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Copula parameter estimation using nonlinear quantile regression

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Abstract

The aim of this paper is to propose a semiparametric method for the estimation of the copula parameters based on a nonlinear quantile regression model. The estimation of the dependence parameter has been selected as the value that minimizes the distance between one of the pseudo samples and the inverse of the quantile regression. A simulation study is performed to measure the performance of this method. The simulation results are compared to the maximum pseudo-likelihood (MPL) method and minimum pseudo-Hellinger distance (MPHD) method for well-known bivariate copula models. These results show that the proposed method based on the copula quantile regression model has a good performance in small sample sizes.

Keywords: Quantile regression; Copula parameter estimation; Minimum distance; Semiparametric Estimation.
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1 Introduction

The copulas describe the dependence between random vector components. Unlike marginal and joint distributions that are clearly observable, the copula of a random vector is a hidden dependence structure

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that connects the joint distribution with its margins. The copula parameter captures the inherent dependence between the marginal variables and it can be estimated by either parametric or semiparametric methods. Maximum likelihood estimation (MLE), which is used to estimate the parameter of any type of model, is the most effective method. It can also be applied to copula, but the problem becomes complicated as the number of parameters and dimension of copula increases, because the parameters of the margins and copula are estimated simultaneously. Therefore, MLE is highly affected by misspecification of marginal distributions.

Inference functions for margins (IFM) method is another method for estimating the copula parameter introduced by Joe (2005). In this method, similar to MLE method, the margins of the copula are important, because the parameter estimation is dependent on the choice of the marginal distributions. In IFM method, the parameters are estimated in two stages. In the first stage, parameters of margins are estimated and then the parameters of copula will be evaluated given the values from the first step.

Genest et al. (1995) introduce a semiparametric method, known as maximum pseudo likelihood (MPL) estimation, similar to MLE. The only difference between this method and MLE is that the data must be converted to pseudo observations. The consistency and asymptotic normality of this method is established in their paper. They established that this method is efficient for independent copula. The results of an extensive simulation studied by kim at al. (2007) show that MLE and IFM methods are non-robust against misspecification of the marginal distributions, and that MPL estimation method performs better than ML and IFM methods, overall.

The minimum distance (MD) method attains one of the most attractive alternatives to the MLE method, because the non-parametric estimator of MD has nice robustness properties. The MD method for copulas has attracted only a little attention in contrast to the MPL and IFM methods. Tsukahara (2005) explores the empirical asymptotic behaviour of CvM and KS distances between the hypothesised and empirical copula in a simulation study. He finds that the MPL estimator should be preferred to the MD estimator. Weiß (2011) presented a comprehensive Monte Carlo simulation study on the performance of minimum distance and maximum likelihood estimators for bivariate parametric copulas. Mohammadi et al. (2020) introduced a new minimum distance estimator based on Hellinger distance and their simulation results showed that the minimum pseudo Hellinger distance method has good performance in small sample size and weak dependency.

The authors present a quantile regression-based semiparametric technique for estimating the copula parameter. They use a technique called "Minimum Pseudo Copula Quantile Regression" ($MPCQR$) to estimate the minimum distance between one of the pseudo samples and the inverse of the quantile regression. The purpose of this paper is to provide a comprehensive simulation study of its performance for bivariate copulas.

As is common in the literature, we focus on the bivariate case. The rest of the paper is arranged as follows. In Section 2, the preliminaries for copulas and MPL method are described. The minimum pseudo Hellinger distance method is provided in Section 3. In Section 4, the parameter estimation using copula quantile regression is introduced. The simulation results are provided to compare the MPL, MPHD, and MPCQR methods in Section 5.

