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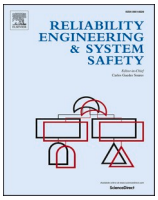

AK-SESC: a novel reliability procedure based on the integration of active learning kriging and sequential space conversion method

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ABSTRACT

To deal with evaluating small failure probabilities, AK–SESC: a novel approach integrating an active learning Kriging meta-model (AK-MCS) and the SESC, a sequential space conversion method, is suggested. The efficiency of the proposed approach relies on the advantages of the AK-MCS and its updating feature to evaluate the actual performance function and the superiority of SESC in estimating small failure probabilities. Although there are effective methods for small probabilities, the beauty of this approach is that it is derived from the probability integral with no simplifications while providing results of high accuracy.

Different problems were solved to study the AK–SESC applicability. The main effort of this method is reducing the function call numbers of the original SESC while reaching the same accuracy as Monte Carlo Simulation (MCS). The reliability analysis results were compared with the main reliability methods of the Importance Sampling (IS), Subset Simulation (SubSim), Line Sampling (LS), First and also Second-Order Reliability Method (FORM and SORM). The solved problems indicate that the proposed approach provides accurate answers with much fewer function calls than SESC. So, it can be a promising method for reliability analyses involving nonlinear or high-dimensional performance functions with small failure probabilities.

1. Introduction

In structural engineering, random features inherent to computational models inevitably induce uncertainties in other parameters. The characterization and propagation of uncertainties across computational models to learn reliability has been essential aspects of structural design. In recent decades, structural reliability analysis has developed in the form of a rational instrument to analyze engineering system design under uncertainties [1]. In this field, some interesting investigations on the reliability assessment of corroded pipelines made of high-strength steel [2, 3], and also fuzzy reliability analysis of nanocomposite zinc oxide beams considering buckling failure mode [4] were recently conducted. In another research to estimate the probabilistic features of different fiber-reinforced polymers used in confined concrete, several distributions were studied [5]. Not only does it make safety estimations of structural systems, but it may also be employed to make likelihood estimations of rare events in other contexts [6, 7]. As a result, the structural reliability theory has a broader range of purposes than its originally planned application range. The determination of the failure likelihood integral is an essential part of reliability analysis. However, this is commonly a challenging task because of many random variables and large nonlinearity in real-life problems. In recent several decades, this challenge has led to the development and improvement of several estimation techniques concerning structural reliability.

Several approaches, e.g., the first-order reliability method (FORM) [8] and second-order reliability method (SORM) [9], apply limit state function expansion at the most probable point (MPP) or the design point through Taylor expansion, ignoring higher-order terms for the purpose of approximating the failure likelihood. Thus, it is required to obtain a derivative-grounded iterative search procedure for locating MPP. Despite the rationally precise solutions of such techniques to a number of real-life problems, the numerical difficulty and inaccuracy of MPP searching may appear for an implicit nonlinear limit state function. Also, moment-grounded approaches can be classified in this category [10, 11].

The crude Monte Carlo simulation (MCS) is the most robust and accurate technique and produces random samples on the grounds of the statistical random variables and estimates the failure probability (P_f) of

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a system using the law of large numbers [12]. However, due to its high computational costs, MCS cannot be claimed to be the most competent method for time-consuming problems (such as finite element ones) and the likelihood estimation of rare events [13].

Despite several improved MCS variants, including the modified importance sampling (MIS) method [14], subset simulation (SS) [15], line sampling method (LS) [16, 17], weighted average simulation method [18, 19], asymptotic sampling [20, 21], and thermodynamic integration and parallel tempering (TIPT)[22], for further enhancing efficiency, such techniques may require significantly high computational costs to yield rationally-accurate results, particularly for the assessment of time-consuming models, e.g., a finite element model. Furthermore, it should be pointed out that some of such approaches have limited applicability, such as small-sized situations. Thus, it was required to develop techniques to make failure probability estimates by a smaller number of calls to costly performance functions. Sampling approaches grounded on variance reduction may yield as accurate estimation as MCS but by a dramatically lower number of limit state functions [23]. However, a large number of limit state function assessments are still required to ensure that the probability estimates of failure are expectedly accurate.

To further reduce the number of costly assessments of limit state functions, metamodel-based approaches are developed for failure probability estimation. In such approaches, the meta-model undergoes training for replacing the real time-consuming limit state function. Next, the metal model can be employed to make a failure probability estimate in place of the real implicit limit state function. The response surface method [24], support vector machines [25], neural networks [26], and Kriging [2729] are extensively employed as metamodel-based techniques for reliability.

Kriging has been of increasing interest in light of advantages over the other meta-model approaches to estimate failure probability. Such advantages stem from the appropriate characteristics of a Gaussian process. [29] suggested that the basic variant of Kriging is built via some training samples. Therefore, accuracy in Kriging is strongly dependent on the training sample-derived information. Kriging would have insufficient prediction accuracy when few samples are employed to diminish the computation cost. On the other hand, using a great number of samples would ensure sufficient accuracy in the prediction; however, this would add to the computation cost, particularly concerning models with heavy computations. As a result, researchers argued that combining Kriging and active learning strategies would bring a trade-off between prediction accuracy and computational efficiency.

An active learning strategy is exploited Kriging–based approaches to sequentially select training samples to refine the Kriging model and obtain the pre-specified accuracy in approximation. The combination of active learning strategies brings almost informative training samples within the Kriging model to substantially enhance the prediction accuracy. Hence, the use of the minimum number of training samples leads to a rapid iteration convergence rate in constructing the Kriging model. In general, learning functions determine the success of such active learning procedures in Kriging-grounded techniques. The expected feasibility function (EFF) [30], U-learning function [31], H-learning function[32], and the least improvement function (LIF) [33] are extensively employed as learning functions in failure probability estimation.

Pioneer studies [3436] demonstrated the U-learning function to require a high cost for easy implementation. Also, they suggested that training sample selection in the U-learning is performed at the maximum misjudge likelihood for updating the Kriging model. The integration of the active learning Kriging and MCS technique, known as AK-MCS [31], can lead to a considerable decline in the computational cost for failure probability cost compared to the MCS method accuracy. For failure probability estimation, AK-MCS builds a Kriging model of the real limit state function in an iterative process within the pool of MCS samples via a learning function to meet the convergent criterion. Then, the well-trained Kriging model is employed to detect failure samples in the MCS-generated sample pool. Eventually, the failure samples are exploited to make a failure probability estimate.

For the efficiency improvement of AK-MCS failure probability estimation, it can be coupled with a sampling technique grounded on variation reduction, which commonly saves more time as compared to AK-MCS due to the substantially smaller sample pool than the one in MCS. The AK-IS [34], AK-SS [37], and AK integration with stochastic sampling and density approximation (AK-SSD) [38] fall into this class of strategies. AK-IS constructs a Kriging model in an iterative process within the IS PDF-generated sample pool, which is of greater efficiency as compared to AK-MCS; however, it solely suits problems with one MPP or a single dominated MPP.

