Development of a numerical formulation for n-dimensional heat equation

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Abstract: Shearlet frames are used to solve N-dimensional heat equation numerically. By this approach the coefficients of the shearlet frame expansion are obtained via separate time independent partial differential equations.

Key words and phrases. Shearlets, Heat equation, Numerical solution.
2010 Mathematics Subject Classification. Primary 42C15; Secondary 42C40.

Introduction and Preliminaries

Heat equation is categorized as parabolic second order partial differential equations. This kind of equations appear in different scientific fields. Several numerical techniques have so far been developed for solution of transient heat transfer problems, such as finite difference methods [7], finite element methods [9], spectral methods [4], wavelet and curvelet methods [2, 8], etc. can be mentioned. Shearlets are newer representation systems that are equipped with a rich mathematical structure similar to wavelets [6]. Compared with wavelets, the continuous shearlet transform has a coherent matrix structure for n-dimensions so that it is useful for solving the higher dimensional PDE [1]. In this paper, an approach for solution of heat equation in the general case of n-dimensions is presented. The unknown function is expanded by using shearlet frames. The shearlet coefficients are obtained by solving far simpler separate time independent partial differential equations. The paper is organized as follows. In the rest of this section, we present some necessary definitions and theorems. Section 2 is devoted to the development of n-dimensional formulation. The conclusions and merits of the approach are concisely discussed in Section 3. Firstly, we present required notation and definitions about shearlets.

Let $\{\psi_{j,k,m}(\cdot)\}_{j,k,m}$ be a family of shearlets in n-dimensions as

$$\psi_{j,k,m}(\cdot) = |\det A_{2^j}|^{-\frac{1}{2}}\psi(A_{2^j}^{-1}S_k^{-1}(\cdot - m)), \tag{1}$$

where $j \in \mathbb{Z}, k \in \mathbb{Z}^{n-1}, m \in \mathbb{Z}^n$ and $\psi \in L^2(\mathbb{R}^n)$ is admissible in the sense that

$$C_\psi = \int_{\mathbb{R}^n} \frac{|\hat{\psi}(\xi)|^2}{|\xi_1|^2} d\xi_n \cdots d\xi_1 < \infty, \tag{2}$$

and

$$A_{2^j} = \begin{bmatrix} 2^j & 0_{n-1}^T \\ 0_{n-1} & 2^{\frac{j}{2}} I_{n-1} \end{bmatrix}, \qquad S_k = \begin{bmatrix} 1 & k^T \\ 0_{n-1} & I_{n-1} \end{bmatrix}. \tag{3}$$

Definition 1. Let $\psi_1 \in L^2(\mathbb{R})$ be a admissible wavelet with $\widehat{\psi}_1 \in C^\infty(\mathbb{R})$ and supp $\widehat{\psi}_1 \subseteq [-2, -\frac{1}{2}] \cup [\frac{1}{2}, 2]$. Consider $\psi_2 \in L^2(\mathbb{R}^{n-1})$ be such that $\widehat{\psi}_2 \in C^\infty(\mathbb{R}^{n-1})$ and supp $\widehat{\psi}_2 \subseteq [-1, 1]^{n-1}$, then the function $\psi \in L^2(\mathbb{R}^n)$ defined by

$$\widehat{\psi}(\xi) = \widehat{\psi}(\xi_1, \tilde{\xi}) = \widehat{\psi}_1(\xi_1) \cdot \widehat{\psi}_2(\frac{\tilde{\xi}}{\xi_1}),$$

where $\tilde{\xi} = (\xi_2, \cdots, \xi_n)$ is a continuous shearlet.

Let $\psi_1 \in L^2(\mathbb{R})$ satisfies the discrete Calderon's condition

$$\sum_{j \in \mathbb{Z}} |\widehat{\psi}_1(2^{-j}\xi)|^2 = 1,$$

with $\widehat{\psi}_1 \in C^\infty(\mathbb{R})$ and supp $\widehat{\psi}_1 \subseteq [-2,-\frac{1}{2}]\cup[\frac{1}{2},2]$. Consider $\psi_2 \in L^2(\mathbb{R}^n)$ is a bump function such that for all $\xi \in [-1,1]^{n-1}$,

$$\sum_{k=-1}^{1} |\widehat{\psi}_2(\xi+k)|^2 = 1,$$

where $\widehat{\psi}_2 \in C^\infty(\mathbb{R}^{n-1})$ and supp $\widehat{\psi}_2 \subseteq [-1,1]^{n-1}$. The shearlet ψ as defined in above is called a classical shearlet [6].

