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Ex-post favoring ranks: a fairness notion for the random assignment problem

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Abstract
We introduce two notions of ex-post fairness, namely *ex-post favoring ranks* (EFR) and *robust ex-post favoring ranks*, which consider whether objects are received by those agents who have the highest rank for them. We examine their compatibility with standard properties of random assignments and state some impossibility theorems. We also propose and formalize a revised version of the Boston mechanism and prove that it provides an EFR random assignment.

Keywords Random assignment problem · Favoring higher ranks · Interim favoring ranks · Ex-post favoring ranks · Robust ex-post favoring ranks

JEL Classification D61 · D63 · D82

1 Introduction

An assignment problem seeks to allocate a finite set of indivisible objects, without recourse to money, to a set of agents who have reported their ordinal preferences over objects, and each is entitled to at most one of them. One of the main properties of an assignment is fairness, which has different variations. Nevertheless, no matter how we define or understand fairness, it is quite difficult, if not impossible, to fairly assign indivisible objects, ex-post, in a deterministic setting. Therefore, to attain fairness, at least from an ex-ante perspective, lotteries have been used to make a random assignment. However, since in real-world settings, agents could not observe the full random assignment, but rather a deterministic assignment from one of its decomposition, fairness concern mostly matters for the deterministic outcome ex-post.

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Our contribution in this paper is that we introduce two notions of ex-post fairness, namely *ex-post favoring ranks* (EFR) and *robust ex-post favoring ranks* (REFR), which, in contrast to the recent papers, are not based on an approximation of fairness, but rather an established fairness notion, called *favoring higher rank* (FHR), introduced by Kojima and Ünver (2014). In deterministic assignments, the typical notion of fairness, i.e., envy-freeness, realized when each agent (weakly) prefers her own object to the object of any other agent, no matter how others appreciate their assigned object. However, favoring higher rank takes into account whether the envy is justified, in the sense that which agent is more rightful to get (has a higher rank for) a contested object.

We assess how our proposed notions of fairness are related to different notions of efficiency to give a big picture of how different notions of fairness and efficiency in random assignment problems are associated with each other. We show that (robust) ex-post favoring ranks is a refinement of (robust) ex-post Pareto efficiency, and it is not logically related to neither ordinal efficiency nor (weak) envy-freeness. These relationships give a hierarchy of different notions of fairness for random assignments.

The Boston mechanism, studied by Abdulkadiroğlu and Sönmez (2003), has been already used to allocate objects. Doğan and Klaus (2018) formalized its desirable properties when respecting the relative rank of a school among the students' preferences is essential. It is easily seen that the FHR is a characteristics feature of the Boston mechanism (Kojima and Ünver 2014; Chen 2016). Last but not the least, we propose and formalize a revised version of the Boston mechanism and prove that it provides an ex-post favoring ranks random assignment.

The literature on the assignment of indivisible goods is concentrated on ex-ante fairness (e.g., Nesterov 2017; Basteck 2018). While there is experimental evidence that people demonstrate concern for ex-post fairness in a probabilistic assignment (e.g., Karni et al. 2008), it only recently attracted more attention. For instance, Freeman et al. (2020) designed an efficient algorithm that is exactly fair ex-ante and approximately fair ex-post simultaneously. Aziz (2020) also presented an algorithm that computes an ex-ante envy-free lottery over deterministic allocations which are envy-free up to one item.

The paper is organized as follows. In Sect. 2, we review the standard model and axioms of random assignments. In Sect. 3, we introduce (robust) ex-post favoring ranks. Section 4 examines its relationship with other notions of efficiency, strategy-proofness, and fairness, especially those that care for favoring higher ranks and prove some impossibility theorems. In Sect. 5, we prove that a revised version of the Boston mechanism is ex-post favoring ranks.

2 Model

Let A be a finite set of objects which should be assigned to a finite set of agents, N, with $|A| = |N| = n$. Each agent $i \in N$ has a complete, transitive, and anti-symmetric *strict preference relation* $\succ_i$ over A. We denote a *preference profile by* $\succ \equiv (\succ_i)_{i \in N}$ and the domain of those preferences by $\mathcal{F}$. Each agent $i \in N$ has a ranking over any object $a \in A$, which we represent by $rk(a, \succ_i)$, where $rk(a, \succ_i) = |\{b \in A | b \succ_i a\}| + 1$.

We represent a *random assignment* by a *bistochastic matrix*[1] $P = [p_{ia}]_{i\in N, a\in A}$, with agents on rows and objects on columns, where p_{ia} is the probability of assigning object a to agent i. We denote the domain of random assignments by R. A *random allocation* for some agent $i \in N$, P_i, is a probabilistic distribution over all objects in A where the sum of probabilities of assigning objects to the agent i equals to 1. A *deterministic assignment*, $\Pi = [\Pi_{ia}]_{i\in N, a\in A}$, is a particular case of random assignment where its entries are all either 0 or 1. The Birkhoff–von Neumann Theorem states that all random assignments can be decomposed as a probability distribution over deterministic assignments and can be implemented in practice.

An important feature of any assignment is efficiency, which has different notions: *Pareto efficiency* (PE) means that there are no Pareto-improving *trading cycles*[2] such that each agent improves her assigned object. Given a preference profile $\succ$, a random assignment P, and for all a,$b \in A$, we define a binary relation τ in A as $a\tau\,(P, \succ)\,b$ if and only if there exists $i \in N$ such that $a \succ_i b$ and $p_{ib} > 0$. Bogomolnaia and Moulin (2001) (Lemma 3, page 304) proved that the random assignment P, $P \in R$, is *ordinally efficient* at a profile $\succ$ if and only if the relation $\tau\,(P, \succ)$ is acyclic, i.e., there is no Pareto improving trading cycle in probability shares in a random assignment.

Put it differently, upon enumerating objects in A for agent i from best to worst according to $a_{i,1} \succ_i a_{i,2} \succ_i a_{i,3} \succ_i \ldots \succ_i a_{i,n}$, where $a_{i,k}$ is the kth best object of agent i, we define $u^P_{ir} = \sum_{k=1}^{r} p_{ia_{i,k}}$ to be the summation of probabilities of receiving the first r best objects of agent i in the random assignment P. Given a preference ordering $\succ_i$ on A, the stochastic dominance relation associated with $\succ_i$is denoted by $\succ^{sd}_i$, where $P_i \succ^{sd}_i Q_i$ if and only if $u^P_{ir} \geq u^Q_{ir}$ for $r = 1, \ldots, n$.

Ex-post Pareto (EP) efficiency for a random assignment requires that there exists a lottery over deterministic assignments where all of them are Pareto efficient. An assignment is *robust ex-post Pareto* (REP) efficient if and only if any of its decomposition is a lottery over Pareto optimal deterministic assignments (Aziz et al. 2015). REP is a weaker notion than ordinal efficiency but stronger than ex-post Pareto efficiency.

A mechanism, given preferences of all agents, provides us with a procedure to assign objects to agents. More formally, a mechanism $\mu(.)$ is a function from F^n into R, that associates each preference profile with some random assignment. When an agent truthfully reports her preferences, she should find her allocation at least as desirable as any allocation she might obtain by misreporting her preferences. More precisely, a mechanism $\mu(.)$ is *strategy-proof (SP)* whenever for all preference profile $\succ \equiv \left(\succ_j\right)_{j\in N}$, and for each $i \in N$, $\mu_i(\succ_i, \succ_{-i}) \succ^{sd}_i \mu_i(\succ'_i, \succ_{-i})$ for all $\succ'_i \neq \succ_i$. A mechanism $\mu(.)$ is *weakly strategy-proof (WSP)* whenever for all preference profile $\succ \equiv (\succ_i)_{i\in N}$, and for each $i \in N$, if $\mu_i(\succ'_i, \succ_{-i}) \succ^{sd}_i \mu_i(\succ_i, \succ_{-i})$ then $\mu_i(\succ'_i, \succ_{-i}) = \mu_i(\succ_i, \succ_{-i})$, for all $\succ'_i \neq \succ_i$.