Nevertheless, two brand new methods named ALK-EMO-IS [39] and ALK-MAIS-TCR [40], have significantly dealt with the mentioned drawback, and in another study, to reduce the time of building the Kriging model in the combination of IS, ALK-KDE-IS was proposed [41]. The method of AK-SS iteratively constructs a Kriging model within the MCS sample pool to meet the convergent criterion. Next, SubSim is employed to perform failure probability estimation via the Kriging model rather than the actual limit state function. Also, AK-SSD integrates AK and SubSim in order to make an IS PDF approximation before utilizing IS for failure probability estimation on the ground of the Kriging model.

SASM [42], the Subset Active Subspace Method, is recently proposed. This procedure uses a hybrid of the subset simulation and the kriging method to predict the high dimensional reliability problems with a rare failure event.

MLS [43], a Metamodel Line Sampling approach, and Meta-IS [44], a surrogate model-based importance sampling method, integrate the importance sampling with active learning Kriging methods used in comparison in some of the solved problems in this study. These methods aim to reduce the computational cost of the reliability analysis.

Generally, efficiency in AK-coupled numerical simulations of failure probability estimation could be basically improved by learning function improvement and the construction alternation of the candidate sample pool. In the latter strategy, another SS-AK integration is employed to make failure probability estimates [45]. This approach utilizes SubSim for translating small failure probability estimates into some greater conditional failure probabilities of suitable intermediate failure events. Next, it iteratively builds the Kriging model of the actual limit state function within the SubSim sample pool to obtain the entire conditional failure probabilities via the trained Kriging model. Then, one can estimate the failure probability by multiplying the conditional failure probabilities.

SubSim is a more capable variance reduction–grounded sampling approach than other reliability techniques due to the use of a decent idea for the probability estimation of rare events. Thus, many studies were conducted on SubSim [22]. Through intermediate failure event introduction, SubSim uses small values of P_f as a product of greater conditional probabilities and employs a smaller number of function calls than the basic MSC variant to solve problems [46]. However, SubSim is too dependable on the geometry of the performance function that may mislead the samples towards the wrong failure region.

To address this issue, SESC, a sequential space conversion reliability method, is newly introduced to solve misleading and some other challenging performance functions, which are not easy for famous reliability methods. This method is inspired by SubSim but in a completely different procedure represented the upcoming sections. *Using metamodels increases the computational time of analysis but highly reduces the function evaluation of analyses.* In this paper, integration of SESC and active learning Kriging is proposed to considerably reduce the number of function calls of the SESC approach while maintaining high precision reliability results. The rest sections of this study comprise a background of the used and inspired methods in Section 2, a comprehensive demonstration of the proposed method in Section 3, numerical examples, and the conclusion in Sections 4 and 5, respectively.

2. Background

2.1. Description of SubSim

SubSim [37, 47] refers to an adaptive Markov Chain Monte Carlo (MCMC) [48] method for the efficient calculation of small failure probabilities (i.e., of rare events). Let F be the ultimate failure event and $F_1, F_2 \ldots F_m$ denote a sequence of intermediate failure events, for which $F_1 \supset F_2 \supset \ldots \supset F_m = F$. Based on the failure event definition in the analysis of reliability, $F_i = \{G(\mathbf{x}) \leq G_i\}$, in which G_i represents the corresponding failure threshold $(G_1 > G_2 > \ldots > G_m = 0)$, as illustrated in Fig. 1.

As a result, one can represent the failure probability by multiplying $P(F_1)$ by some conditional probabilities as:

$$P_F = P(F_m) = P(F_m|F_{m-1})P(F_{m-1}) = \ldots = P(F_1)\prod_{i=1}^{m-1} P(F_{i+1}|F_i) \tag{1}$$

Let $\{\mathbf{x}_k^{(1)} : k = 1, \ldots, N_1\}$ denote independent samples of an identical distribution (i.i.d.) that are simulated based on the probability density PDF $q(\mathbf{x})$. Also, $I_{F_1}(\mathbf{x}_k^{(1)})$ stands for the indicator function. Thus, $P(F_1)$ is calculated using direct MCS:

$$P_1 = P(F_1) = \frac{1}{N}\sum_{k=1}^{N_1} I_{F_1}\left(\mathbf{x}_k^{(1)}\right) \tag{2}$$

To calculate conditional probabilities $P(F_{i+1}|F_i)$ $(i = 1, \ldots, m-1)$, the samples of $(i+1)$th subset are produced by samples in i *th* subset, which are in the failure region F_i as:

$$\mathbf{X}^i : \; G(\mathbf{x}) < G_i, \; i = 1, \ldots, m-1 \tag{3}$$

The selection of failure thresholds G_i $(i = 1, 2, \ldots, m)$ is typically performed to allow for equal partial failure probabilities p_0 of each subset [8]. Considering the samples in F_i, the target distribution $\pi(\mathbf{x}|F_i)$ is employed to generate conditional samples. Also, Eq. (2) is utilized to evaluate $P(F_{i+1}|F_i)$:

$$\pi(\mathbf{x}|F_i) = q(\mathbf{x})I_{F_i}(\mathbf{x})/P(F_i) \tag{4}$$

$$P_{i+1} = P(F_{i+1}|F_i) = \frac{1}{N_{i+1}}\sum_{k=1}^{N_{i+1}} I_{F_{i+1}}\left(\mathbf{x}_k^{(i+1)}\right) \tag{5}$$

The present study employed a modified Metropolis algorithm variant [15] in MCMC simulation to produce conditional samples. SubSim continues as:

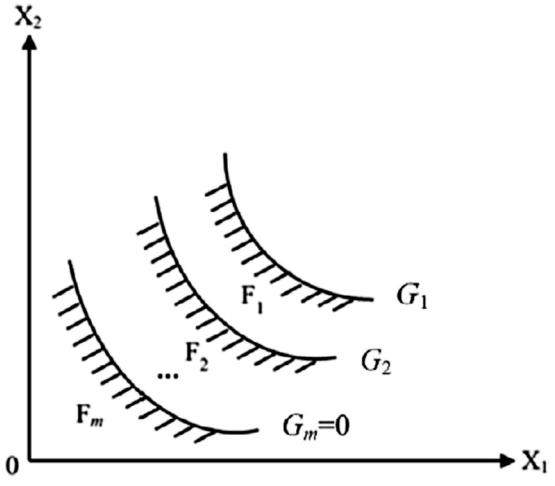

Fig. 1. Subset simulation [45].

(1) Produce N_1 i.i.d samples $(\mathbf{x}_k^{(1)} : k = 1, 2, \ldots, N_1)$ based on the PDF $q(\mathbf{x})$; $\mathbf{x}_k^{(1)} : k = 1, 2, \ldots, N_1$.