For case of n, we will show that the classical shearlet can be defined as follows

$$\hat{\psi}(\xi_1,\tilde{\xi}) = \hat{\psi}_1(\xi_1)\hat{\psi}_2(\frac{\tilde{\xi}}{\xi_1}),$$

where $\hat{\psi}_1, v(x), b(\xi)$ is the same of example in [3] and consider $I = \{i_1, i_2, \cdots, i_{n_i}\} \subseteq \{2,3,\cdots,n\}$; $J = \{j_1, j_2, \cdots, j_{n_j}\} \subseteq \{2,3,\cdots,n\}$ be such that $I \cap J = \emptyset$. Now, we defined $\hat{\psi}_2(\tilde{\xi})$ as follows

$$\hat{\psi_2}^2(\tilde{\xi}) = \hat{\psi_2}^2(\xi_2,\xi_3,\cdots,\xi_n) = v(1-\xi_{i_1})\cdots v(1-\xi_{i_{n_i}})v(1+\xi_{j_1})\cdots v(1+\xi_{j_{n_j}})$$

where v is the function defined in [3], $\xi_{i_{n'}} \geq 0$ for $n' = 1,2,\cdots,n_i$, $\xi_{i_{n''}} \leq 0$ for $n'' = 1,2,\cdots,n_j$, and $n_i + n_j = n-1$ It can be shown that $\hat{\psi}_2$ are satisfied in the conditions mentioned in definition 1.

In the following, we will show that $\hat{\psi}_2$ satisfies the conditions in Definition 1. Consider $\tilde{k} = (k_2,k_3,\cdots,k_n)$ where $k_\alpha \in \mathbb{Z}$, $\alpha \in \{2,3,\cdots,n\}$, specially we could consider $k_\alpha \in [-2^j, 2^j]$ where j is the scale parameter. In the following we show that

$$\sum_{k_2=-2^j}^{2^j}\sum_{k_3=-2^j}^{2^j}\cdots\sum_{k_n=-2^j}^{2^j} \hat{\psi}_2(\tilde{\xi}+\tilde{k})|^2 = 1. \tag{4}$$

By definition of $\hat{\psi}_2(\tilde{\xi})$ we can write

$$\begin{aligned}&\textstyle\sum_{k_2}\sum_{k_3}\cdots\sum_{k_n} |\hat{\psi}_2(\tilde{\xi}+\tilde{k})|^2\\ &\qquad= \textstyle\sum_{k_{i_1}} v(1-\xi_{i_1}+k_{i_1})\cdots\sum_{k_{j_{n_j}}} v(1+\xi_{j_{n_j}}+k_{j_{n_j}}) = 1,\end{aligned}$$

(see Theorem 2.5 in [3]) Moreover by (4) and Theorem 2.2 in [3] it can be easily seen that

$$\sum_{j,k} |\hat{\psi}_{j,k}(\xi)|^2 = 1.$$

where $\xi = (\xi_1,\cdots,\xi_n)$, $k = (k_1,\cdots,k_n)$.

In the next step, we have the following proposition whose proof is similar to [6, Section 5.1, Proposition 2] and so is omitted.

