose that H and K are compact subgroups of the locally compact group G and that μ is a relatively invariant measure on $K \setminus G/H$. Then the Banach algebra $L^1(K \setminus G/H, \mu)$ possesses a right (left) approximate identity.*

Proof. Let $\{\beta_\alpha\}_{\alpha\in I}$ be an approximation identity for $L^1(G)$; see [3]. For all $\alpha \in I$, let $\psi_\alpha = Q_\rho(\beta_\alpha)$. Now using Proposition 2.1, for each $\varphi \in L^1(K \setminus G/H, \mu)$, we have

$$\begin{aligned}
\lim_{\alpha\in I} \|\varphi \sharp \psi_\alpha - \varphi\|_{L^1(K\setminus G/H,\mu)} &= \lim_{\alpha\in I} \|Q_\rho(\varphi_\rho \underset{N}{*} \beta_\alpha - \varphi_\rho)\|_{L^1(K\setminus G/H,\mu)} \\
&= \lim_{\alpha\in I} \|\varphi_\rho \underset{N}{*} \beta_\alpha - \varphi_\rho\|_{L^1(G)} = 0.
\end{aligned}$$

Similarly, one can show that $L^1(K \setminus G/H, \mu)$ has a left approximate identity. □

Lemma 2.6. *Suppose that H and K are compact subgroups of the locally compact group G and that μ is a relatively invariant measure on $K \setminus G/H$ that arises from the rho-function ρ. Then for all $\varphi, \psi \in L^1(K \setminus G/H, \mu)$, we have*

(i) $\varphi \sharp \psi(\ddot{x}) = \rho(e) \int_N \frac{\varphi(n)}{\rho(n)} \psi(n^{-1}\ddot{x}) d\omega(n)$, *for μ-almost all $\ddot{x} \in K \setminus G/H$,*

(ii) $\|L_n\varphi\|_1 = \frac{\rho(n)}{\rho(e)} \|\varphi\|_1$.

Proof. (i) First, let $\varphi, \psi \in C_c(K \setminus G/H)$. Then

$$\begin{aligned}
\varphi\sharp\psi(\ddot{x}) &= Q_\rho(\varphi_\rho \underset{N}{*} \psi_\rho)(\ddot{x}) \\
&= \int_{K\setminus G/H}\int_N \frac{\varphi_\rho(n)\psi_\rho(n^{-1}k^{-1}xh)}{\rho(k^{-1}xh)} d\omega(n) d(\nu_1 \times \nu_2)(h,k) \\
&= \rho(e)\int_N \frac{\varphi_\rho(n)}{\rho(n)} \int_{K\setminus G/H} \frac{\psi_\rho(k^{-1}n^{-1}xh)}{\rho(k^{-1}n^{-1}xh)} d(\nu_1 \times \nu_2)(h,k) d\omega(n) \\
&= \rho(e)\int_N \frac{\varphi_\rho(n)}{\rho(n)} Q_\rho(\psi_\rho)(n^{-1}\ddot{x}) d\omega(n) \\
&= \rho(e)\int_N \frac{\varphi_\rho(n)}{\rho(n)} \psi(n^{-1}\ddot{x}) d\omega(n).
\end{aligned}$$

Since $C_c(K \setminus G/H)$ is dense in $L^1(K \setminus G/H, \mu)$, we conclude that

$$\varphi\sharp\psi(\ddot{x}) = \rho(e)\int_N \frac{\varphi_\rho(n)}{\rho(n)} \psi(n^{-1}\ddot{x}) d\omega(n),$$

for μ-almost all $\ddot{x} \in K \setminus G/H$.

(ii) Let $n \in N$ and let $\varphi \in L^1(K \setminus G/H, \mu)$; then

$$\begin{aligned}
\|L_n\varphi\|_1 &= \int_{K\setminus G/H} |L_n\varphi(\ddot{x})| d\mu(\ddot{x}) \\
&= \int_{K\setminus G/H} |\varphi(n^{-1}\ddot{x})| d\mu(\ddot{x}) \\
&= \int_{K\setminus G/H} |\varphi(Kn^{-1}xH)| d\mu(\ddot{x}) \\
&= \int_{K\setminus G/H} \frac{\rho(n)}{\rho(e)} |\varphi(\ddot{x})| d\mu(\ddot{x}) \\
&= \frac{\rho(n)}{\rho(e)} \|\varphi\|_1,
\end{aligned}$$

and the proof is complete.