[1] A bistochastic matrix is a square matrix with nonnegative real numbers that the summation of elements in each of its rows and columns equals to 1.

[2] The trading cycle is a sequence of agent/object pairs, where each agent prefers (the fraction of) the object of the next pair to her, in a (probabilistic) deterministic setting. Therefore, each agent is willing to trade (a probability of getting) her object, in a (probabilistic) deterministic setting, with the agent in the successor pair. For the very last pair, the next pair is defined to be the very first one.

If an agent finds her allocation at least as desirable as an allocation of another one, he does not envy. In other words, for each $\succ \in \digamma^n$ and all $i, j \in N$, a random assignment $P \in R$ is *envy-free (EF)* if $P_i \succ_i^{sd} P_j$. If an agent does not find an allocation of another one strictly better than her own allocation, he does not envy in the weak sense. More precisely, for each $\succ \in \digamma^n$ and all $i, j \in N$, a random assignment $P \in R$ is *weakly envy-free (WEF)* if $P_j \succ_i^{sd} P_i$ then $P_i = P_j$.

Equal treatment of equals (ETE) requires that for each $i, j \in N$ with $\succ_i = \succ_j$ the allocation of both agents i and j are identical: $P_i = P_j$. Nesterov (2017) introduced a stronger notion of fairness called *strong equal treatment of equals (SETE).* An assignment satisfies SETE if any two agents with identical preferences from the top object down to some particular one receive identical assignments from the top down to that object. More formally, a random assignment P satisfies SETE, whenever for every two arbitrary agents i and j, if the first kth best object of agent i and j are the same, then for all objects a with rank less than or equal to k in preference of agent i, i.e. $rk(a, \succ_i) \leq k$, we have $p_{ia} = p_{ja}$.

3 Favoring ranks in random assignment problems

We introduce two new notions of ex-post fairness, namely *ex-post favoring ranks* (EFR) and *robust ex-post favoring ranks* (REFR). We consider a random assignment fair if it admits a convex decomposition over *favoring higher ranks (FHR)* deterministic assignments: A random assignment is ex-post favoring ranks if it is equal to a lottery of some FHR deterministic assignments. We also say a mechanism (rule) is EFR whenever for every preference profile, it outputs an EFR random assignment. A random assignment is REFR if all its decompositions are into FHR deterministic assignments. When a random assignment is REFR, it is guaranteed that any of its possible realizations leads to deterministic assignments that all favor higher ranks.

This idea of FHR was first codified by Kojima and Ünver (2014): A deterministic assignment Π is favoring higher ranks when for some agent j and object b, Π assigns b to j, i.e., $\Pi_{jb} = 1$, while there is another agent i who prefers b more, i.e., $rk(b, \succ_i) < rk(b, \succ_j)$, then Π does not assign to i any inferior object, such as c that $rk(c, \succ_i) > rk(b, \succ_i)$, i.e., $\Pi_{ic} = 0$. Note that the notion of FHR differs from EF. Lemma 1 depicts this difference.

Lemma 1 *For deterministic assignments, EF implies FHR, but the reverse does not hold.*

Proof Suppose that Π is EF and assigns b to j, i.e., $\Pi_{jb} = 1$, while there is another agent i who prefers b more, i.e., $rk(b, \succ_i) < rk(b, \succ_j)$. Assume that for some object a, we have $\Pi_{ia} = 1$. Since Π is EF, agent i does not envy agent j and thus $rk(a, \succ_i) < rk(b, \succ_i)$, and therefore Π does not assign to i any inferior object, such as c that $rk(c, \succ_i) > rk(b, \succ_i)$, i.e., $\Pi_{ic} = 0$. Note that the reverse does not hold. For these preferences

$$\begin{array}{l} 1 : a \succ_1 b \\ 2 : a \succ_2 b \text{'} \end{array} \tag{1}$$

the deterministic assignment that gives a to agent 1 and b to agent 2 favors higher rank while it is not envy-free since agent 2 envies agent 1.

The *interim favoring ranks* (IFR) concept of Harless (2018)[3] could be considered as one of the possible extensions of FHR for random assignments: A random assignment P is *interim favoring ranks* whenfor every agent j and object b, if P assigns b to j, i.e., $p_{jb} > 0$, while there is another agent i who prefers b more, i.e., $rk(b, \succ_i) < rk(b, \succ_j)$, then P does not assign to i any inferior object, such as c that $rk(c, \succ_i) > rk(b, \succ_i)$, i.e., $p_{ic} = 0$. The next Lemma shows that the notion of IFR is different from EF and WEF, as none of them could imply the other one.

Lemma 2 *There exists an IFR assignment that fails to be EF or WEF, and there exists an EF and WEF assignment that fails to be IFR.*

Proof The random assignment

$$P = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix},$$

for these preferences

$$\begin{array}{l} 1: a \succ_1 b \\ 2: b \succ_2 a \end{array},$$

is both EF and WEF but not IFR since there is a chance that the random assignment gives object b to agent 1, i.e., $p_{1b} > 0$, and while agent 2 prefers b more, $rk(b, \succ_2) < rk(b, \succ_1)$, P assigns her the inferior object a, $p_{2a} > 0$. Moreover, the random assignment

$$P = \begin{pmatrix} 1/4 & 3/4 \\ 3/4 & 1/4 \end{pmatrix},$$

for preferences (1), is IFR but it is neither EF since agent 1 envies agent 2, i.e., $P_2 \succ_1^{sd} P_1$, nor WEF since $P_2 \succ_1^{sd} P_1$ but $P_2 \neq P_1$.

Example 1 shows that IFR is different from EFR as well.

Example 1 A random assignment which is not IFR but it is a lottery of FHR deterministic assignments.

Let $N = \{1, 2, 3, 4\}$ *be the set of agents and* $A = \{a, b, c, d\}$ *be the set of objects. For these preferences,*

$$\begin{array}{lccccccc} 1: & a & \succ_1 & b & \succ_1 & d & \succ_1 & c \\ 2: & d & \succ_2 & a & \succ_2 & b & \succ_2 & c \\ 3: & a & \succ_3 & d & \succ_3 & b & \succ_3 & c \\ 4: & d & \succ_4 & b & \succ_4 & a & \succ_4 & c \end{array}, \tag{2}$$

[3] Harless (2018) named it *Respect for Rank* However, to make the connection with Kojima and Ünver (2014) clearer, we rename this concept to *in*terim favoring rank.

we could think of a random assignment

$$P = \overbrace{\begin{pmatrix} 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \end{pmatrix}}^{\text{Not IFR}} = \frac{1}{2}\overbrace{\begin{pmatrix} 0\,0\,1\,0 \\ 0\,0\,0\,1 \\ 1\,0\,0\,0 \\ 0\,1\,0\,0 \end{pmatrix}}^{\text{FHR}} + \frac{1}{2}\overbrace{\begin{pmatrix} 1\,0\,0\,0 \\ 0\,1\,0\,0 \\ 0\,0\,1\,0 \\ 0\,0\,0\,1 \end{pmatrix}}^{\text{FHR}}, \tag{3}$$

which is not IFR since agent 1 *ranksb higher than agent* 2 *does, i.e.,* $rk(b, \succ_1) < rk(b, \succ_2)$*, but* P assigns b to agent 2 with positive probability, i.e., $p_{2b} > 0$, while gives agent 1 her inferior object c, i.e., $rk(c, \succ_1) > rk(b, \succ_1)$, with positive probability, i.e., $p_{1c} > 0$. *However, P can be represented as a lottery over two deterministic assignments, both favor higher ranks. Therefore, for this realization of the lottery, no agent could have a valid rank-based complaint that what she prefers more, has been given to another agent who has a lower rank for it in her preference.*

Example 1 gives us another reason to be interested in EFR or REFR. We believe that IFR is quite strict, as the random assignment in Example 1 is not IFR, but it could be decomposed into deterministic assignments that all favor higher ranks. Moreover, we typically face impossibilities where various desiderata are not compatible with each other. A very fair mechanism, in the sense of IFR, might come at the cost of losing other properties. Therefore, as REFR and EFR are less restricted fairness notions, they make it more probable to fulfill other desirable properties.