(2) Calculate N_1 responses: $\{G(\mathbf{x}_k^{(1)}) : k = 1, 2, \ldots, N_1\}$. The failure threshold G_1 is chosen as quantity $N_1 p_0$ of $G(\mathbf{x})$ in ascending order, in which $p_0 = P_1 = P(F_1)$ is an already-assigned sufficient value;

(3) Begin with $p_0 N_i$ samples in the failure region F_i $(i = 1, \ldots, m-1)$ and produce other conditional samples $(N_{i+1} - p_0 N_i)$ based on $q(\mathbf{x}|F_i)$;

(4) Likewise, *calculate* $G(\mathbf{x}) : \{G(\mathbf{x}_k^{(i+1)}) : k = 1, 2, \ldots, N_{i+1}\}$ to obtain responses N_{i+1} . Determine the failure threshold G_{i+1} as quantity $N_{i+1}p_0$ of $G(\mathbf{x})$ in ascending order. Then, $P(F_{i+1}|F_i) = p_0$ and $P(F_{i+1}) = p_0^{i+1}$.

(5) Iterate the above steps to meet the criterion $G(\mathbf{x}_{p_0 N_{i+1}}^{(m)}) < 0$. Then, G_m is set to zero, and $P(F_m|F_{m-1})$ and P_F are computed. As suggested in [15], the coefficient of variation (CV) of $P_1 = P(F_1)$ is $\delta_1 = \sqrt{1 - P_1/P_1 N_1}$, whereas the CV of P_i for subset i $(i = 2, \ldots, m)$ is calculated as:

$$\delta_i = \sqrt{\frac{1 - P_i}{P_i N_i}(1 + \gamma_i)}, \quad i = 2, \ldots, m \tag{6}$$

$$\gamma_i = 2\sum_{k=1}^{N_i/N_{ci}-1}\left(1 - \frac{kN_{ci}}{N_i}\right)\rho_i(k) \tag{7}$$

where $\rho_i(k)$ represents the correlation coefficient. Also, one can estimate the CV of P_F as:

$$\delta_{P_{F(SS)}} = \sqrt{\sum_{i=1}^{m}\delta_i^2} \tag{8}$$

Remember that the MCS-obtained CV of P_F is $\delta_{P_{F(\mathrm{MCS})}} = \sqrt{(1 - P_F)/(P_F N_{MCS})}$. At a very small $P_F$value, $\delta_{P_{F(\mathrm{sS})}}$ is considerably lower than $\delta_{P_{F(\mathrm{MCS})}}$ for an equivalent size of samples.

2.2. Basic theory of kriging

Kriging has been of great interest in the construction of meta-model models in recent years. For k sample points, $\boldsymbol{x}_i \in R^n$, and k responses $\mathbf{Y} = [G(\mathbf{x}_1), \ldots, G(\mathbf{x}_k)]^{\mathrm{T}}$, in which n denotes the dimension of $\mathbf{x}_i$. As a strong meta-model of interpolation, Kriging considers the deterministic response $G(\mathbf{x})$ to be a random function realization, which involves a centered stochastic process and a regression procedure [49, 50].

$$\widehat{G}(\mathrm{x}) = F(\mathrm{x}, \beta) + z(\mathrm{x}) = \mathrm{f}^T(\mathrm{x})\beta + z(\mathrm{x}) \tag{9}$$

in which $F(x, \beta)$ denotes the deterministic term of a regression model representing the Kriging trend and yielding a response estimate in the form of the mean. $f^T(x) = \{f_1(x), \ldots, f_k(x)\}$stands for the primary polynomial function vector, whereas $\beta^T = \{\beta_1, \ldots, \beta_k\}$ denote the regression coefficient vector. Also, $z(x)$ is a zero-mean stationary Gaussian process and covariance between two points that are defined as:

$$COV\left(Z(\mathrm{x}_i), Z(\mathrm{x}_j)\right) = \sigma_z^2 R_\theta(\mathrm{x}_i, \mathrm{x}_j) \tag{10}$$

in which σ_z^2 represents the process variance of $Z(x)$, while R_θ denotes the Gaussian correlation function with its set of parameters corresponding to the number of random variables $\theta^T = \{\theta_1, \theta_2 \ldots, \theta_n\}$. Several models exist for defining the correlation function in Kriging, which performs the smoothness management of the model. Then, the anisotropic Gaussian function is employed as:

$$R_\theta(x_i, x_j) = \prod_{k=1}^{n} \exp\left[-\theta_k\left(x_i^{(k)} - x_j^{(k)}\right)^2\right] \tag{11}$$

One can represent the design of experiments (DoE) points as $x = [x^{(1)}, x^{(2)}, \ldots, x^{(p)}]$, in which vector $x^{(i)} \in \mathbb{R}^n$ stands for test i. Then, one can express the responses as $Y^{(i)} = G(x^{(i)}) \in \mathbb{R}$. Also, one may calculate scalar β and process variance σ_z^2 as:

$$\widehat{\beta} = \left(1^T \mathbb{R}_\theta{}^{-1} 1\right)^{-1} 1^T \mathbb{R}_\theta{}^{-1} Y \tag{12}$$

$$\widehat{\sigma}_z{}^2 = \frac{1}{p}\left(Y - 1\widehat{\beta}\right)^T \mathbb{R}_\theta{}^{-1}\left(Y - 1\widehat{\beta}\right) \tag{13}$$

in which the matrix of correlation the DoE point pairs is obtained by $R_{\theta i,j} = R_\theta(x^{(i)}, x^{(j)})$, and 1 is a unit vector of length p. Eqs. (12 13) are based on the correlation parameter θ_i, which could be found using maximum likelihood estimation (MLE):

$$\theta = \arg\min_\theta (\det \mathbb{R}_\theta)^{\frac{1}{p}} \widehat{\sigma}^2{}_z \tag{14}$$

Best linear unbiased prediction (BLUP) $\widehat{G}(x)$ may be calculated at an unknown point x as:

$$\widehat{G}(X) = \beta + r(X)\mathbb{R}_\theta{}^{-1}\left(Y - 1\widehat{\beta}\right) \tag{15}$$

in which $r(X) = \{R_\theta(X, X^{(i)})\}_{i=1,\ldots,p}$correlates X and p observed points. The Kriging variance $\widehat{\sigma}_{\widehat{G}}{}^2$ is obtained as the minimum mean squared error (MSE) between $G(x)$ and predicted response $\widehat{G}(x)$ as:

$$\sigma^2_{\widehat{G}}(x) = \sigma_z^2\left[1 + u(x)^T\left(1^T \mathbb{R}_\theta{}^{-1} 1\right)^{-1} u(x) - r(x)^T \mathbb{R}_\theta{}^{-1} r(x)\right] \tag{16}$$

in which $u(x)$ is;

$$u(x) = 1^T \mathbb{R}_\theta{}^{-1} r(x) - 1 \tag{17}$$

The predicted Kriging response $\widehat{G}(x)$ at point x has a normal distribution:

$$\widehat{G}(x) \sim N\left(\widehat{G}(x), \sigma^2{}_{\widehat{G}(X)}\right) \tag{18}$$

2.3. AK-based on U-learning function

Given that several studies have been conducted on Kriging and the U-learning function, the present study avoids replicating pioneer studies. [31, 32, 34, 51] provided detailed descriptions of Kriging and the U-learning function. The present work solely introduces their kernels.