2 Preliminaries

Some definitions related to a copula function will be briefly reviewed. Sklar (1959) was the primary to display the fundamental concept of the copula. Let (X, Y) be a continuous random variable with joint cumulative distribution function (cdf) F, then copula C corresponding to F defined as:

$$F(x, y) = C(F_X(x), F_Y(y)), \qquad (x, y) \in R^2, \tag{1}$$

where F_X and F_Y are the marginal distributions of X and Y, respectively. A bivariate copula function C is a cumulative distribution function of random vector (U, V), defined on the unit square $[0, 1]^2$, with

Table 1: Some well-known bivariate copulas

Copula	$C(u,v;\theta)$	Parameter Space	Kendall's tau
Clayton	$(u^{-\theta}+v^{-\theta}-1)^{-1/\theta}$	$\theta\in(-1,+\infty)-\{0\}$	$\frac{\theta}{\theta+2}$
Gumbel	$\exp\Big\{-\Big[(-\ln u)^{\theta}+(-\ln v)^{\theta}\Big]^{1/\theta}\Big\}$	$\theta\in[1,+\infty)$	$\frac{\theta-1}{\theta}$
Frank *	$\frac{-1}{\theta}log\Big\{1+\frac{(e^{-u\theta}-1)(e^{-v\theta}-1)}{e^{-\theta}-1}\Big\}$	$\theta\in(-\infty,+\infty)-\{0\}$	$1+\frac{4}{\theta}(D_1(\theta)-1)$
Gaussian †	$\Phi_2(\Phi^{-1}(u),\Phi^{-1}(v);\theta)$	$\theta\in[-1,+1]$	$\frac{2}{\pi}arcsin(\theta)$
T ‡	$t_{2,\nu}(t_{\nu}^{-1}(u),t_{\nu}^{-1}(v);\theta)$	$\theta\in[-1,+1],\nu>1$	$\frac{2}{\pi}arcsin(\theta)$

uniform marginal distributions as $U=F_X(X)$ and $V=F_Y(Y)$.

The authors shall write $C(u,v;\theta)$ for a family of copulas indexed by the parameter θ. If $C(u,v;\theta)$ is an absolutely continuous copula distribution on $[0,1]^2$, then its density function is $c(u,v;\theta)=\frac{\partial^2 C(u,v;\theta)}{\partial u\partial v}$. As a result, the relationship between the copula density function (c) and the joint density function (f) of (X,Y) according to equation (1) can be represented as

$$f(x,y)=c(F_X(x),F_Y(y);\theta)f_X(x)f_Y(y),\qquad (x,y)\in R^2, \tag{2}$$

where f_X and f_Y are the marginal density functions of X and Y, respectively.

Table 1 presents summary information of some well-known bivariate copulas such as the parameter space and Kendall's tau (τ) of them. In this table, Clayton, Gumbel, and Frank copulas belong to the class of Archimedean copulas and Gaussian and T copulas belong to the class of Elliptical copulas. The copula-based Kendall's tau association for continuous variables X and Y with copula C is given by $\tau=4\int_{[0,1]^2}C(u,v)dC(u,v)-1$.

2.1 Semiparametric maximum likelihood estimation

In view of (2), the log-likelihood function takes the form

$$\mathcal{L}(\theta)=\sum_{i=1}^{n}log\big(c(F(x),G(y);\theta)\big)+\sum_{i=1}^{n}log\big(f(x)\big)+\sum_{i=1}^{n}log\big(g(y)\big).$$

Hence the MLE of θ, which is denoted by $\hat{\theta}_{ML}$, is the global maximizer of $\mathcal{L}(\theta)$ and $\sqrt{n}(\hat{\theta}_{ML}-\theta)$ converges to a Gaussian distribution with zero mean, where θ is the true value. Since it is assumed that the model is correctly specified and hence $\mathcal{L}(\theta)$ is the correct log-likelihood, it follows that the MLE enjoys some optimality properties and so, it is the preferred first option. If the model is not correctly specified so that $\mathcal{L}(\theta)$ is not the correct log-likelihood, then the maximizer of $\mathcal{L}(\theta)$ is not the MLE and hence it may lose its preferred status.

In MPL method, the marginal distributions have unknown functional forms. Estimation of marginal distributions are estimated non parametrically by their sample empirical distributions. Then, θ is estimated by the maximizer of the pseudo log-likelihood as

$$\hat{\theta}_{MPL}=\arg\max_{\theta}\sum_{i=1}^{n}log\big(c(\tilde{U}_i,\tilde{V}_i;\theta)\big), \tag{3}$$

where $\tilde{U}_i=n\hat{F}_X(x_i)/(n+1)$, $\tilde{V}_i=n\hat{F}_Y(y_i)/(n+1)$ for $i=1,\cdots,n$, are the pseudo observations and $\hat{F}_X$ and $\hat{F}_Y$ are the empirical cumulative distribution function of the observation X_i and Y_i, respectively.

