From shift invariant to Gabor- type invariant spaces

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Abstract: The theory of shift invariant subspaces on locally compact abelian groups has been extremely grown in last two decades. Let G be a locally compact abelian group with a closed, discrete and cocompact subgroup Γ. A closed subspace V of $L^2(G)$ is said to be shift invariant if it is invariant under shifts by elements of Γ. The omission of some conditions (discrete and cocompact) on Γ generalizes shift invariant spaces to translation invariant spaces. In this paper we briefly review this generalization. We also prove similar results on modulation invariant spaces which lead to characterization of Gabor- type invariant spaces.

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Introduction

For a locally compact abelian (LCA) group G, a closed subgroup Γ of G is called uniform lattice if it is discrete and cocompact (i.e. the quotient group G/Γ is compact). A closed subspace V of $L^2(G)$ is said to be shift invariant if $T_\gamma V \subseteq V$ for all $\gamma \in \Gamma$, where T_γ is the shift operator defined as $T_\gamma : L^2(G) \longrightarrow L^2(G)$, $T_\gamma f(x) = f(x-\gamma)$. A closed subspace W of $L^2(G)$ is defined to be modulation invariant with respect to Λ (Λ- modulation invariant) if it is invariant under modulations by elements of a closed subgroup Λ (which is not necessarily discrete or cocompact) of the dual group $\widehat{G}$. Our goal in this paper is to review the history of concept of shift invariant spaces which is studied in the last two decades. By a similar approach we define modulation invariant spaces and characterize these spaces in terms of range functions. We have organized the article as follows. The rest of this section is devoted to stating some required preliminaries on LCA groups and shift invariant spaces which were studied in [7]. Section 2 consists of two generalizations of shift invariant spaces. Following an idea of [3], using two transformations of $L^2(G)$ into a vector valued space, we find a correspondence between translation invariant spaces of $L^2(G)$ and multiplicatively invariant subspaces of the vector valued space, which yields the desired characterization. Finally in section 3, we characterize modulation invariant and Gabor- type invariant spaces in terms of range functions.

Assume that G is a second countable LCA group, Γ is a uniform lattice in G, and Ω is a fundamental domain for Γ^* in $\widehat{G}$ with a measure $d\xi$ on it. In the following proposition which is [7, Proposition 2.1], it is shown that $L^2(G)$ is isometrically isomorphic to the space $L^2(\Omega, l^2(\Gamma^*))$ of square integrable functions from Ω to $l^2(\Gamma^*)$.

Proposition 1. *The mapping $\mathcal{T} : L^2(G) \longrightarrow L^2(\Omega, l^2(\Gamma^*))$, defined by $\mathcal{T}f(\xi) = (\widehat{f}(\xi\eta))_{\eta\in\Gamma^*}$ is an isometric isomorphism, between Hilbert spaces $L^2(G)$ and $L^2(\Omega, l^2(\Gamma^*))$.*

A range function is a mapping

$$J : \Omega \to \{\text{closed subspaces of } l^2(\Gamma^*)\}.$$

J is called measurable if the associated orthogonal projections $P(\omega) : l^2(\Gamma^*) \to J(\omega)$ are measurable i.e. $\omega \mapsto < P(\omega)a, b >$ is measurable for each $a, b \in l^2(\Gamma^*)$. The main result of this section is the following characterization theorem in $L^2(G)$([7, Theorem 3.1]. The following theorem characterizes shift invariant spaces in terms of range functions. Note that in this case the subgroup Γ is closed, discrete and cocompact.