The posterior distribution of the Kriging model $g_K(x)$ (where the subscript "K"stands for "Kriging") at point x is $g_K(x) \sim N(\mu_{\widehat{g}}(x),\ \sigma^2_{\widehat{g}}(x))$, in which $\mu_{\widehat{g}}(x)$ denotes the mean of the Kriging prediction, while $\sigma^2_{\widehat{g}}(x)$ indicates the variance of the Kriging prediction. In practice, the mean prediction $\mu_{\widehat{g}}(x)$ is considered to be the value of the Kriging prediction at x, whereas the variance of the prediction $\sigma^2_{\widehat{g}}(x)$ indicates the prediction uncertainty.

To build a Kriging model of adequate accuracy for the strategy of the intermediate limit state $g(x) = b_k (k = 1, 2, \cdots, m)$ via the minimum number of training samples, it is required to adopt an active learning strategy to select new training samples with the maximum impact on the quality of prediction for the step-by-step refining of the Kriging model. The U-learning function is employed to select informative training samples:

$$U(x) = \frac{\left|\mu_{\widehat{g}}(x) - b_k\right|}{\sigma_{\widehat{g}}(x)} \tag{19}$$

The basis of the U-learning function is the fact that failure probability estimation only requires the sign of the limit state function. It suggests an index of misjudging risk for the sign of $g(x)$. A lower $U(x)$ represents significantly larger uncertainty of the sign of $g(x)$. Hence, for refining the Kriging model, the best new training point x_{new} is selected as:

$$x_{new} = \underset{x \in S_k}{argmin}[U(x)] \tag{20}$$

in which S_k is the candidate sample pool of intermediate failure event k, and a $U(x)$ equal to 2 suggests $1 - \Phi(-2) = 97.7\%$ confidence in judging the sign of $g(x)$. Therefore, a discontinuation Kriging updating criterion is defined as:

$$\min_{x \in S_k}[U(x)] \geq 2 \tag{21}$$

2.4. SESC

SubSim is the primary estimation approach of small failure probabilities (i.e., rare events) [46]. It conveys the samples from the origin toward the failure domain of the maximum probability, estimating small probabilities through several failure domains on the ground of the performance function geometry and MCMC. Nevertheless, given that it assumes performance function geometry tracking to move the MCMC samples toward the failure region of the maximum probability, SubSim cannot be generally employed for solving reliability problems. It may be accurate enough solely concerning particular problems to which this assumption applies. Therefore, to tackle this drawback, many studies [47] recently conducted detailed examinations.

As a result, sequential space conversion (SESC) was recently introduced for solving complicated and high-dimensional problems concerning rare events[52]. Although the formulation of SubSim is grounded on Bayes' theorem, the SESC formulation is obtained by the control variate method. In this approach, an estimate of rapid inaccurate failure probability is made. Then, the estimation is improved by the refining of the terms. A set of scaled limit state functions analogous to the original variant but with larger failure probabilities is built to be exploited as control variables. Finally, the MCMC samples are directed toward the essential region of failure. Thus, in contrast to SubSim, SESC does not operate based on the geometry of a performance function positioned distant from the limit state surface.

Reducing the function calls in SESC, the present work recommends its integration with active learning Kriging. The suggested procedure is described in the next section.

3. Proposed method

In the analysis of reliability, one can derive failure probability by computing the integral:

$$P_f = P_{g(x)\leq 0} = \int_\Omega q(x)dx \tag{22}$$

in which Ω is the failure domain ($g(x) \leq 0$), $x = [x_1, \ldots, x_n]$ is the vector of the system input variables under uncertainty, and $q(x)$ represents the joint probability density function. The performance function $g(x)$, which is also referred to as the limit state function, implies a failure for $g(x) \leq 0$ and safety for $g(x) > 0$.

The non-use of crude MCS in the analysis of reliability due to the assumptions of other reliability techniques leads to inaccurate and incorrect probability estimates. Thus, one can rewrite P_f as:

$$P_f = P_{f,in} + \varepsilon \tag{23}$$

in which $P_{f,in}$ denotes the imprecise probability result of a reliability method while ε stands for the estimation error. Eq. (23) can be solved using CV for probability error estimation via a small sample size

simulation [23, 53].

A reliability problem with a complicated LSF can be solved using an instrumental PDF function $h(x)$ representing the CV of the original PDF function (q) [54]:

$$P_f = \int_{\Omega} h(x)dx + \int_{\Omega} (q(x) - h(x))dx \tag{24}$$

A similar $h(x)$ to $q(x)$ with an m-fold higher standard deviation produces a more significant number of samples within the failure region as compared to $q(x)$. Therefore, the probability of a rare event rises by m times.

As a result, it is possible to easily make an inaccurate P_f estimate as:

$$P_{f,in} = \int_{\Omega} h(x)dx \tag{25}$$

Also, the estimation error ε is computed by integrating the difference between the two PDFs within the failure domain:

$$\varepsilon = \int_{\Omega} (q(x) - h(x))dx \tag{26}$$

Fig. 2 illustrates a schematic of the proposed procedure in a one-dimensional setting. This procedure can be claimed to be a high-versatility variant of the well-known IS approach. It allows for utilizing advanced variance reduction methods for the solving of problems. Furthermore, one may estimate the first probability term through an efficient technique (such as inaccurate analytical ones), and the error term could be approximated by performing simulations within the importance failure domain [23]. For analysis, the present study adopted the linear CV and the regression parameter of α as [13]:

$$\begin{aligned} P_f &= \alpha \int_{\Omega} h(x)dx + \int_{\Omega} (q(x) - \alpha h(x))dx \\ &= \alpha P_{f,in} + \int_{\Omega} (q(x) - \alpha h(x))dx \end{aligned} \tag{27}$$

After approximating the inaccurate probability $P_{f,in}$, the importance failure region is found, and one can accordingly estimate the second term in the equation by using sampling.

For high-dimensional problems, space conversion is recommended as it offers a general method for solving these problems. This approach decomposes the integral problem in Eq. (27) into two different problems within separate spaces (i.e., original and augmented). Then, these two particular problems are mapped into a new equivalent standard normal space $\mathbb{Z}$, the process of which resembles FORM but yields an entirely different result; two distinct LSFs will exist in the $\mathbb{Z}$-space. The mapping of the LSFs leads to the mathematical reformulation of Eq. (27) by space conversion as:

$$P_f = \alpha P_{f,in} + \int_{\Omega} \left(I_g(\mathrm{z}) - \alpha \pi_{\widehat{g}}(\mathrm{z})\right)\varphi(\mathrm{z})dz \tag{28}$$

in which $I_{\widehat{g}}(\mathrm{z})$ and $\pi_{\widehat{g}}(\mathrm{z})$ represent the indicator functions of the primary and scaled failure domains in the $\mathbb{Z}$-space, respectively. Also, $\varphi(\mathrm{z})$ represents the standard normal PDF.So, an integral problem with a failure domain and two PDFs is transformed into a problem with two LSFs and a PDF within the standard normal space by space conversion.