Proposition 1. *The shearlet system $\{\psi_{j,k,m}\}_{j,k,m}$ defined as (1) with ψ as in Definition 1, is a Parseval frame for $L^2(\mathbb{R}^n)$.*

Now, we define $\tilde{\psi}_{j,k,m}$ as follows [5]

$$\tilde{\psi}_{j,k,m} : \mathbb{R}^n \times \mathbb{R} \to \mathbb{C}$$

$$\hat{\tilde{\psi}}_{j,k,m}(\xi,t) = \overline{\hat{\psi}}_{j,k,m}(\xi)e^{\pm i|\xi|ct}, \tag{5}$$

where ψ is a classical shearlet. Then $\{\hat{\tilde{\psi}}_{j,k,m}(\cdot)\}_{j,k,m}$ is also a Parseval frame for $L^2(\mathbb{R}^n)$. This can be proved in a very similar fashion as Proposition 1. Since $\{\tilde{\psi}_{j,k,m}\}_{j,k,m}$ is a Parseval frame for $L^2(\mathbb{R}^n)$, then we can write $f \in L^2(\mathbb{R}^n)$ as follows

$$f(x,t) = \sum_{j,k,m} \langle f, \tilde{\psi}_{j,k,m}\rangle \tilde{\psi}_{j,k,m}(x,t). \tag{6}$$

For later use, we state the forthcoming lemma which is [1, Lemma 2.2].

Lemma 1. *Let $\{f_i\}_i$ be a Parseval frame for $L^2(\mathbb{R}^n)$ and for all $f \in L^2(\mathbb{R}^n)$, $\sum_i \langle f, f_i\rangle f_i = 0$. Then we have $\langle f, f_i\rangle = 0$ for all $i \in \mathbb{Z}$.*

Proof. See [1]. $\square$

Solution procedure by shearlet frames

In this section, we present a method for solving n-dimensional heat equation using shearlet frames. First, we consider n-dimensional heat equation as

$$u_t(x,t) = {c'}^2 \Delta u(x,t),\ x \in \mathbb{R}^n,\ 0 \le x_i \le a_i,\ i = 1, \cdots, n, \tag{7}$$

where $\Delta u = \sum_{i=1}^{n} (\frac{\partial^2 u}{\partial x_i^2})$ and c' is a positive coefficient called the diffusivity of the medium. Consider

$$\begin{aligned} u(x,t) &= \textstyle\sum_{j,k,m} C_{j,k,m} \tilde{\psi}_{j,k,m}(x,t), \\ \Delta u(x,t) &= \textstyle\sum_{j,k,m} C^{\Delta}_{j,k,m} \tilde{\psi}_{j,k,m}(x,t), \end{aligned} \tag{8}$$

in which

$$C_{j,k,m} = \langle u(x,t), \tilde{\psi}_{j,k,m}(x,t) \rangle, \quad C^{\Delta}_{j,k,m} = \langle \Delta u(x,t), \tilde{\psi}_{j,k,m}(x,t) \rangle. \tag{9}$$

Substituting (8) to (7) and then applying the Fourier transform and noticing the Fourier transform of the drivative, we have

$$(i|\xi_{n+1}|) \sum_{j,k,m} C_{j,k,m} \hat{\tilde{\psi}}_{j,k,m}(\xi,t) = {c'}^2 \sum_{j,k,m} C^{\Delta}_{j,k,m} \hat{\tilde{\psi}}_{j,k,m}(\xi,t). \tag{10}$$

So

$$\sum_{j,k,m} \left[i|\xi_{n+1}| C_{j,k,m} - c'^2 C^{\Delta}_{j,k,m} \right] \hat{\tilde{\psi}}_{j,k,m} = 0. \tag{11}$$

By definition of $C_{j,k,m}$ and $C^{\Delta}_{j,k,m}$, we obtain

$$\sum_{j,k,m} \left[\langle i|\xi_{n+1}| u - c'^2 \Delta u, \hat{\tilde{\psi}}_{j,k,m} \rangle \right] \hat{\tilde{\psi}}_{j,k,m} = 0. \tag{12}$$

Since $\{\hat{\tilde{\psi}}_{j,k,m}\}$ is a Parseval frame, by Lemma 1, we can conclude

$$i|\xi_{n+1}| C_{j,k,m} - c'^2 C^{\Delta}_{j,k,m} = 0. \tag{13}$$

Using the Plancherel theorem, we get

$$\begin{aligned} C_{j,k,m} &= \langle u, \tilde{\psi}_{j,k,m} \rangle = \langle \hat{U}, \hat{\tilde{\psi}} \rangle = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{U}(\xi,t).\overline{\hat{\tilde{\psi}}_{j,k,m}(\xi,t)} d\xi \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{U}^{\pm}(\xi).\overline{\hat{\psi}^{\pm}_{j,k,m}(\xi)} d\xi, \\ C^{\Delta}_{j,k,m} &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \widehat{\Delta U}.\overline{\hat{\tilde{\psi}}}_{j,k,m}(\xi,t) d\xi \\ &= -\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} (\xi_1^2 + \cdots + \xi_n^2) \hat{U}^{\pm}(\xi).\overline{\hat{\psi}^{\pm}_{j,k}(\xi)} e^{i\langle m,\xi \rangle} d\xi. \end{aligned} \tag{14}$$