Theorem 1. *Suppose G is a second countable LCA group, Γ is a uniform lattice in G, and Ω is a fundamental domain for Γ^* in $\widehat{G}$. A closed subspace $V \subseteq L^2(G)$ is shift invariant (with respect to the uniform lattice Γ) if and only if $V = \{f \in L^2(G),\ \mathcal{T}f(\omega) \in J(\omega)\ for\ a.e\ \omega \in \Omega\}$, where J is a measurable range function and $\mathcal{T}$ is the mapping as in Proposition 1. The correspondence between V and J is one to one under the convention that the range functions are identified if they are equal a.e. Moreover, if $V = S(\mathcal{A})$ for some countable set $\mathcal{A} \subseteq L^2(G)$ then*

$$J(\omega) = \overline{span}\{\mathcal{T}\varphi(\omega);\ \varphi \in \mathcal{A}\}. \tag{1}$$

Main results

The goal of this section is to study generalizations of shift invariant spaces to translation invariant spaces(the case that Γ is not discrete or cocompact). To this end, we first consider vector valued spaces and multiplicatively invariant subspaces, and we show that there is a one to one correspondence between translation invariant spaces on $L^2(G)$ and multiplicatively invariant spaces on the vector valued space.

Let (Ω, m) be a σ- finite measure space and $\mathcal{H}$ be a separable Hilbert space. A range function is a mapping $J : \Omega \longrightarrow \{$ closed subspaces of $\mathcal{H}$ $\}$. We write $P_J(\omega)$ for the orthogonal projections of $\mathcal{H}$ onto $J(\omega)$. A range function J is measurable if the mapping $\omega \mapsto \langle P_J(\omega)(a), b\rangle$ is measurable for all $a, b \in \mathcal{H}$. Consider the space $L^2(\Omega, \mathcal{H})$ of all measurable functions ϕ from Ω to $\mathcal{H}$ such that $\|\phi\|_2^2 = \int_\Omega \|\phi(\omega)\|^2_{\mathcal{H}} dm(\omega) < \infty$ with the inner product $\langle \phi, \psi\rangle = \int_\Omega \langle \phi(\omega), \psi(\omega)\rangle_{\mathcal{H}} dm(\omega)$. It can be shown that $L^2(\Omega, \mathcal{H})$ is isometrically isomorphic to $L^2(\Omega) \otimes \mathcal{H}$, where $\otimes$ denotes the tensor product of Hilbert spaces. A subset $\mathcal{D}$ of $L^\infty(\Omega)$ is said to be a determinig set for $L^1(\Omega)$, if for any $f \in L^1(\Omega)$, $\int_\Omega fg dm = 0$ for all $g \in \mathcal{D}$ implies that $f = 0$. A closed subspace W of $L^2(\Omega, \mathcal{H})$ is called multiplicatively invariant with respect to a determining set $\mathcal{D}$, if for each $\phi \in W$ and $g \in \mathcal{D}$, one has $g\phi \in W$. Bownik and Ross in [3, Theorem 2.4], showed that there is a correspondence between multiplicatively invariant spaces and measurable range functions as follows.

Proposition 2. *Suppose that $L^2(\Omega)$ is separable, so that $L^2(\Omega, \mathcal{H})$ is also separable. Then for a closed subspace W of $L^2(\Omega, \mathcal{H})$ and a determining set $\mathcal{D}$ for $L^1(\Omega)$ the following are equivalent.*
(1) W is multiplicatively invariant with respect to $\mathcal{D}$.
(2) W is multiplicatively invariant with respect to $L^\infty(\Omega)$.
(3) There exists a measurable range function J such that

$$W = \{\phi \in L^2(\Omega, \mathcal{H}) : \phi(\omega) \in J(\omega)\ , \quad a.e.\ \omega \in \Omega\}.$$

Identifying range functions which are equivalent almost everywhere, the correspondence between $\mathcal{D}$- multiplicatively invariant spaces and measurable range functions is one to one and onto.