To simplify Eq. (28), one can calibrate α so that the second term becomes zero. Thus, it is possible to rewrite the failure probability as:

$$P_f = \alpha . P_{f,in} \tag{29}$$

in which $\alpha = \frac{\int I_g(\mathrm{z})\ \varphi(\mathrm{z})\ d\mathrm{z}}{\int \pi_{\widehat{g}}(\mathrm{z})\ \varphi(\mathrm{z})\ d\mathrm{z}}$. Then, one can employ an instrumental quasi-optimal function $\varphi^*(\mathrm{z}) = \varphi(\mathrm{z})\ / P_{f,in}$ for the estimation of α as:

$$\alpha = \mathbb{E}_{\varphi^*}\left(I_g(\mathrm{z})\right) \Big/ \mathbb{E}_{\varphi^*}\left(\pi_{\widehat{g}}(\mathrm{z})\right) \tag{30}$$

The use of a conventional component-wise MCMC may yield $\mathbb{E}_{\varphi^*}(\pi_{\widehat{g}}(\mathrm{z})) = \int \pi_{\widehat{g}}(\mathrm{z})\ \varphi(\mathrm{z})\ d\mathrm{z} = 1$ when producing the entire samples within the failure domain $\widehat{g}$. Therefore, based on Eq. (29), one can represent P_f as:

$$P_f = \mathbb{E}_{\varphi^*}\left(I_{g\leq 0}(\mathrm{z})\right).P_{f,in} \tag{31}$$

Concerning the use of the presented space conversion approach, CV properly estimates P_f when $g = 0$ and $\widehat{g} = 0$ are close; however, for a large m, $g = 0$ and $\widehat{g} = 0$ are not close, the estimation of $\alpha = \mathbb{E}_{\varphi^*}(I_g(\mathrm{z}))$ requires a large sample size.

Therefore, it is suggested that one should reduce the estimate variance by using a reducing vector $\boldsymbol{m} = \{m_1, m_2, m_3, \ldots, m_n\}$, where $m_1 > m_2 > m_3 > \ldots > m_n$ for the iterative design of scaled LSFs between $\widehat{g}_1 = 0$ and $g = 0$. Next, for $\sigma_i = m_i\sigma_f$ within LSF conversion, a sequential CV approach is adopted to solve the problem. Hence, the basic LSF lies next to some scaled LSFs close to the basic LSF but have failure probabilities. Also, the failure probability is estimated as:

$$\begin{aligned} P_f &= P_f^{\widehat{g}_1} \cdot P_f^{\widehat{g}_2} \cdot \ldots \cdot P_f^{\widehat{g}_{n-1}} \cdot P_f^{\widehat{g}_n} \\ &= \prod_{i=1}^{n} \cdot P_f^{\widehat{g}_i} \end{aligned} \tag{32}$$

Assuming $\widehat{g}_1 = 0$ to be the initial LSF with the largest failure probability, MCS is employed to approximate the inaccurate probability $P_{f,in} = P_f^{\widehat{g}_1}$ as:

$$P_{f,in} = P_f^{\widehat{g}_1} = \int \pi_{\widehat{g}_1 \leq 0}(\mathrm{z})\ \varphi(\mathrm{z})\ dz = \mathbb{E}_{\varphi}\left(\pi_{\widehat{g}_1}(\mathbf{z})\right) \tag{33}$$

To avoid a Monte Carlo simulation and the assessment of the entire population on performance function, the present work recommends using AK-MCS [31] as it separates positive performance function predictions from negative ones for the Monte Carlo population. The sign at each point is derived in light of the Kriging predictions on the ground of some assessed points. Hence, the first stage is the production of a Monte

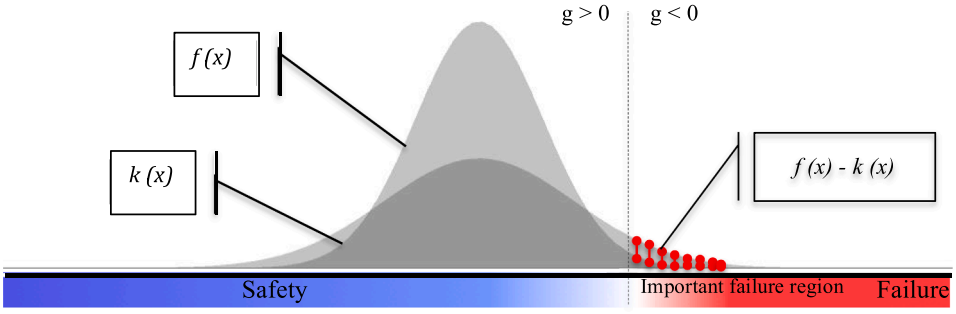

Fig. 2. $h(x)$ as the CV of $q(x)$ in a one-dimensional setting [53].

Carlo population S. It involves N_{MCS} sample points of input variables within the design space via the Monte Carlo technique as $\mathbf{x}_{MCS} = \{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_{n_{MCS}}\}$. The sampling procedure is performed based on the PDF of the problem random variables. Also, it is not necessary to compute the performance function at this stage.

Then, there should be an initial DoE to carry out Kriging, which involves randomly selecting N_0 points from the population to be assessed on the performance function and employed as the initial DoE for the Kriging model. It is preferable to define a small initial DoE and add the point solely with the most considerable improvement contribution to the meta-model in a stepwise manner. Then, the DACE toolbox is utilized to construct the Kriging model [55]. A Gaussian correlation model is selected for computing Kriging predictions of the entire population. Also, the regression model is selected to be fixed; this represents ordinary Kriging.

To identify the best next training point $\boldsymbol{x}^*$and assess its performance function, it must implement a learning function. Thus, the U-learning function is chosen. For each of the points within the population, the Kriging variance $\widehat{\sigma}_{\widehat{G}}^2$is to be obtained by Eq. (16). Also, Eq. (19) is applied to estimate the U-learning function for $b_k = 0$. Since solely the sign of the limit state function is of importance within a Monte Carlo simulation, the learning function is aimed at the identification of highly uncertain points on their signs. These hazardous points may contain a large Kriging variance, suggesting significant uncertainty or closeness to the limit state or such attributes. U represents a reliability index on the misjudging risk of the sign of response predictions. Then, Eq. (20) is applied to identify the best training sample with the smallest value of U.

As mentioned, $\underset{\boldsymbol{x}\in S_k}{min}[U(\boldsymbol{x})] \geq 2$ is the first discontinuation criterion to find the best next point $\boldsymbol{x}^*$. The learning process continues to perform, and the true performance function of the best point $\boldsymbol{x}^*$that was identified in the previous step is assessment until the discontinuation criterion is met. Next, the best point is added to the DoE to be exploited to estimate a new Kriging model. Once the discontinuation criterion is met, the learning process stops, and the Kriging model is sufficiently accurate. Then, to ensure the sufficient largeness of the Monte Carlo population in the first step for obtaining a small CoV in the Kriging failure probability prediction, the following criterion is to be met:

$$C.O.V_{\widehat{P}_f} = \sqrt{\frac{1-\widehat{P}_f}{\widehat{P}_f n_{MCS}}} < 0.05 \tag{34}$$

In which $\widehat{P}_f = \frac{\widehat{N}_{G\leq 0}}{N_{MCS}}$. When this criterion is not met, the sample count of the population needs to be raised using the newly generated samples of another MCS population, likewise the first step. The active learning Kriging model is constructed once the entire criteria have been met. Next, for performance function evaluation, the updated Kriging-derived training points are applied to $h(x)$ as the initial set of training samples rather than producing MCS sample points on the ground of a new PDF $h(x)$. Despite the substantial decline in the sample size due to the replacement of $q(x)$ with $h(x)$, the use of AK-MCS training samples induces even a sharper reduction in the number of performance function assessments.