□

At the end, we give a necessary and sufficient condition on a closed subspace of $L^1(K \setminus G/H, \mu)$ to be a left ideal, where μ is an N-invariant measure on $K \setminus G/H$. However, first consider the following remark.

Remark 2.7. Let H and K be compact subgroups of G and let μ be an N-invariant measure on G.

Then

$$\begin{aligned} L_n(\varphi\sharp\psi) &= L_n(Q_\rho(\varphi_\rho *_N \psi_\rho)) \\ &= Q_\rho(L_n(\varphi_\rho *_N \psi_\rho)) \\ &= Q_\rho(L_n(\varphi_\rho) *_N \psi_\rho) \\ &= Q_\rho((L_n\varphi)_\rho *_N \psi_\rho) \\ &= L_n\varphi\sharp\psi, \end{aligned}$$

for all $n \in N$ and $\varphi, \psi \in L^1(K \setminus G/H, \mu)$. Therefore

$$L_n(\varphi\sharp\psi) = L_n\varphi\sharp\psi. \tag{2.3}$$

We conclude it by the characterization of the closed ideal in $L^1(K \setminus G/H, \mu)$, where μ is N-invariant measure on the double coset space $K \setminus G/H$.

Theorem 2.8. *Suppose that μ is an N-invariant measure on $K\backslash G/H$ and that I is a closed subspace of $L^1(K \setminus G/H, \mu)$. Then I is a left ideal if and only if it is closed under the left N-translation.*

Proof. Suppose that I is a left ideal, that $\{\psi_U\}_{U\in\mathcal{U}}$ is an approximate identity, and that $\varphi \in I$. Then, for all $n \in N$, by applying Lemma 2.7, we obtain $L_n\varphi = \lim_{U\to\{e\}} L_n(\psi_U\sharp\varphi) = \lim(L_n\psi_U)\sharp\varphi$, which shows that $L_n\varphi \in I$.
For the converse, suppose that I is closed under the left N-translation. According to Lemma 2.6, for all $\varphi \in L^1(K \setminus G/H, \mu)$ and $\psi \in I$, we have $\varphi\sharp\psi$ which is a member of the closed linear span of the left N-translation of ψ; therefore $\varphi\sharp\psi \in I$. □

Remark 2.9. Note that if $K = H$, then $L^1(G//H, \mu)$ has a Banach structural, and this space is a hypergroup and all the results achieved through are true.

References

[1] F. Fahimian, R.A. Kamyabi-Gol and F. Esmaelzadeh, N-relatively invariant and N-invariant measure on double coset spaces, *Bull. Iran. Math. Soc.*, **45**(2) (2019), 515–525.

[2] F. Fahimian, R.A. Kamyabi-Gol and F. Esmaelzadeh, N-strongly quasi-invariant measure on double coset space, *arXiv: 1807.00132[mathRT].*

[3] G.B. Folland, *A Course in Abstract Harmonic Analysis*, Boca Raton, CRC press, Inc., 1995.

[4] G.B. Folland, *Real Analysis*, John Wiley and Sons, New York, 1999.

[5] A. Ghaani Farashahi, A class of abstract linear representations for convolution function algebras over homogeneous spaces of compact group, *The Canadian Journal of Mathematics (CJM)*, **70**(1) (2018), 97–116.

[6] R.A. Kamyabi-Gol and N. Tavallaei, Convolution and homogeneous spaces, *Bull. Iran. Math. Soc.*, **35**(1) (2009), 129–146.

[7] T.S. Liu, Invariant Measure on Double Coset Spaces, *University of Pennsylvania and university Massachusetts*, 1965.

[8] Kh. Trimeche, *Generalized Wavelets and Hyper Groups*, Gordon and Breach Science Publishers, Amsterdam , 1997.