* $D_k(\theta)=\frac{k}{\theta^k}\int_0^{\theta}\frac{t^k}{e^t-1}dt$.

† Φ^{-1} is the inverse of the standardized univariate Gaussian distribution and Φ_2 is the standardized bivariate Gaussian distribution with correlation parameter θ.

‡ t_{ν}^{-1} is the inverse of the standardized univariate Student's t distribution with ν degree of freedom and $t_{2,\nu}$ is the standardized bivariate Student's t distribution with correlation coefficient θ and ν degree of freedom.

The authors shall refer to (3) as the maximum pseudo likelihood (MPL) estimator of θ. Genest et al. (1995) and Tsukahara (2005) showed that $\hat{\theta}_{MPL}$ is consistent estimator. This non-linear optimization problem can easily be solved by Statistical programming language R or Mathematica.

3 Minimum Hellinger distance estimation

Initially, Chernoff (1952) proposed the Alpha-Divergence, which is a generalization of the KL divergence. The Alpha-Divergence ($\mathcal{AD}$) between two probability density functions f_1 and f_2 of a continuous random variable can be defined as:

$$\mathcal{AD}_\alpha(f_1 \parallel f_2) = \frac{1}{\alpha(\alpha-1)}\Big(\int_{[0,1]^2} f_1^{\alpha}(x)\ f_2^{1-\alpha}(x)dx - 1\Big), \qquad \alpha \in R \setminus \{0,1\}. \tag{4}$$

The $\mathcal{AD}$ divergence is non-negative and true equality to zero holds if and only if $f_1(x) = f_2(x)$. If $\alpha \to 1$, the Kullback-Leibler divergence (KLD) can be obtained from equation (4). The well-known Hellinger distance (HD) and Neyman (Neyman Chi-square) divergence (ND) can be obtained from equation (4) for $\alpha = 0.5$ and $\alpha = 2$,

The Alpha-Divergence between copula density estimation $\hat{c}(u, v)$ and true copula density $c(u, v; \theta)$ based on pseudo observation can be obtained MPAD estimation defined as $\hat{\theta}_{MPAD} = \arg\min_\theta \mathcal{AD}(\hat{c}||c)$. It is well known that maximizing the likelihood is equivalent to minimizing the KL divergence. So, the minimum pseudo KL divergence (MPKLD) between copula density estimation $\hat{c}(u, v)$ and true copula density $c(u, v; \theta)$ as a special case of MPAD estimator, is equivalent to the MPL estimator.

The minimum pseudo Hellinger distance (MPHD) is given by

$$\begin{aligned}\hat{\theta}_{MPHD} &= \arg\min_\theta HD(\hat{c}||c) = \arg\min_\theta \frac{1}{2}\int_{[0,1]^2} (\sqrt{\hat{c}(u,v)} - \sqrt{c(u,v;\theta)})^2\ dudv \\ &= \arg\min_\theta \int_{[0,1]^2} \Big(1 - \sqrt{\frac{c(u,v;\theta)}{\hat{c}(u,v)}}\Big)^2 dC_n(u,v) \\ &= \arg\min_\theta \frac{1}{n}\sum_{i=1}^{n} \Big(1 - \sqrt{\frac{c(\tilde{U}_i,\tilde{V}_i;\theta)}{\hat{c}(\tilde{U}_i,\tilde{V}_i)}}\Big)^2, \end{aligned} \tag{5}$$

where $\tilde{U}_i$ and $\tilde{V}_i$ for $i = 1, \cdots, n$, are the pseudo observations. Mohammadi et al. (2020) showed that the Hellinger distance had better performance than Neyman Divergence in almost always for some bivariate Archimedean and Elliptical copulas. Thus, in this paper, the author only consider Hellinger distance as special cases of Alpha-Divergence based on pseudo observations to obtain the copula parameter estimation.