4. Numerical examples

To assess the robustness and efficiency of the proposed method and to verify the results, four benchmark problems with implicit performance functions, which are widely used in the literature, are solved. The obtained results are compared to those of some other common reliability procedures. The efficiency of different reliability methods is compared in terms of N_{call} which is the number of function evaluation, and e_f. e_f is considered the percentage error of the problem failure probability in comparison with those of MCS as the reference value.

4.1. Example I

The first example is a high dimension paraboloid problem with standard normal variables:

$$g(\boldsymbol{X}) = \alpha + \gamma \sum_{i=2}^{D} X_i^2 - X_1 \tag{35}$$

the geometrical features of the limit state function are defined by the constant parameters of α, γ and D. This example was solved with the setting of the N_f to 50 and the dimension of the problem (D) assumed to be 10. The reliability analysis results are shown in Table 1. The reference value of MCS is 7.74 $\times 10^{-4}$ with a number of 518,360 function calls.

The reliability results demonstrated in Table 1 show that the most accurate method is AK-SESC, and the least one is MLS with the relative error of 0.65% and 2.58%, respectively. Even though the function calls of the MLS are 44.4% of the ones in AK-SESC, the precision of the proposed method is 25% more than the MLS approach. SubSim method with the huge number of function calls still has a 2.06% relative failure probability error compared with the reference value of MCS. LS indicates a small error but its requirement to 41,893 N_{call} cannot be negligible. In the case of reducing the SESC function calls, which is the aim of this paper, the results show that the AK-SESC method reaches the 64.6% of N_{call} required by SESC with a reduction of 63.1% in relative error. Fig. 3

4.2. Example II

In this section, a performance function with a very small failure probability is studied. The PDF of two random variables X_1 and X_2 are normally distributed, and the performance function is defined as:

$$g(\boldsymbol{X}) = X_1 X_2 - 146.14 \tag{36}$$

The corresponding performance function is illustrated in Fig. 4, and the Statistical Properties of the random input variables are shown in Table 2.

400 samples are considered as the initial ones to build the Kriging model and the N_f is set to 100. The result of IS is extracted from ref. [23]. As shown in Table 3, FORM is not accurate enough, and SORM fails in finding the solution. The original SESC has almost the same number of function calls comparing to SubSim, but the estimation of the P_f is much more accurate. The AK–SESC proposed method outperforms the original SESC method in terms of N_{call} by reduction of around 72% in function calls and also with a better accuracy. Fig. 5 illustrated the effect of the different numbers of considered initial samples on the reliability index.

4.3. Example III

The next example is a Metaball function:

$$g(X) = \frac{30}{\left(\frac{4(x_1+2)^2}{9} + \frac{x_2^2}{25}\right)^2 + 1} + \frac{20}{\left(\frac{(x_1-2.5)^2}{4} + \frac{(x_2-0.5)^2}{25}\right)^2 + 1} - 5 \tag{37}$$

The two random variables are considered standard normal ones. The

Table 1
The reliability results of example I (considering $\alpha = 0$, $\gamma = 1$) [43].

Method	P_f	N_{call}	e_f(%)
MCS	7.74 $\times 10^{-4}$	518,360	–
SubSim	7.58 $\times 10^{-4}$	10^5	2.06
LS	7.67 $\times 10^{-4}$	41,893	0.90
MLS	7.94 $\times 10^{-4}$	1201	2.58
SESC	7.82 $\times 10^{-4}$	4186	1.03
AK-SESC	7.79 $\times 10^{-4}$	2705	0.65

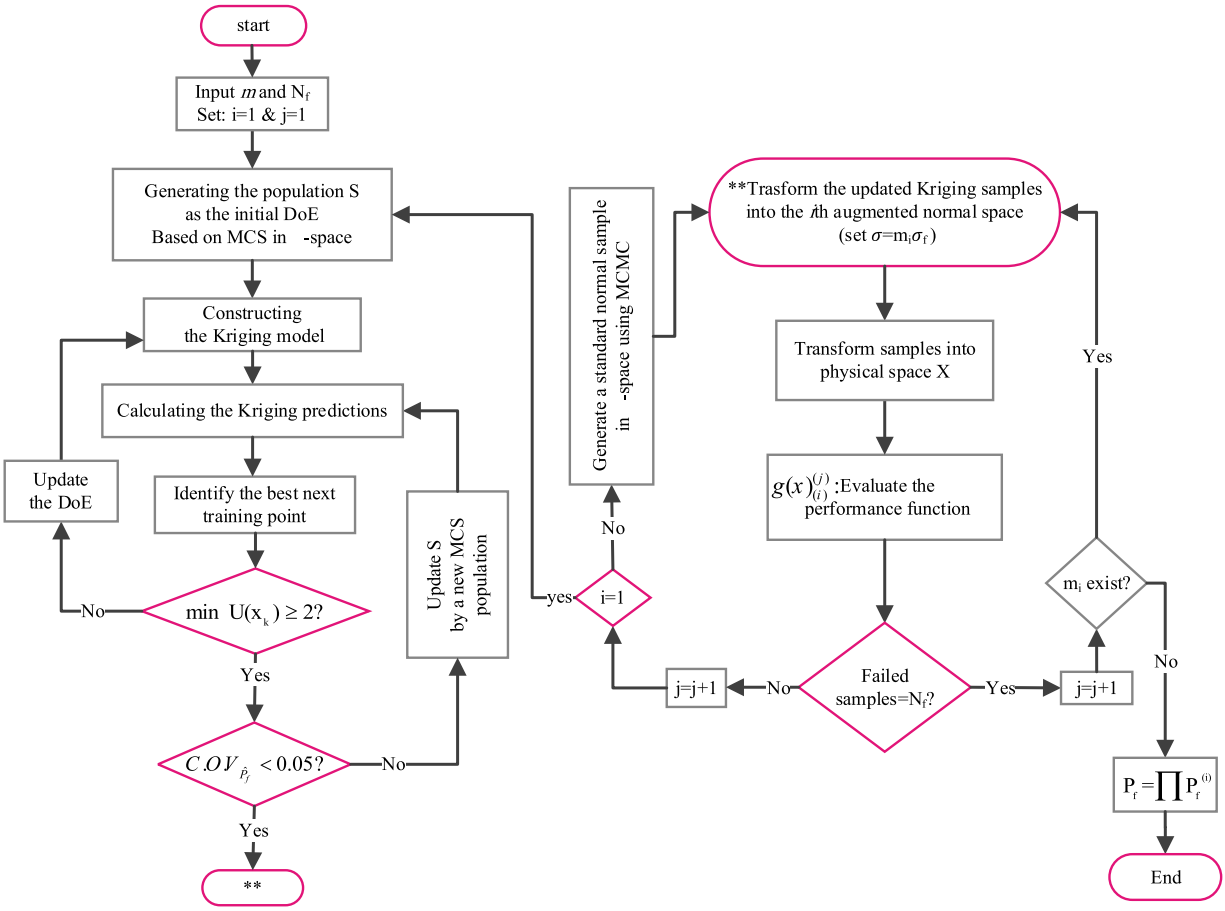

Fig. 3. Flowchart of the proposed AK-SESC method.