Changing the variables ξ in (14) to $S^T_{-k} A_{2^{-j}} \xi$, we have

$$C_{j,k,m} = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} A_{2^j} S^T_k \left(\hat{U}^{\pm}(\xi).\overline{\hat{\psi}^{\pm}}_{j,k}(\xi) \right) e^{i\langle m,\xi \rangle} d\xi,$$

$$C^{\Delta}_{j,k,m} = -\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} |A_{2^j} S^T_k \xi|^2 A_{2^j} S^T_k \left(\hat{U}^{\pm}(\xi).\overline{\hat{\psi}^{\pm}_{j,k}}(\xi) \right) e^{i\langle m,\xi \rangle} d\xi.$$

For simplicity, we consider $\Gamma := A_{2^j} S^T_k \left(\hat{U}^{\pm}(\xi).\overline{\hat{\psi}^{\pm}_{j,k}}(\xi) \right)$. Hence $C^{\Delta}_{j,k,m}$ can be rewritten as

$$C^{\Delta}_{j,k,m} = -\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} |A_{2^j} S^T_k \xi|^2 \Gamma e^{i\langle m,\xi \rangle} d\xi, \tag{15}$$

where

$$|A_{2^j} S_k^T \xi|^2 = 2^j[(2^j + k^2(n-1))\xi_1^2 + \xi_2^2 + \cdots + \xi_n^2 + 2k\xi_1\xi_2 + \cdots + 2k\xi_1\xi_n]. \tag{16}$$

Denoting the right hand side of (16) by $\Theta(\xi)$, then $C^{\Delta}_{j,k,m}$ can be rewritten as

$$C^{\Delta}_{j,k,m} = -\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}[\Theta(\xi)]\Gamma e^{i\langle m,\xi\rangle}d\xi.$$

It can be seen that $C^{\Delta}_{j,k,m}$ is a combination of the following terms

$$\begin{aligned} C^{\Delta}_{k_1} &= 2^j[(2^j + k^2(n-1))\frac{\partial^2 C_{j,k,m}}{\partial m_1^2} + \frac{\partial^2 C_{j,k,m}}{\partial m_2^2} + \cdots + \frac{\partial^2 C_{j,k,m}}{\partial m_n^2}], \\ C^{\Delta}_{k_1k_n} &= 2^{j+1}k[\frac{\partial^2 C_{j,k,m}}{\partial m_1\partial m_2} + \frac{\partial^2 C_{j,k,m}}{\partial m_1\partial m_3} + \cdots + \frac{\partial^2 C_{j,k,m}}{\partial m_1\partial m_n}]. \end{aligned} \tag{17}$$

Replacing (17) in (13) leads to

$$i|\xi_{n+1}|C_{j,k,m} - {c'}^2\left[C^{\Delta}_{k_1} + C^{\Delta}_{k_1k_n}\right] = 0. \tag{18}$$

For each j, k, m, (18) is a time independent PDEs, which can be solved by some common methods such as finite difference and pseudo-spectral methods to find the coefficients $C_{j,k,m}$.

Conclusion

A method for solution of n-dimensional transient heat equation by making use of shearlet frames is presented. This approach is general and can be employed for other PDE problems such as Poisson and wave equation. As it was shown, in this approach the unknown function is approximated by an expansion via shearlet frame expansion and the coefficients of this expansion are obtained by employing Fourier transformation and Planchere theorem. The main merit of this approach is that for finding the unknown coefficients there is no need to solve a system of simultaneous algebric equations and each of the expansion coefficients can be obtained from a separte time independent differential equation.

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