While the notion of FHR captures a new kind of fairness, it fails to provide a fair division in a setting where some agents have the same preference over a set of objects. Randomization over different possible deterministic assignments could give these agents more chance to get their favorite objects and increase fairness. IFR is one way to make lottery over some deterministic assignments that prioritizes entitlements to objects according to their rank. The main concern in IFR is to eliminate ex-ante dissatisfaction of each agent due to receiving a less preferred object while her more preferred one has been handed to another agent who prefers it less. However, EFR addresses this concern from the ex-post perspective.

4 Characteristics of (robust) ex-post favoring ranks

As Fig. 1 illustrates, there is a hierarchy of relationships between different notions of efficiency, while so far there has not been similar notions for fairness in the literature to build an analogous hierarchy upon. In this Section, we characterize EFR and REFR and deduce their relationship with different notions of efficiency, according to Fig. 1, as well as strategy-proofness and fairness notions

We already know that OE implies REP. Could we draw the same logical relationship for IFR and REFR as well? The first Proposition confirms that we have the same resemblance.

Proposition 1 *IFR implies REFR but not conversely.*

Proof See the Appendix. □

	Efficiency Notions		**Fairness Notions**
Deterministic Assignments	Pareto Efficiency (PE)	$\Longleftarrow$	Favoring Higher Ranks (FHR)
	Ordinal Efficiency (OE)	*Harless (2018)* $\Longleftarrow$	Interim Favoring Ranks (IFR)
	$\Downarrow$ *Aziz et al. (2015)*		$\Downarrow$?
Probabilistic Assignments	Robust Ex-post Pareto (REP)	$\overset{?}{\Longleftarrow}$	*Robust Ex-post Favoring Ranks (REFR)*
	$\Downarrow$ *Aziz et al. (2015)*		$\Downarrow$?
	Ex-post Pareto (EP)	$\overset{?}{\Longleftarrow}$	*Ex-post Favoring Ranks (EFR)*

Fig. 1 Analogy between notions of efficiency and fairness

Now it is clear that IFR implies REFR (Proposition 1) and OE (Harless 2018), and REFR implies EFR by definition, and REP (since a REFR random assignment could be decomposed into FHR deterministic assignments which are all Pareto efficient). Moreover, REP is a logical consequence of OE. In order to have a complete picture of how all these different notions are associated, we should examine, on the one hand, how REFR and EFR are related to OE, and on the other hand, how EFR is linked to REP. Proposition 2 addresses all these relationships.

Proposition 2 *There exist*

(i) *REFR assignments that fail to be OE, and there exist OE assignments that fail to be REFR. A random assignment that satisfies both OE and REFR is not necessarily IFR.*
(ii) *EFR assignments that fail to be OE, and there exist OE assignments that fail to be REFR. A random assignment that satisfies both OE and EFR is not necessarily IFR.*
(iii) *EFR assignments that fail to be REP, and there exist REP assignments that fail to be EFR. A random assignment that satisfies both REP and EFR is not necessarily REFR.*

Proof See the Appendix. □

Proposition 2 shows that neither REFR nor OE implies the other one, and their combination is not necessarily IFR. Moreover, because REFR implies EFR, an IFR random assignment is EFR as well. While IFR random assignment is also OE, the question arises whether EFR or OE might indicate the other one and if not whether their combination always implies IFR. Proposition 2 shows that none of these two conjectures are indeed the case.

It is also trivial that EFR implies EP (since an EFR random assignment has a decomposition into FHR deterministic assignments which all are Pareto efficient).

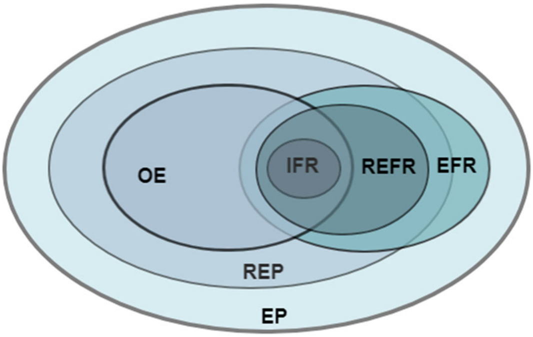

Fig. 2 Relationship of EFR and REFR with notions of efficiencies

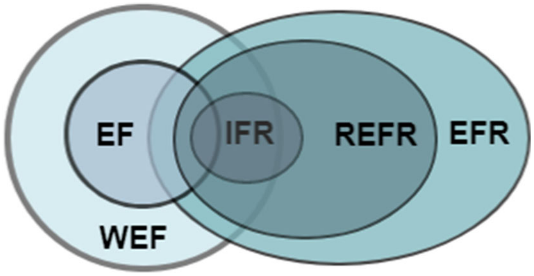

Fig. 3 Relationship of EFR and REFR with EF and WEF

Therefore, we could summarize our characteristics of EFR and REFR with respect to IFR and different notions of efficiency in Fig. 2.

The following impossibility result states that EFR is not logically related to EF since, for some profile of preferences, no mechanism could provide a random assignment that meets both EFR and EF.

Proposition 3 *EFR and EF are logically unrelated.*

Proof See the Appendix. □

As it is mentioned in Lemma 2, IFR is compatible with neither EF nor WEF. Therefore, EFR and REFR are new fairness concepts that are logically distinct from IFR, EF, and WEF. Figure 3 gives us a broad picture of how to associate these different notions of fairness.

Bogomolnaia and Moulin (2001) (Theorem 2, page 310) proved that no mechanism meets ordinal efficiency, strategy-proofness, and ETE. However, their proposed Probabilistic Serial (PS) mechanism satisfies ordinal efficiency together with EF and WSP. As Nesterov (2017) showed that EF implies SETE, OE is possible with SETE and WSP as well. Our following impossibility result states that EFR, SETE, and WSP are not possible. The impossibility also proves that SP mechanisms do not necessarily provide an EFR random assignment.

Theorem 1 (i) *For at least four agents, EFR, SETE, and Weak Strategy-proof are not compatible.*

(ii) *For at least three agents, there is no strategy-proof mechanism that always yields an EFR assignment.*

Proof See the Appendix. □

We already showed impossibility results with strategy-proof, which is a demanding property to fulfill. There might also be other (im)possibilities in light of other possible combinations of axioms without strategy-proofness. While there is no strategy-proof mechanism that meets OE and ETE, one could examine whether there is a non-strategy-proof mechanism that satisfies OE, EFR, and EF (or any weaker notions such as SETE or ETE).

5 The revised boston mechanism

The two best-known mechanisms for the assignment problem, namely the Probabilistic Serial (PS) mechanism of Bogomolnaia and Moulin (2001), and the Random Priority (RP) mechanism of Zhou (1990), satisfy EF and WEF, respectively. Could these mechanisms provide a fair random assignment, in the sense of EFR or REFR as well? Example 2 shows that PS and RP do not care whether objects are received by those agents who have the highest ranks for them.

Example 2 The PS and RP mechanisms are neither EFR nor REFR.