In practice, instead of $\hat{c}$ in equation (5), the local likelihood probit transformation estimation of copula density ($\hat{c}_n^{(\mathcal{LLPT})}$) will be used. Let $(U_i, V_i)_{i=1,\ldots,n}$ are independent and identically distributed observations from the bivariate copula C and the purpose is to estimate the corresponding copula density function. Denote Φ as the standard Gaussian distribution and ϕ as its first order derivative. Then $(S_i, T_i) = (\Phi^{-1}(U_i), \Phi^{-1}(V_i))$ is a random vector with Gaussian margins and copula C. According to (2), the corresponding density function can be written as $f(s,t) = c(\Phi(s), \Phi(t))\phi(s)\phi(t)$. Thus, an estimation of the copula density function can be given by

$$\hat{c}_n^{(\mathcal{PT})}(u,v) = \frac{\hat{f}_n(\Phi^{-1}(u), \Phi^{-1}(v))}{\phi(\Phi^{-1}(u))\phi(\Phi^{-1}(v))}, \qquad (u,v) \in (0,1)^2. \tag{6}$$

This kernel estimator has asymptotic problems at the edges of the distribution support. To remedy this problem, local likelihood probit transformation ($\mathcal{LLPT}$) method was recently suggested by Geenens et al. (2017). Instead of applying the standard kernel estimator, they locally fit a polynomial to the log-density of the transformed sample. Recently, Nagler (2018) with a comprehensive simulation study has shown that $\mathcal{LLPT}$ method for copula density estimation yields very good.

4 Minimum copula quantile regression method

When the distribution function of the variables is not normal, the conditional expectation $E(Y|X)$ does not suffice to give full information on the conditional distribution function. In such cases, the quantile regression is used. Copula-quantile regression, which is introduced by Bouye and Salmon (2009) is a nonlinear form of quantile regression. They studied properties and application of this model.

To apply quantile regression, one needs to know the conditional copula distribution which is given by $C_{V|U}(v|u) = \frac{\partial C(u,v)}{\partial u}$. It can be proved that $0 \leq C_{V|U}(v|u) \leq 1$. Now, the p^{th} quantile curve of v conditional on u is defined by

$$C_{V|U}(v|u) = p, \tag{7}$$

and rearranging with respect to v the copula-quantile regression is given by

$$v = g(u, p; \theta). \tag{8}$$

If $p = 0.5$ is considered, the median regression is obtained. Bouye and Salmon (2009) finding and established the expression of quantile regression for four most applicable copulas, i.e., Clayton, Frank, Gaussian, and T copulas as

(i) Clayton copula: $v = ((p^{-\frac{\theta}{1+\theta}} - 1)u^{-\theta} + 1)^{-\frac{1}{\theta}}$;

(ii) Frank copula: $v = -\frac{1}{\theta} \log\left(1 - (1 - e^{-\theta})(1 + e^{-\theta u}(p^{-1} - 1))^{-1}\right)$;

(iii) Gaussian copula: $v = \Phi\left(\theta\Phi^{-1}(u) + \sqrt{1-\theta^2}\Phi^{-1}(p)\right)$;

(iv) T-copula: $v = t_\nu\left(\theta t_\nu^{-1}(u) + \sqrt{(1-\theta^2)(1+\nu)^{-1}(\nu + t_\nu^{-1}(u)^2)}t_{\nu+1}^{-1}(p)\right)$.

It is worth noting that the Gumbel copula does not have a closed form for the copula-quantile regression method. Therefore, its copula-quantile regression has to be found numerically.

Now, the authors define the minimum pseudo copula quantile regression (MPCQR) method as

$$\hat{\theta}_{MPCQR} = \arg\min_{\theta} \sum_{i=1}^{n} \left(\tilde{V}_i - g(\tilde{U}_i, p; \theta)\right)^2 \tag{9}$$

where $\tilde{U}_i$ and $\tilde{V}_i$ for $i = 1, \cdots, n$, are the pseudo observations.

5 Simulation study

A simulation study was performed to compare the MPL and MPHD estimators to the MPCQR estimator. The aim of this simulation study is to compare the true parameter θ with the parameter estimate $\hat{\theta}$, under the assumption that the copula's parametric form is correctly selected. This aim is accomplished by comparing the Bias and mean square error (MSE) of the three approaches of copula parameter estimations.