Now assume that G is a second countable LCA group and Γ is a closed cocompact subgroup of G. Assume that Γ^* is the annihilator of Γ in $\widehat{G}$. Also suppose that Ω is a measurable section for the quotient $\widehat{G}/\Gamma^*$ and C is a measurable section for the quotient G/Γ. For $\gamma \in \Gamma$ we denote by X_γ the associated character on $\widehat{G}$, i.e. $X_\gamma(\chi) = \chi(\gamma)$ for $\chi \in \widehat{G}$. One can see that the set $\mathcal{D} = \{X_\gamma|_\Omega : \gamma \in \Gamma\}$ is a determining set for $L^1(\Omega)$. A closed subspace $V \subseteq L^2(G)$ is called Γ-translation invariant space, if $T_\gamma V \subseteq V$ for all $\gamma \in \Gamma$. We say that V is generated by a countable subset $\mathcal{A}$ of $L^2(G)$, when $V = S^\Gamma(\mathcal{A}) = \overline{span}\{T_\gamma f : f \in \mathcal{A}, \gamma \in \Gamma\}$. Using Proposition 1, one can characterize translation invariant spaces in terms of range functions. The following theorem which is proved in [3, Theorem 3.8], characterizes translation invariant spaces in the case that Γ is a closed and cocompact subgroup(not necessarily discrete). Notice that the condition Γ is discrete is not used in the proof of Proposition 1.

Theorem 2. *Let $V \subseteq L^2(G)$ be a closed subspace and $\mathcal{T}$ be as in Proposition 1. Then the following are equivalent.*
(1) V is a Γ- translation invariant space.
(2) $\mathcal{T}(V)$ is a multiplicatively invariant subspace of $L^2(\Omega, l^2(\Gamma^*))$ with respect to the determining set $\mathcal{D} = \{X_\gamma|_\Omega : \gamma \in \Gamma\}$.
(3) There exists a measurable range function $J : \Omega \longrightarrow \{closed\ subspaces\ of\ l^2(\Gamma^*)\}$ such that

$$V = \{f \in L^2(G) : \mathcal{T}(f)(\omega) \in J(\omega),\ \textit{for a.e. } \omega \in \Omega\}.$$

Identifying range functions which are equivalent almost everywhere, the correspondence between Γ- translation invariant spaces and measurable range functions is one to one and onto.

Now we omit the condition Γ is cocompact from the definition of translation invariant spaces. Assume G is a second countable LCA group and Γ is an arbitrary closed subgroup of G which is necessarily discrete or cocompact. A closed subspace $V \subseteq L^2(G)$ is called Γ- generalized translation invariant space, if $T_\gamma V \subseteq V$ for all $\gamma \in \Gamma$. In the rest of this section, following ideas of [1] to characterize generalized translation invariant spaces in terms of range functions.

In [1, Proposition 6.4] it is shown that there exists an isometric isomorphism between $L^2(G)$ and $L^2(\Omega, L^2(C))$, namely $Z : L^2(G) \longrightarrow L^2(\Omega, L^2(C))$ satisfying

$$Z(T_\gamma \phi) = X_\gamma|_\Omega Z(\phi). \tag{2}$$

Proposition 3. *Let $V \subseteq L^2(G)$ be a closed subspace and Z be as above. Then the following are equivalent.*
(1) V is a Γ- generalized translation invariant space.
(2) $Z(V)$ is a multiplicatively invariant subspace of $L^2(\Omega, L^2(C))$ with respect to the determining set $\mathcal{D} = \{X_\gamma|_\Omega : \gamma \in \Gamma\}$.
(3) There exists a measurable range function $J : \Omega \longrightarrow \{closed\ subspaces\ of\ L^2(C)\}$ such that

$$V = \{f \in L^2(G) : Z(f)(\omega) \in J(\omega),\ \textit{for a.e. } \omega \in \Omega\}.$$

Identifying range functions which are equivalent almost everywhere, the correspondence between Γ- *generalized translation invariant spaces and measurable range functions is one to one and onto.*

In the sequel of this section, we investigate modulation invariant spaces and Gabor invariant spaces on locally compact abelian groups.