For the preferences profile

$$\begin{array}{l} 1: a \succ_1 b \succ_1 c \\ 2: a \succ_2 c \succ_2 b, \\ 3: b \succ_3 a \succ_3 c \end{array} \tag{4}$$

we have the following random assignments from running PS and RP:

$$PS = \begin{pmatrix} 1/2 & 1/4 & 1/4 \\ 1/2 & 0 & 1/2 \\ 0 & 3/4 & 1/4 \end{pmatrix} = \frac{1}{4}\overbrace{\begin{pmatrix} 0&1&0 \\ 1&0&0 \\ 0&0&1 \end{pmatrix}}^{\text{Not FHR but Pareto}} + \frac{1}{4}\overbrace{\begin{pmatrix} 0&0&1 \\ 1&0&0 \\ 0&1&0 \end{pmatrix}}^{\text{FHR and Pareto}} + \frac{1}{2}\overbrace{\begin{pmatrix} 1&0&0 \\ 0&0&1 \\ 0&1&0 \end{pmatrix}}^{\text{FHR and Pareto}}$$

$$RP = \begin{pmatrix} 1/2 & 1/6 & 1/3 \\ 1/2 & 0 & 1/2 \\ 0 & 5/6 & 1/6 \end{pmatrix} = \frac{1}{6}\overbrace{\begin{pmatrix} 0&1&0 \\ 1&0&0 \\ 0&0&1 \end{pmatrix}}^{\text{Not FHR but Pareto}} + \frac{1}{3}\overbrace{\begin{pmatrix} 0&0&1 \\ 1&0&0 \\ 0&1&0 \end{pmatrix}}^{\text{FHR and Pareto}} + \frac{1}{2}\overbrace{\begin{pmatrix} 1&0&0 \\ 0&0&1 \\ 0&1&0 \end{pmatrix}}^{\text{FHR and Pareto}}.$$

The first deterministic matrix in both decompositions is not FHR, since agent 1 *gets object* b *which agent* 3 *prefers it more and nevertheless agent* 3 *has been given a less preferable object* c. *Because both random assignments have a deterministic assignment in their support which does not favor higher ranks, the output of the PS and RP mechanisms do not satisfy EFR. Moreover, as their decomposition is unique, these mechanisms do not provide REFR as well.*

We introduce a revised Boston mechanism[4] with single tie-breaking that satisfies EFR. Let us think of objects as schools and agents as students, where there is only

[4] The Boston mechanism has different variations, which are different, especially in terms of strategy-proofness (Mennle and Seuken, 2018; Dur, 2019). In its standard version, first, each school reports its strict priority ordering of students and each student submits his preference ranking over schools. Then, at Step 1,

one copy of each object (only one seat in each school), and objects (schools) do not have any priorities over agents (students). As they are indifferent over their applicants, they will randomize between them in case of a tie. Here is the formalization of our proposed revised Boston mechanism:

Given an object, say $a \in A$, a subset of all agents, say $B \subseteq N$, and a preference profile $\succ$, we demonstrate the most rightful agents to get a, i.e., those agents who have the highest rank for it, by

$$M(a, B, \succ) = \left\{ i \in B \mid \forall j \in B: \ rk(a, \succ_i) \leq rk(a, \succ_j) \right\}.$$

An ordering (line-up) of N is a one-to-one mapping σ from $\{1, 2, \ldots, n\}$ to N. We refer to the set of all orderings by θ. For the set of objects A, we recursively define

- $A^1 = \{a \in A \mid \exists i \in N \ rk(a, \succ_i) = 1\}$,
- $A^2 = \{a \in A \mid \exists i \in N \ rk(a, \succ_i) = 2\} \setminus A^1$,
- $\ldots$
- $A^k = \{a \in A \mid \exists i \in N \ rk(a, \succ_i) = k\} \setminus \bigcup_{1 \leq t \leq k-1} A^t$.

Informally, A^k is the set of k-best objects of some agents which are not the t-best objects of any other agent for any $t < k$. One may note that this recursive definition is partitioning A, as $A = \bigcup_{1 \leq t \leq n} A^t$ and $A^k \cap A^l = \emptyset$ for $k \neq l$. Hence, for each object $a \in A$, we define $s(a)$ to be a number that $a \in A^{s(a)}$.

A higher-rank ordering (line-up) of A is a one-to-one mapping δ from $\{1, 2, \ldots, n\}$ to A where for every objects a, b we have $s(a) < s(b)$ if and only if $\delta^{-1}(a) < \delta^{-1}(b)$. we refer to the set of all higher-rank orderings by Δ.

Given a line $\sigma \in \theta$ of agents, a line $\delta \in \Delta$ of objects, and a preference profile $\succ$, we assign the kth object in line δ to $L(k)$th agent in line σ where $L(k)$ is inductively defined as follows

- $L(1) = \min\{t \mid \sigma(t) \in M(\delta(1), N, \succ)\}$,
- $\ldots$
- $L(k+1) = \min\{t \mid \sigma(t) \in M(\delta(k+1), N \setminus \{\sigma(L(1)), \sigma(L(2)), \ldots, \sigma(L(k))\}, \succ)\}$.

We refer to this deterministic assignment by $\Pi^{(\sigma,\delta,\succ)}$ where $\Pi^{(\sigma,\delta,\succ)}(\delta(k)) = \sigma(L(k))$ for all $1 \leq k \leq n$, which assigns the kth object in the line δ to the $L(k)$th agent in the line σ. The summation of all these matrixes weighted by their relative probability of incidence, gives us the final random assignment matrix, denoted by $P(\succ)$. We define it as $P(\succ) = \dfrac{1}{|\theta|} \dfrac{1}{|\Delta|} \sum_{\sigma \in \theta, \delta \in \Delta} \Pi^{(\sigma,\delta,\succ)}$.

It is obvious that given σ and δ, and a preference profile $\succ$, $\Pi^{(\sigma,\delta,\succ)}$ is a unique deterministic assignment. As $P(\succ)$ is a convex combination of all $\Pi^{(\sigma,\delta,\succ)}$ for all

Footnote 4 continued
each student applies to his first best school while schools admit students, following their priority order, until there is left neither a seat nor a student who has listed that school as his first choice. At Step k, each of remaining student who has been rejected at step $k-1$, applies to his kth best school even though it might not have any seat left. Each school considers students who have listed it as their kth choice and assigns remaining seats to these students, one at a time following their priority order, until there is left neither a seat nor a student who has listed that school as his kth choice. The algorithm terminates when no student applies to a school.

$\sigma \in \theta$, $\delta \in \Delta$, $P(\succ)$ is also unique. Therefore, our proposed mechanism is well-defined.

Informally, our mechanism works as follows: we randomly choose a line-up of agents σ from θ, and a higher-rank line-up of objects δ from Δ. Then, the first object in the line δ, i.e., $\delta(1)$, is assigned to the first agent in σ, say i, that $rk(\delta(1), \succ_i) < rk(\delta(1), \succ_j)$ for all other agents j . The agent i leaves the line σ with her assigned $\delta(1)$. Then, second object in line δ, i.e., $\delta(2)$, is assigned to the first agent in the remaining σ who ranks $\delta(2)$ higher than all other agents in the line, and so on. In the end, we are left with a deterministic assignment matrix that satisfies FHR.

Example 3 demonstrates how this revised Boston mechanism works.

Example 3 *For the preference profile* (4), with $N = \{1, 2, 3\}$*, we have* 6 *possible line-ups. Given* $A = \{a, b, c\}$*, we recursively define*

- $A^1 = \{a \in A \mid \exists i \in N \; rk(a, \succ_i) = 1\} = \{a, b\}$,
- $A^2 = \{a \in A \mid \exists i \in N \; rk(a, \succ_i) = 2\} \backslash A^1 = \{a, b, c\} \setminus \{a, b\} = \{c\}$,
- $A^3 = \{a \in A \mid \exists i \in N \; rk(a, \succ_i) = k\} \backslash \bigcup_{1 \leq t \leq k-1} A^t = \{b, c\} \setminus (\{a, b\} \cup \{c\}) = \varnothing$.