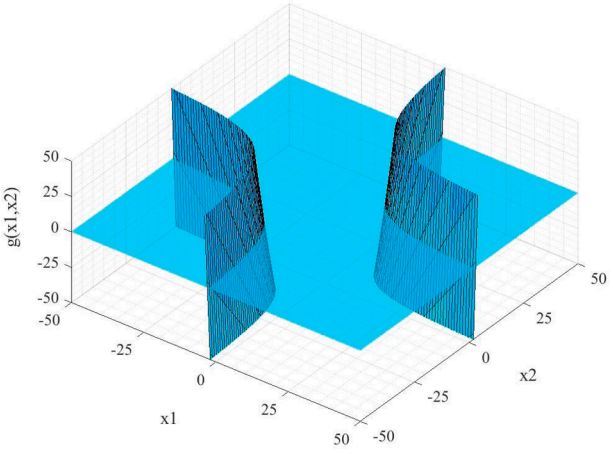

Fig. 4. Performance function of example II.

Table 2
Statistical properties of example II.

Variable	Distribution	Mean	Standard Deviation
X_1	Normal	7.80644×10^{4}	1.17097×10^{4}
X_2	Normal	1.04×10^{-2}	1.56×10^{-3}

Table 3
The reliability results of example II.

Method	P_f	B	N_{call}
MCS	1.34×10^{-7}	5.14	3×10^{8}
IS	1.41×10^{-7}	5.14	7.0×10^{3}
FORM	2.86×10^{-8}	5.43	200
SORM	Failed	Failed	–
SubSim	5.55×10^{-8}	5.31	13,841
LS	8.21×10^{-8}	5.24	100
SESC	1.29×10^{-7}	5.15	13,717
AK-SESC	1.32×10^{-7}	5.15	3869

Metaball functions can be used to create problems where the topological structure of the domain of failure varies. The performance function of this example is depicted in Fig. 6.

As mentioned in ref. [52], the geometry of this performance function misleads MCMC samples to an incorrect failure region of the metaball, so SubSim fails to reach the result. The results demonstrated in Table 4 indicate that the proposed method solved the problem with fewer function calls but still with high accuracy. Comparing to SESC, the AK-SESC approach has 18.7% fewer function calls. It is noteworthy that the suggested method just needs 100 initial samples. The results of example III with a different number of initial samples are shown in Fig. 7.

4.4. Example IV

A parallel system, including a high nonlinear performance function,

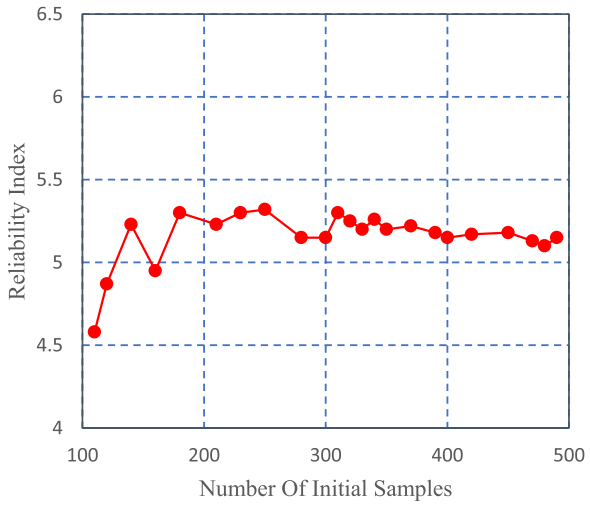

Fig. 5. The results of Example II with a different number of initial samples.

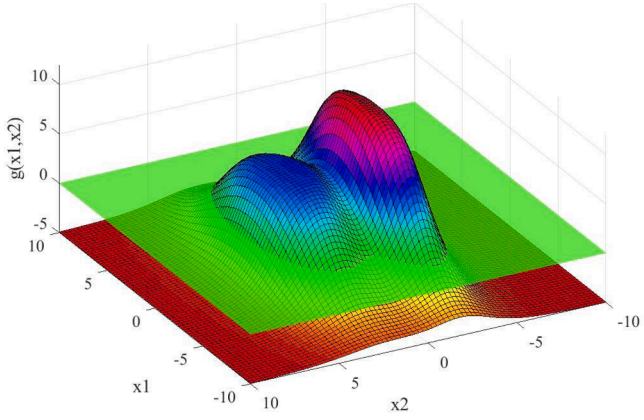

Fig. 6. Performance function of example III.

Table 4
The reliability results of example III.

Method	P_f	N_{call}	e_f(%)
MCS	1.55×10^{-5}	10^7	–
SubSim	Failed	–	–
SESC	1.50×10^{-5}	7321	3.23
AK-SESC	1.59×10^{-5}	5954	2.58

is investigated in this problem [44]:

$$g(X) = min\{ exp\left(\frac{-x_1^2}{10}\right) + \left(\frac{x_1}{5}\right)^4 - x_2 + 4 \\ -x_1 x_2 - 12.5 \quad (38)$$

The two independent random variables have standard normal distributions.

Same as the other examples, the reference value to compare the reliability analysis answers is the failure probability of the problem, evaluated by MCS. According to the results shown in Table. 5, the failure probability accuracy obtained by SubSim, is far from the reference value.

Although the precision and function calls of the Meta-IS and SESC methods do not have significant differences, the Meta-IS reveals better reliability answers which are comprising 17% less N_{call} ,and 1.37 better accuracy.

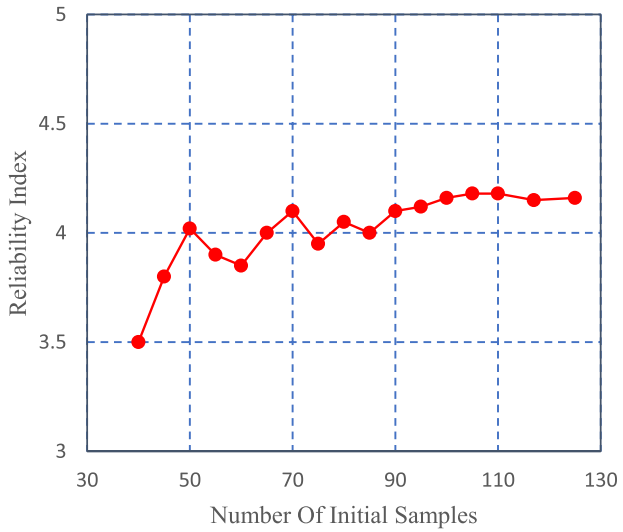

Fig. 7. The results of example III with a different number of initial samples.