Let Λ be a closed subgroup of $\widehat{G}$. Assume that Λ^* is the annihilator of Λ in G, i.e. $\Lambda^* = \{x \in G : \lambda(x) = 1,\ \lambda \in \Lambda\}$. In addition, suppose that Π is a measurable section for the quotient G/Λ^* and D is a measurable section for the quotient $\widehat{G}/\Lambda$. For $\lambda \in \Lambda$ we denote by X_λ the corresponding character on G. One can see that the set $\mathcal{D} = \{X_\lambda|_\Pi : \lambda \in \Lambda\}$ is a determining set for $L^1(\Pi)$. A closed subspace $W \subseteq L^2(G)$ is called Λ- modulation invariant space, if $M_\lambda W \subseteq W$ for all $\lambda \in \Lambda$, where M_λ is the modulation operator defined as $M_\lambda : L^2(G) \longrightarrow L^2(G)$, $M_\lambda f(x) = \lambda(x) f(x)$. We say that W is generated by a countable subset $\mathcal{A}$ of $L^2(G)$, when $W = M^\Lambda(\mathcal{A}) = \overline{span}\{M_\lambda f : f \in \mathcal{A}, \lambda \in \Lambda\}$.

Let $\mathcal{F}$ denote the Fourier transform and Z be the Zak transform. We define an isometric isomorphism as

$$\tilde{Z} : L^2(G) \longrightarrow L^2(\Pi, L^2(D)),\ \tilde{Z} := Z\ o\ \mathcal{F}. \tag{3}$$

In the next theorem, we show that $\tilde{Z}$ turns Λ- modulation invariant spaces in $L^2(G)$ into multiplicatively invariant spaces in $L^2(\Pi, L^2(D))$ and vice versa. Further we establish a characterization of Λ- modulation invariant spaces in terms of range functions. The main idea of the proof is that the Fourier transform maps Λ- modulation invariant subspaces of $L^2(G)$ to Λ-translation invariant subspaces of $L^2(\widehat{G})$. Note that in the following theorem we assume that Λ is a closed subgroup which is not necessarily discrete and cocompact.

Theorem 3. *Let $W \subseteq L^2(G)$ be a closed subapace and $\tilde{Z}$ be as in* (3)*. Then the following are equivalent.*
(1) W is a Λ- modulation invariant space.
(2) $\tilde{Z}(W)$ is a multiplicatively invariant subspace of $L^2(\Pi, L^2(D))$ with respect to the determining set $\mathcal{D} = \{X_\lambda|_\Pi : \lambda \in \Lambda\}$.
(3) There exists a measurable range function $J : \Pi \longrightarrow \{closed\ subspaces\ of\ L^2(D)\}$ such that

$$W = \{f \in L^2(G) : \tilde{Z}(f)(x) \in J(x),\ \text{for a.e. } x \in \Pi\}. \tag{4}$$

Identifying range functions which are equivalent almost everywhere, the correspondence between Λ- *modulation invariant spaces and measurable range functions is one to one and onto.*

Remark 1. *Gabor-type invariant spaces are translation invariant spaces that have the extra condition to be also invariant under modulations. Due to the important role of the Gabor theory in mathematical analysis and its applications, it is important to study Gabor-type invariant spaces. Combining the results on translation invariant spaces in [1] and our results on modulation invariant spaces, we can give a characterization of Gabor-type invariant subspaces of $L^2(G)$. Let $\mathcal{W}$ be a Gabor-type invariant space. Transforming $\mathcal{W}$ into a certain vector valued space, gives a corresponding of Gabor-type invariant spaces and multiplicatively invariant spaces. By a similar approach as in the proof of Theorem 3, we can use Proposition 2 to characterize Gabor-type invariant subspaces of $L^2(G)$. Then the result of [5] is a special case of our result (when closed subgroups with respect to translation and modulation are both discrete and cocompact).*

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