Therefore $s(a) = s(b) = 1$ *and* $s(c) = 2$*. There are two possible higher-rank orderings in this case, i.e.,* $\Delta = \{[a, b, c], [b, a, c]\}$*. For now, let us take* $\delta_1 = [a, b, c]$.

We arbitrarily line up all agents, e.g., $\sigma_1 = [1, 2, 3]$*, that means agent* 1 *is the first one in the line, agent* 2 *is the second one in the line, and agent* 3 *is the third one in the line. We also define inductively*

- $L(1) = \min\{t \mid \sigma_1(t) \in M(a, N, \succ)\} = 1$,
- $L(2) = \min\{t \mid \sigma_1(t) \in M(b, N\backslash \{L(1)\}, \succ)\} = 3$
- $L(3) = \min\{t \mid \sigma_1(t) \in M(c, N\backslash\{L(1), L(2)\}, \succ)\} = 2$.

We could then define the corresponding deterministic assignment, as $\Pi^{(\sigma,\delta,\succ)}(a) = \sigma(1)$, $\Pi^{(\sigma,\delta,\succ)}(b) = \sigma(3)$*, and* $\Pi^{(\sigma,\delta,\succ)}(c) = \sigma(2)$*:*

$$\Pi^{(\sigma_1,\delta_1,\succ)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

As long as agent 1 *stands before agent* 2 *in the line, e.g., for* $\sigma_2 = [1, 3, 2]$*, and* $\sigma_3 = [3, 1, 2]$*, we get the same deterministic assignment* $\Pi^{(\sigma_1,\delta_1,\succ)}$*. For* $\sigma_4 = [3, 2, 1]$*, we define*

- $L(1) = \min\{t \mid \sigma_4(t) \in M(a, N, \succ)\} = 2$,
- $L(2) = \min\{t \mid \sigma_4(t) \in M(b, N\backslash \{L(1)\}, \succ)\} = 1$
- $L(3) = \min\{t \mid \sigma_4(t) \in M(c, N\backslash\{L(1), L(2)\}, \succ)\} = 3$.

We could then define the corresponding deterministic assignment, as $\Pi^{(\sigma_4,\delta_1,\succ)}(a) = \sigma(2)$, $\Pi^{(\sigma_4,\delta_1,\succ)}(b) = \sigma(1)$*, and* $\Pi^{(\sigma_4,\delta_1,\succ)}(c) = \sigma(3)$*:*

$$\Pi^{(\sigma_4,\delta_1,\succ)} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

As long as agent 2 *stands before agent* 1 *in the line, e.g., for* $\sigma_5 = [2, 1, 3]$*, and* $\sigma_6 = [2, 3, 1]$*, we get the same deterministic assignment* $\Pi^{(\sigma_4,\delta_1,\succ)}$.

Now, let us take $\delta_2 = [b, a, c]$*. It is easy to show that* $\Pi^{(\sigma_l,\delta_1,\succ)} = \Pi^{(\sigma_l,\delta_2,\succ)}$*, for all* $l = 1, 2, \ldots, 6$*, i.e., all possible line-ups of agents. Note that we had two line-ups for objects and six line-ups for agents, i.e.,* $|\Delta| = 2$ *and* $|\theta| = 6$ *. Therefore, we could define the final random assignment, as*

$$P(\succ) = (1/2)(1/6)\sum 6\begin{pmatrix}1\,0\,0\\0\,0\,1\\0\,1\,0\end{pmatrix} + 6\begin{pmatrix}0\,0\,1\\1\,0\,0\\0\,1\,0\end{pmatrix} = \begin{pmatrix}1/2 & 0 & 1/2\\1/2 & 0 & 1/2\\0 & 1 & 0\end{pmatrix}.$$

We could now prove that our proposed mechanism, in the deterministic setting, provides FHR matrixes.

Proposition 4 *Given* $\sigma \in \theta$*,* $\delta \in \Delta$ *and a preference profile* $\succ$*, the deterministic assignment* $\Pi^{(\sigma,\delta,\succ)}$ *satisfies FHR.*

Proof For simplicity of notation, we refer to the matrix of the assignment $\Pi^{(\sigma,\delta,\succ)}$ simply by Π. Let for some arbitrary agent j and arbitrary object b, $\Pi_{jb} = 1$ (i.e., $\Pi^{(\sigma,\delta,\succ)}(b) = j$). Also, suppose that for some agent i, $rk(b, \succ_i) < rk(b, \succ_j)$. Let also c be an object where $rk(b, \succ_i) < rk(c, \succ_i)$. To prove that $\Pi^{(\sigma,\delta,\succ)}$ satisfies FHR, we should show that $\Pi_{ic} = 0$ (i.e., $\Pi^{(\sigma,\delta,\succ)}(c) \neq i$). Suppose that it is not the case, i.e., $\Pi_{ic} = 1$.

Let $\delta^{-1}(b) = v_1$, $\delta^{-1}(c) = v_2$, $\sigma^{-1}(j) = z_1$, and $\sigma^{-1}(i) = z_2$. Since $\Pi_{ic} = 1$, by definition of $\Pi^{(\sigma,\delta,\succ)}$, we have

$$L(v_2) = \min\{t \mid \sigma(t) \in M(c, N - \{\sigma(L(1)), \sigma(L(2)), \ldots, \sigma(L(v_2 - 1))\}, \succ)\} = z_2,$$

which implies

$$i \in N - \{\sigma(L(1)), \sigma(L(2)), \ldots, \sigma(L(v_2 - 1))\}. \tag{5}$$

Similarly as $\Pi_{jb} = 1$, we have

$$L(v_1) = \min\{t \mid \sigma(t) \in M(b, N - \{\sigma(L(1)), \sigma(L(2)), \ldots, \sigma(L(v_1 - 1)\}, \succ)\} = z_1,$$

which implies

$$j \in M(b, N - \{\sigma(L(1)), \sigma(L(2)), \ldots, \sigma(L(v_1 - 1)\}, \succ).$$

As $rk(b, \succ_i) < rk(b, \succ_j)$, if $i \in N - \{\sigma(L(1)), \ldots, \sigma(L(v_1 - 1)\}$, by definition of M, $j \notin M(b, N - \{\sigma(1), \ldots, \sigma(v_1 - 1)\}, \succ)$. Thus, $i \in \{\sigma(L(1)), \ldots, \sigma(L(v_1 - 1)\}$. Moreover, from (5), we know $i \notin \{\sigma(L(1)), \ldots, \sigma(L(v_2 - 1))\}$. Therefore, we must have $v_2 < v_1$, and since δ is a higher-rank line up, we should have $s(c) < s(b)$ which contradicts with $rk(b, \succ_i) < rk(c, \succ_i)$, and thus $\Pi_{ic} = 0$. □

The outcome of our proposed mechanism is simply EFR, by definition. Corollary 1 formalizes this idea which is trivial to prove.