The data are generated from three Archimedean copulas such as Clayton, Gumbel, and Frank and two Elliptical copulas such as Gaussian and T (ν=2 and ν=10) copulas with Kendall's tau 0.2, 0.4, and 0.6 that are presented in Table 1. These copulas cover different dependence structures. Gaussian and Frank copulas exhibit symmetric and weak tail dependence in both lower and upper tails. The Clayton copula exhibits strong left tail dependence and the Gumbel copula has strong right tail dependence. In T copula with positive dependency and small degrees of freedom ($\nu < 10$) tail dependency occurs in both lower and upper tails and as the degree of freedom increases, dependency in the tail areas decreases (see Demarta and McNeil (2005)). Moreover, 1000 Monte Carlo samples of sizes $n = 30$ and 150 are generated from each type of copulas and the three estimates are computed: MPL, MPHD, and MPCQR.

Results of the simulation study are presented in Tables 2-5. These tables present the Bias and MSE relative to the three estimators of the respective copulas for different values of sample sizes and Kendall's tau. The simulation procedure was performed for the positive and negative values of Kendall's tau and

Table 2: estimated Bias of the estimators for Archimedean copulas

Copula	τ	$n=30$			$n=150$		
		$\hat{\theta}_{MPL}$	$\hat{\theta}_{MPHD}$	$\hat{\theta}_{MPCQR}$	$\hat{\theta}_{MPL}$	$\hat{\theta}_{MPHD}$	$\hat{\theta}_{MPCQR}$
Clayton	0.2	0.0288	-0.0180	-0.0173	0.0107	-0.0129	-0.0582
	0.4	0.0624	-0.0516	-0.0425	0.0181	-0.0228	-0.1133
	0.6	0.0807	-0.2256	-0.0754	0.0347	-0.1119	-0.2790
Gumbel	0.2	0.0373	-0.0219	-0.0203	0.0021	-0.0076	-0.0213
	0.4	0.0460	-0.0414	-0.0322	0.0028	-0.0106	-0.0432
	0.6	0.0730	-0.2323	-0.0525	0.0045	-0.1357	-0.1427
Frank	0.2	0.1222	-0.1032	-0.0961	0.0685	-0.0737	-0.0850
	0.4	0.1436	-0.1247	-0.1135	0.0894	-0.0918	-0.1169
	0.6	0.1588	-0.2594	-0.1232	0.1208	-0.2004	-0.2127

Table 3: estimated Bias of the estimators for Elliptical copulas

Copula	τ	$n=30$			$n=150$		
		$\hat{\theta}_{MPL}$	$\hat{\theta}_{MPHD}$	$\hat{\theta}_{MPCQR}$	$\hat{\theta}_{MPL}$	$\hat{\theta}_{MPHD}$	$\hat{\theta}_{MPCQR}$
Gaussian	0.2	-0.0188	-0.0146	-0.0127	-0.0081	-0.0095	-0.0126
	0.4	-0.0215	-0.0192	-0.0175	-0.0023	-0.0116	-0.0296
	0.6	-0.0164	-0.0326	-0.0152	-0.0010	-0.0227	-0.0297
$T(\nu=2)$	0.2	-0.0230	-0.0214	-0.0181	-0.0101	-0.0124	-0.0329
	0.4	-0.0158	-0.0483	-0.0101	-0.0129	-0.0162	-0.0669
	0.6	-0.0148	-0.0516	-0.0126	-0.0088	-0.0326	-0.0761
$T(\nu=10)$	0.2	0.0065	-0.0042	-0.0038	0.0005	-0.0024	-0.0125
	0.4	0.0030	-0.0384	-0.0023	0.0003	-0.0124	-0.0236
	0.6	-0.0025	-0.0460	-0.0015	0.0007	-0.0194	-0.0317

Table 4: estimated MSE of the estimators for Archimedean copulas

Copula	τ	$n=30$			$n=150$		
		$\hat{\theta}_{MPL}$	$\hat{\theta}_{MPHD}$	$\hat{\theta}_{MPCQR}$	$\hat{\theta}_{MPL}$	$\hat{\theta}_{MPHD}$	$\hat{\theta}_{MPCQR}$
Clayton	0.2	0.0944	0.0689	0.0514	0.0232	0.0216	0.0298
	0.4	0.1092	0.0818	0.0756	0.0341	0.0525	0.0737
	0.6	0.2121	0.2925	0.1824	0.0834	0.1753	0.2002
Gumbel	0.2	0.0349	0.0226	0.0217	0.0086	0.0079	0.0159
	0.4	0.0486	0.0342	0.0203	0.0121	0.0216	0.0278
	0.6	0.1077	0.1185	0.0954	0.0254	0.0537	0.0640
Frank	0.2	0.5950	0.5167	0.4932	0.2554	0.2611	0.2997
	0.4	0.6116	0.5691	0.5135	0.2693	0.2918	0.3487
	0.6	0.6642	0.6984	0.5924	0.3207	0.4379	0.5157

according to the symmetry of the obtained results, the results have been reported only for positive values of Kendall's tau. As the results for the sample sizes greater than 150 were in line with our expectation that the increase in sample size will improve the parameter estimation, the corresponding results were omitted from the tables for brevity. Also, the results show that the MPL method outperforms MPHD and MPCQR for sample sizes greater than 150. The results for the T copula with 4 and 7 degrees of

Table 5: estimated MSE of the estimators for Elliptical copulas

Copula	τ	$n=30$			$n=150$		
		$\hat{\theta}_{MPL}$	$\hat{\theta}_{MPHD}$	$\hat{\theta}_{MPCQR}$	$\hat{\theta}_{MPL}$	$\hat{\theta}_{MPHD}$	$\hat{\theta}_{MPCQR}$
Gaussian	0.2	0.0270	0.0161	0.0116	0.0070	0.0068	0.0108
	0.4	0.0220	0.0141	0.0129	0.0048	0.0062	0.0117
	0.6	0.0085	0.0101	0.0068	0.0015	0.0032	0.0048
$T(\nu=2)$	0.2	0.0372	0.0305	0.0287	0.0122	0.0160	0.0266
	0.4	0.0324	0.0276	0.0227	0.0088	0.0142	0.0217
	0.6	0.0173	0.0248	0.0183	0.0035	0.0089	0.0174
$T(\nu=10)$	0.2	0.0275	0.0245	0.0215	0.0091	0.0115	0.0159
	0.4	0.0242	0.0226	0.0201	0.0066	0.0090	0.0138
	0.6	0.0096	0.0178	0.0079	0.0032	0.0076	0.0111

freedom were omitted as well as the results did not differ from those for the two other T copulas with 2 and 10 degrees of freedom.

The results given in Tables 2-5 show that estimated Bias and MSE of parameter estimation of the Archimedean and Elliptical copulas decrease as sample size increases and parameter estimates improve. The estimated Bias and MSE of parameter estimation increase with increasing Kendall's tau for Archimedean copulas. Also, estimated MSE of parameter estimation decrease with increasing Kendall's tau, whereas estimated Bias of parameter estimation has no clear trend for Elliptical copulas.

The results given in Tables 2-5 show that the MPL yields the best results for the large sample size ($n \geq 100$) and high dependency ($\tau \geq 0.5$). For the small sample size ($n < 100$), MPCQR method outperforms MPL method. Among the MPHD and MPCQR estimators, the results show that $\hat{\theta}_{MPCQR}$ is better than $\hat{\theta}_{MPHD}$ based on MSE in small sample size. This advantage for $\hat{\theta}_{MPCQR}$ is clearer in Archimedean copulas than in Elliptical copulas. In addition to these results, the estimated bias seem to be considerably higher for Archimedean copulas than for Elliptical copulas. Finally, it is necessary to note that although the time required to compute the MPCQR method is longer than the MPL method, but the MPCQR method has accurate and acceptable results for small sample size.

Discussion and Resuls

In this paper, a new method of copula parameter estimation based on quantile regression was presented for some bivariate Archimedean and Elliptical copulas. This method was compared by MPHD and MPL methods to obtain the copula parameter estimation based on pseudo observations. The simulation results suggests that the minimum pseudo copula quantile regression (MPCQR) method has good performance in small sample sizes.

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