Corollary 1 *The random assignment* $P(\succ) = \frac{1}{|\theta|}\frac{1}{|\Delta|}\sum_{\sigma\in\theta,\delta\in\Delta}\Pi^{(\sigma,\delta,\succ)}$ *satisfies EFR.*

Proof Since for all $\sigma \in \theta$, $\delta \in \Delta$, and a preference profile $\succ$, the deterministic assignment $\Pi^{(\sigma,\delta,\succ)}$ satisfies FHR, thus the random assignment $P(\succ) = \frac{1}{|\theta|}\frac{1}{|\Delta|}\sum_{\sigma\in\theta,\delta\in\Delta}\Pi^{(\sigma,\delta,\succ)}$ which is convex combinations of $\Pi^{(\sigma,\delta,\succ)}$ satisfies EFR. $\square$

6 Conclusion

In a deterministic assignment of indivisible objects, it is quite difficult to make a fair assignment, ex-post, regardless of how we define fairness. Therefore, lotteries have been used to make a fair random assignment. While the literature on the random assignment of indivisible goods is concentrated on ex-ante fairness, in this paper, we introduce two notions of ex-post fairness, namely ex-post favoring ranks (EFR) and robust ex-post favoring ranks (REFR), which are based on a fairness notion, called favoring higher rank (FHR), introduced by Kojima and Ünver (2014). A random assignment is EFR if it is equal to a lottery over some FHR deterministic assignments. A random assignment is REFR if all its decompositions are lotteries over FHR deterministic assignments. Therefore, REFR is a stronger notion of fairness that makes sure that all of its possible deterministic realizations favor higher ranks.

We assess how our proposed notions of fairness are related to different notions of efficiency. We found that (robust) ex-post favoring ranks is a refinement of (robust) ex-post Pareto while it is not logically related to ordinal efficiency. Moreover, ex-post favoring ranks is not logically associated with robust ex-post Pareto. We demonstrated that the logical relationship between ex-post favoring ranks, robust ex-post favoring ranks, interim favoring ranks (IFR), in the sense of (Harless 2018), and favoring higher rank is similar to the logical relationship between ex-post Pareto efficiency, robust ex-post Pareto efficiency, ordinal efficiency, and deterministic Pareto efficiency.

Furthermore, we showed that IFR is a refinement of REFR while the latter is not logically related to (weak) envy-freeness. We also proved two impossibility results that, for at least three agents, firstly, EFR, strong equal treatment of equals, and weak strategy-proof are not compatible. Secondly, there is no strategy-proof mechanism that always yields an EFR random assignment. Finally, we proposed and formalized a revised version of the Boston mechanism and proved that it provides an EFR random assignment.

Appendix: omitted proofs

Proof of Proposition 1 *We prove that every decomposition of an IFR random assignment is a lottery over deterministic assignments that favor higher ranks. An IFR* *random assignment P could be decomposed into deterministic assignments, $P = \sum_{l=1}^{k} \lambda_l \Pi_l$ where $\sum_{l=1}^{k} \lambda_l = 1$. Suppose for some l, where $\lambda_l \neq 0$, Π_l does not favor higher ranks. Then, in this deterministic assignment, there are agents i and j, such that i could object j in the sense that for some object b and c, $(\Pi_l)_{jb} = 1$ and $rk(b, \succ_i) < rk(b, \succ_j)$, while $(\Pi_l)_{ic} = 1$ where $rk(c, \succ_i) > rk(b, \succ_i)$. Now, consider the random assignment P. Since $(\Pi_l)_{jb} = 1$, we have $p_{jb} \geq \lambda_l (\Pi_l)_{jb} = \lambda_l > 0$, and since P is IFR, we must have $p_{ic} = 0$. However, as $(\Pi_l)_{ic} = 1$, we have $p_{ic} \geq \lambda_l (\Pi_l)_{ic} = \lambda_l > 0$, which is a contradiction.*

We show the other direction by a counterexample: Take a random assignment (3) for the preference profile (2). The support of P has only four following deterministic assignments:

$$\Pi_1 = \begin{pmatrix} 1\,0\,0\,0 \\ 0\,0\,0\,1 \\ 0\,0\,1\,0 \\ 0\,1\,0\,0 \end{pmatrix}, \Pi_2 = \begin{pmatrix} 1\,0\,0\,0 \\ 0\,1\,0\,0 \\ 0\,0\,1\,0 \\ 0\,0\,0\,1 \end{pmatrix},$$
$$\Pi_3 = \begin{pmatrix} 0\,0\,1\,0 \\ 0\,0\,0\,1 \\ 1\,0\,0\,0 \\ 0\,1\,0\,0 \end{pmatrix}, \Pi_4 = \begin{pmatrix} 0\,0\,1\,0 \\ 0\,1\,0\,0 \\ 1\,0\,0\,0 \\ 0\,0\,0\,1 \end{pmatrix},$$

where all favor higher ranks, and therefore P is REFR. However, P is not IFR since while $rk(b, \succ_1) < rk(b, , \succ_2)$ and $p_{2b} > 0$, we have $p_{1c} > 0$ where $rk(c, \succ_1) > rk(b, , \succ_1)$. **QED.**

Proof of Proposition 2 (i) *For the preference profile*

$$\begin{array}{l} 1 : a \succ_1 b \succ_1 c \succ_1 d \\ 2 : a \succ_2 c \succ_2 b \succ_2 d \\ 3 : b \succ_3 c \succ_3 d \succ_3 a \\ 4 : b \succ_4 d \succ_4 a \succ_4 c \end{array}, \tag{6}$$

the random assignment

$$P = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 1/2 & 0 & 1/2 \end{pmatrix} = \frac{1}{2} \overbrace{\begin{pmatrix} 0\,0\,1\,0 \\ 1\,0\,0\,0 \\ 0\,1\,0\,0 \\ 0\,0\,0\,1 \end{pmatrix}}^{\text{FHR}} + \frac{1}{2} \overbrace{\begin{pmatrix} 0\,0\,1\,0 \\ 1\,0\,0\,0 \\ 0\,0\,0\,1 \\ 0\,1\,0\,0 \end{pmatrix}}^{\text{Not FHR}}, \tag{7}$$

is not REFR since it has a decomposition with the deterministic assignment which is not FHR: the second deterministic matrix is not FHR: agent 1 gets object c which agent 3 prefers it more while the latter has been given a less preferable object d.

However, P is OE since it is acyclic: we only have $a\tau\,(P, \succ_1)\,c$ (since $a \succ_1 c$ and $p_{1c} > 0$), $b\tau\,(P, \succ_1)\,c$ (since $b \succ_1 c$ and $p_{1c} > 0$), $b\tau\,(P, \succ_3)\,d$ (since $b \succ_3 d$ and

$p_{3d} > 0$), $c\tau(P, \succ_3)\, d$ (*since* $c \succ_3 d$ *and* $p_{3d} > 0$), *and* $\tau(P, \succ_4)$ (*since* $b \succ_4 d$ *and* $p_{4d} > 0$) *which do not make a cycle.*

For the preference profile

$$\begin{array}{l} 1 : a \succ_1 b \succ_1 c \succ_1 d \\ 2 : c \succ_2 b \succ_2 d \succ_2 a \\ 3 : c \succ_3 d \succ_3 b \succ_3 a\text{'} \\ 4 : a \succ_4 d \succ_4 b \succ_4 c \end{array} \tag{8}$$

the random assignment

$$P = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 & 1/2 \end{pmatrix} = \frac{1}{2}\overbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}^{\text{FHR}} + \frac{1}{2}\overbrace{\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}}^{\text{FHR}} \tag{9}$$

is not OE since P *is not acyclic: we have* $b\tau(P, \succ_2)\, d$ *(since* $b \succ_2 d$ *and* $p_{2d} > 0$*),* $d\tau(P, \succ_3)\, b$ *(since* $d \succ_3 b$ *and* $p_{3b} > 0$*), which make a cycle.*

Yet, P *is REFR since it has a unique decomposition* (9) *into FHR deterministic assignments: We must give a to either to agent* 1 *or agent* 4*, since if we give it to agent* 2 *or agent* 3*, a trade happens as both agents* 2 *and* 3 *know a as their worst choice, while agents* 1 *and* 4 *know a as their first best choice. On the one hand, once we give a to agent* 1 *, as* $p_{2b} = p_{4b} = 0$*, we have to give b to agent c, and as* $p_{4c} = 0$ *we only have one choice to give c to agent* 2 *and finally d to agent* 4*. On the other hand, once we give a to agent* 4 *, as* $p_{1d} = p_{3d} = 0$ *we have to give d to agent* 2*, and since* $p_{1c} = 0$ *we only have one choice to give c to agent* 3 *and finally give b to agent* 1.

For the preference profile

$$\begin{array}{l} 1 : a \succ_1 d \succ_1 c \succ_1 b \\ 2 : c \succ_2 a \succ_2 d \succ_2 b \\ 3 : c \succ_3 d \succ_3 b \succ_3 a\text{'} \\ 4 : a \succ_4 c \succ_4 b \succ_4 d \end{array}$$

the random assignment

$$P = \begin{pmatrix} 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 & 0 \end{pmatrix} = \frac{1}{2}\overbrace{\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}}^{\text{FHR}} + \frac{1}{2}\overbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}}^{\text{FHR}}$$

is OE since it is acyclic: we have $a\tau(P, \succ_1)\, d$ *(since* $a \succ_1 d$ *and* $p_{1d} > 0$ *),* $c\tau(P, \succ_2)\, d$ and $a\tau(P, \succ_2)\, d$ *(since* $c \succ_3 d \succ_3 a$ *and* $p_{2d} > 0$*),* $c\tau(P, \succ_3)\, b$ and $d\tau(P, \succ_3)\, b$ *(since* $c \succ_3 d \succ_3 b$ *and* $p_{3b} > 0$*), and finally* $a\tau(P, \succ_4)\, b$ and $c\tau(P, \succ_4)\, b$ *(since* $a \succ_4 c \succ_4 b$ *and* $p_{4b} > 0$*), which do not make a cycle.*

P *is also REFR since it has a unique decomposition into FHR deterministic assignments: The decomposition of* P *is also unique: We must give a to either to agent* 1 *or* *agent* 4*, since if we give it to agent* 2 *or agent* 3 *, a trade happens. On the one hand, once we give a to agent* 1*, as* $p_{3d} = p_{4d} = 0$*, we have to give d to agent* 2*, and as* $p_{1c} = p_{4c} = 0$ *we only have one choice to give c to agent* 3 *and finally b to agent* 2*. On the other hand, once we give a to agent* 4*, as* $p_{1b} = p_{2b} = 0$ *we have to give b to agent* 3*, and since* $p_{1c} = 0$ *we only have one choice to give c to agent* 3 *and finally give d to agent* 1*.*

However, P is not IFR since $p_{2d} > 0$ *, and while agent* 3 *ranks d higher than agent* 2 does, *i.e.,* $rk(d, \succ_3) < rk(d, \succ_2)$*, P assigns an inferior object b, i.e.,* $rk(b, \succ_3) > rk(d, \succ_3)$*, to agent* 3 with a positive probability.

ii) *For the preference profile* (6)*,the random assignment* (7) *is OE not EFR since in its only possible decomposition into deterministic assignments, there is a deterministic matrix which does not favor higher ranks. For the preference profile* (8)*, the random assignment* (9) *is not OE since it is not acyclic, while it has a decomposition into FHR deterministic assignments.*

We already showed in Example 1 that an assignment (3) *for the preferences profile* (2) *is not IFR while it is EFR. We now show that it is also OE as it is acyclic: we only have* $a\tau\,(P, \succ_1)\,c$*,* $b\tau\,(P, \succ_1)\,c$*,* $d\tau\,(P, \succ_1)\,c$*, (since* $a \succ_1 b \succ_1 d \succ_1 c$ *and* $p_{1c} > 0$*),* $d\tau\,(P, \succ_2)\,b$*,* $a\tau\,(P, \succ_2)\,b$ *(since* $d \succ_2 a \succ_2 b$ *and* $p_{2d} > 0$*),* $a\tau\,(P, \succ_3)\,c$*,* $d\tau\,(P, \succ_3)\,c$*, and* $b\tau\,(P, \succ_3)\,c$*, (since* $a \succ_3 d \succ_3 b \succ_3 c$ *and* $p_{3b} > 0$*),* $d\tau\,(P, \succ_4)\,b$ *(since* $d \succ_4 b$ *and* $p_{4b} > 0$*) which do not make a cycle.*

iii) *We show both directions by counterexamples: For the preference profile* (6)*, the random assignment* (7) *is robust ex-post Pareto efficient since both deterministic assignments in its only decomposition are Pareto efficient. However, it is not EFR as there exists a deterministic assignment in its decomposition, which is not FHR.*

For the preference profile

$$\begin{array}{l} 1 : a \succ_1 d \succ_1 b \succ_1 c \\ 2 : c \succ_2 b \succ_2 d \succ_2 a \\ 3 : c \succ_3 d \succ_3 b \succ_3 a \\ 4 : a \succ_4 b \succ_4 c \succ_4 d \end{array},$$

the random assignment

$$P = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \\ 1/2 & 1/2 & 0 & 0 \end{pmatrix} = \frac{1}{2}\overbrace{\begin{pmatrix} 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \\ 1&0&0&0 \end{pmatrix}}^{\text{FHR}} + \frac{1}{2}\overbrace{\begin{pmatrix} 1&0&0&0 \\ 0&0&0&1 \\ 0&0&1&0 \\ 0&1&0&0 \end{pmatrix}}^{\text{FHR}}$$

$$= \frac{1}{2}\overbrace{\begin{pmatrix} 0&1&0&0 \\ 0&0&0&1 \\ 0&0&1&0 \\ 1&0&0&0 \end{pmatrix}}^{\text{Not Pareto}} + \frac{1}{2}\overbrace{\begin{pmatrix} 1&0&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \\ 0&1&0&0 \end{pmatrix}}^{\text{Pareto}},$$

is EFR since there it has a decomposition into FHR deterministic assignments. However, P is not REP since there is a decomposition of it that admits a non-Pareto optimal deterministic assignment consistent with it.

For the preference profile (6), *the random assignment*

$$P=\begin{pmatrix}1/2 & 0 & 1/2 & 0\\ 1/2 & 0 & 1/2 & 0\\ 0 & 1/2 & 0 & 1/2\\ 0 & 1/2 & 0 & 1/2\end{pmatrix}=\frac{1}{2}\overbrace{\begin{pmatrix}0\,0\,1\,0\\1\,0\,0\,0\\0\,0\,0\,1\\0\,1\,0\,0\end{pmatrix}}^{\text{Not FHR but Pareto}}+\frac{1}{2}\overbrace{\begin{pmatrix}1\,0\,0\,0\\0\,0\,1\,0\\0\,1\,0\,0\\0\,0\,0\,1\end{pmatrix}}^{\text{FHR and Pareto}}$$
$$=\frac{1}{2}\overbrace{\begin{pmatrix}0\,0\,1\,0\\1\,0\,0\,0\\0\,1\,0\,0\\0\,0\,0\,1\end{pmatrix}}^{\text{FHR and Pareto}}+\frac{1}{2}\overbrace{\begin{pmatrix}1\,0\,0\,0\\0\,0\,1\,0\\0\,0\,0\,1\\0\,1\,0\,0\end{pmatrix}}^{\text{FHR and Pareto}},$$

has only two possible decompositions where in both all deterministic assignments are Pareto while there is a deterministic assignment in the first decomposition that does not favor higher ranks. Therefore, P is REP and EFR, but not REFR. **QED.**

Proof of Proposition 3 *Suppose for the preferences profile,*

$$\begin{array}{ccccccc}1 & : & a & \succ_1 & b & \succ_1 & \ldots\\ 2 & : & a & \succ_2 & b & \succ_2 & \ldots\\ \ldots & : & \ldots & \ldots & \ldots & \ldots & \ldots\\ \mathbf{i} & : & \mathbf{b} & \succ_i & \mathbf{a} & \succ_i & \ldots\\ \ldots & : & \ldots & \ldots & \ldots & \ldots & \ldots\\ n & : & a & \succ_n & b & \succ_n & \ldots\end{array},$$

P is a random assignment where it is both EF and EFR with a decomposition $P=\sum_{l=1}^{k}\lambda_l\Pi_l$ where Π_l is FHR for all *l*.

On the one hand, since b is the first best object of only agent i, all deterministic FHR assignments, Π_l, should assign object b to agent i, which implies that $p_{ib}=1$ (hence agent i gets all other objects with zero probability, particularly object a, i.e., $p_{ia}=0$). Moreover, any other agent $j\neq i$ has no chance to get object b, i.e., $p_{jb}=0$; particularly agent 1, $p_{1b}=0$.

On the other hand, since P is EF and object a is the first best of all agents $j\neq i$, the chance of receiving object a must be divided equally among all these $n-1$ agents, i.e., $p_{ja}=p_{1a}=\frac{1}{n-1}$. Now, since

$$p_{1a}+p_{1b}=\frac{1}{n-1}<p_{ia}+p_{ib}=1,$$

the allocation of agent 1*, regarding her preference, does not stochastically dominate the allocation of agent* i*, agent* 1 *envies agent* i*, and* P *is not EF, which is a contradiction.* **QED.**

Proof of Theorem 1 (i) *Suppose* $n = |A| = |N| = 4$*, and mechanism* μ *is EFR and* P *is the output of* μ *on preference profile:*

$$\begin{array}{l} 1 : a \succ_1 b \succ_1 c \succ_1 d \\ 2 : a \succ_2 b \succ_2 c \succ_2 d \\ 3 : b \succ_3 a \succ_3 c \succ_3 d \\ 4 : b \succ_4 a \succ_4 c \succ_4 d \end{array}.$$

We should have $p_{ic} \leq 1/4$ for $i = 1, \ldots 4$. *Without loss of generality, suppose* $p_{1c} \leq 1/4$. *As* μ *is EFR and SETE, we have* $p_{1a} = p_{2a} = 1/2$*, and* $p_{1b} = 0$. *Now, if agent* 1 *misreports to* $\succ_1'$: $a \succ_1' c \succ_1' b \succ_1' d$*, since* μ *is EFR and SETE, we have* $p'_{1a} = p'_{2a} = 1/2$, $p'_{1b} = 0$*, and* $p'_{1c} = 1/2$. *(Since agent* 1 *prefers* c *more than any other agent, in any deterministic FHR assignments, she is assigned either her first best object* a*, or her second best object* c*, i.e.,* $p'_{1a} + p'_{1c} = 1$.*) Now, we have* $p'_{1a} = 1/2 \geq p_{1a} = 1/2$, $p'_{1a} + p'_{1b} = 1/2 \geq p_{1a} + p_{1b} = 1/2$ *, and* $p'_{1a} + p'_{1b} + p'_{1c} = 1 > p_{1a} + p_{1b} + p_{1c} = 3/4$. *Therefore, agent* 1 *is better off misreporting.*

Now, suppose $n > 4$*, and we have additional agents* 5 ,6, . . ., n *and objects* o_5 , o_6, ..., o_n*, where* o_i *is the first-best object of each new agent* i. *Since mechanism* μ *is EFR and* P *is the output of* μ *on preference profile*

$$\begin{array}{llllllllllllllll}
1: & a & \succ_1 & b & \succ_1 & c & \succ_1 & d & \succ_1 & o_5 & \succ_1 & o_6 & \succ_1 & \ldots & \succ_1 & o_n \\
2: & a & \succ_2 & b & \succ_2 & c & \succ_2 & d & \succ_2 & o_5 & \succ_2 & o_6 & \succ_2 & \ldots & \succ_2 & o_n \\
3: & b & \succ_3 & a & \succ_3 & c & \succ_3 & d & \succ_2 & o_5 & \succ_2 & o_6 & \succ_2 & \ldots & \succ_2 & o_n \\
4: & b & \succ_4 & a & \succ_4 & c & \succ_4 & d & \succ_2 & o_5 & \succ_2 & o_6 & \succ_2 & \ldots & \succ_2 & o_n \\
5: & o_5 & \succ_6 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
6: & o_6 & \succ_6 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
n: & o_n & \succ_n & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots
\end{array},$$

for $i > 4$*, we should have* $p_{io_i} = 1$ *and* $p_{jo_i} = 0$ *for* $j \neq i$. *Otherwise there exits a deterministic assignment in any decomposition of P that is not favoring higher ranks. For* $n \geq 4$*, agents* 1 *to* 4 *get the very same allocation as when we had* $n = 4$ *agents and objects. Therefore, with the same argument, agent* 1 *has an incentive to misreport. Thus,* $P'_1 \succ_1^{sd} P_1$ *, and* μ *is not weak-strategy proof.*

ii*) Suppose that the number of agents and objects is at least three, i.e.,* $n = |A| = |N| \geq 3$*, and for all agents, object* a *is the first best and object* b *is the second best. For every EFR mechanism* $\mu(\succ) = P$*, there exists some agent* i *such that,* $p_{ia} + p_{ib} < 1$. *(Since if for all agents* i*, we have* $p_{ia} + p_{ib} = 1$*, as the number of agents is at least three, then* $\sum_{i \in N} p_{ia} + \sum_{i \in N} p_{ib} > 2$*, which contradicts with the fact that matrix* P *is bistochastic.) Let us define* $\varepsilon = 1 - (p_{ia} + p_{ia}) > 0$. *For a utility function,* u_i, *which respects the preference $\succ_i$, let us also assume $u_i(a) = 10+\varepsilon$, $u_i(b) = 10$, and $u_i(c) < \varepsilon/n$ for all $c \in A/\{a,b\}$. We prove that given u_i, agent i has the incentive to misreport her preference as $i : b \succ_i^{'} a \succ_i^{'} \ldots$.*

Since $P^{'} = \mu(\succ_i^{'}, \succ_{-i})$ satisfies EFR and b is the first best object of i in $\succ_i^{'}$, and not the first best object of anyone else, she must get it for sure, i.e., $p_{ib}^{'} = 1$, and $p_{id}^{'} = 0$ for $d \in A/\{b\}$. Therefore, the (expected) utility of agent i in the assignment $P^{'}$ is $U_i(P^{'}) = p_{ib}^{'} u_i(b) + \sum_{d\in A/\{b\}} p_{id}^{'} u_i(d) = 10$. However,

$$\begin{aligned} U_i(P) &= p_{ia}u_i(a) + p_{ib}u_i(b) + \sum\nolimits_{c\in A/\{a,b\}} p_{ic}u_i(c) \\ &= p_{ia}(10+\varepsilon) + 10p_{ib} + \sum\nolimits_{c\in A/\{a,b\}} p_{ic}u_i(c) \\ &= 10(p_{ia}+p_{ib}) + p_{ia}\varepsilon + \sum\nolimits_{c\in A/\{a,b\}} p_{ic}u_i(c) \\ &< 10(p_{ia}+p_{ib}) + p_{ia}\varepsilon + \sum\nolimits_{c\in A/\{a,b\}} p_{ic}(\frac{\varepsilon}{n}) \\ &< 10(p_{ia}+p_{ib}) + p_{ia}\varepsilon + (n-2)(\frac{\varepsilon}{n}) \\ &< 10(1-\varepsilon) + \varepsilon + \varepsilon < 10 = U_i(P^{'}). \end{aligned}$$

1-*Hence, given the utility function u_i, agent i could gain more via misreporting her preference, and thus it is not possible for the mechanism μ to be both SP and EFR.* **QED.**

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