Table 5
The reliability results of example IV.

Method	P_f	N_{call}	e_f(%)
MCS	9.48×10^{-7}	4.2×10^8	–
SubSim	6.55×10^{-7}	7×10^5	30.91
Meta-IS	9.17×10^{-7}	2940	3.27
SESC	9.04×10^{-7}	3550	4.64
AK-SESC	9.67×10^{-7}	2408	2.01

By investigating the values depicted in Table. 5 compared with the MCS results, the AK-SESC indicates a more accurate failure of probability with less number of function calls among the other approaches. The proposed method with the 2408 number of function calls is only 5.7 $\times 10^{-4}$ % of the required ones by MCS with just 2.01% accuracy relative error which is a considerable result. The AK-SESC method with N_f set to 180 and 250 initial samples need 32.17% and 18.10% fewer function calls comparing to SESC and Meta-IS, respectively.

4.5. Example V

This example is a high-dimensional reliability problem with D independent log-normal random variables to evaluate the efficiency of the proposed method to solve a reliability problem with 50 dimensions. The considered performance function is as follows [44]:

$$g(X) = \left(D + \alpha\sigma\sqrt{D}\right) - \sum_{i=1}^{D} x_i \quad (39)$$

where the α parameter is set to 3. The mean and standard deviation of the variables are units mean values and σ= 0.2 respectively. The ob-

Table 6
The reliability results of example V.

Method	P_f	N_{call}	e_f(%)
MCS	1.88×10^{-3}	1.3×10^6	–
FORM	1.54×10^{-4}	166	91.81
IS	1.97×10^{-3}	14,066	4.79
Meta-IS	1.95×10^{-3}	1800	3.72
SESC	1.82×10^{-3}	2145	3.19
AK-SESC	1.93×10^{-3}	1408	2.66

tained reliability results are shown in Table 6.

Despite the fact that it is a single design point reliability problem, the FORM estimation is not accurate. Even with the N_{call} equal to 14,066, the failure probability obtained by IS method holds an almost reasonable accuracy with a 4.79% relative error. The SESC method with 2145 function calls, around 20% more than the ones in Meta-IS, has a slightly better estimation of P_f. The AK-SESC proposed method has the least relative error of 2.66% in Table 6, comparing with the reference value of failure probability acquired by MCS. In the case of N_{call} , using 200 initial samples and N_f equal to 150, the suggested approach caused a 34.4% reduction in SESC function calls.

4.6. Example VI

The real engineering problem to study structural reliability analysis, illustrated in Fig. 8, is a passive vehicle suspension. The considered performance function is the road-holding ability of the vehicle as follows:

$$\left(\frac{\pi mVA}{b_0 kg^2}\right)\cdot\left(\left(\frac{c_k}{m+M}-\frac{c}{M}\right)^2+\frac{c^2}{mM}+\frac{c_k k^2}{mM^2}\right)-1 \tag{40}$$

where the random variables are the spring stiffness c (kg/cm), shock absorber damping coefficient k (kg/cm s) and tire stiffness c_k (kg/cm). The considered constant values and the defined probabilistic features of the model are represented in Tables 7 and 8, respectively.

This applied engineering problem is solved with six reliability methods to compare the performance of the proposed AK-SESC approach. In the case of AK-SESC, N_f is set to 150, and the initial number of samples is 610. The results of failure probability obtained from reliability analysis are presented in Table 9. As the results show, FORM, LS and SubSim are not able to detect the most important failure domain due to the misleading geometry of the performance function. However, the SESC reliability method presents suitable accuracy with 3% function calls compared to the MCS results. Nevertheless, the proposed AK-SESC achieved a better precision with about half of the SESC function calls, making it a promising approach to overcome the reliability analysis challenge of real-world engineering problems.

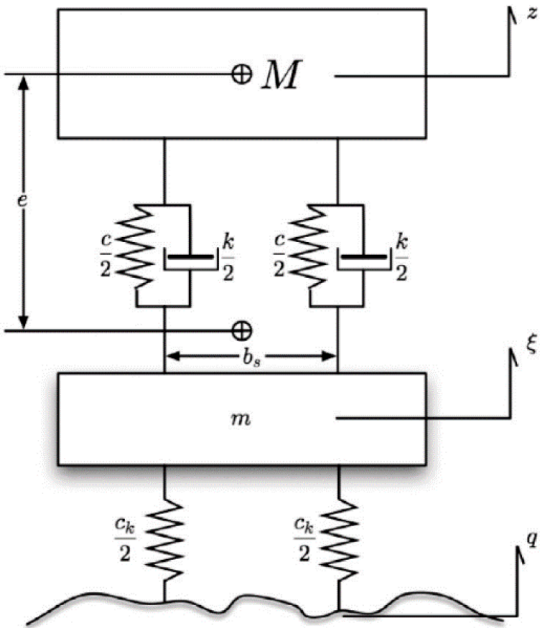

Fig. 8. The model of passive vehicle suspension [52].

Table 7
The performance function constant values of example VI.

parameters	unit	value
V	m/s	10.0
A	$cm^2/cycle.m$	1.0
b_0	–	0.27
m	$kg.s^2/cm$	0.8158
M	$kg.s^2/cm$	3.2633
g	cm/s^2	981

Table 8
The probabilistic features of example VI.

Variable	Description	Unit	Distribution	Mean	Standard deviation
c	spring stiffness	kg/cm	Normal	424.0	10.0
c_k	tire stiffness	kg/cm	Normal	1480.0	10.0
k	damping coefficient	$kg/cm.s$	Normal	47.0	10.0

Table 9
The reliability results of example VI.

Method	MCS	FORM	LS	SubSim	SESC	AK-SESC
P_f	1.47×10^{-6}	4.38×10^{-11}	Failed	4.91×10^{-10}	3.13×10^{-6}	2.05×10^{-6}
N_{call}	10^7	4994	–	7.5×10^4	28,617	15,048

5. Conclusion

This paper introduces the AK-SESC as a novel reliability method to solve the problem with a small failure probability. This method is inspired by SubSim with the difference that SubSim is based on the Bayes theorem, and the area of failure in each subset depends on the previous one. On the contrary, the proposed method inherits the specifications of the newly introduced SESC reliability method, including using a series of scaled performance functions with independent failure domains as the control variate of the original performance function. The proposed method aims to reduce function calls of the SESC approach by integrating with active learning Kriging.

The reliability analysis of the examples indicates that by exploiting the helpful features of the U-learning function, the function calls of the proposed method have much decrease even to 70% comparing to SESC while maintaining high accuracy. However, some other famous reliability methods fail to obtain the proper answer or demand numerous function evaluations to reach the same precision. The results also indicate that when utilizing a large group of samples is not possible during a simulation process, which has always been a matter of concern in practical problems or the ones involving Finite Element Analysis, the AK-SESC proposed method still is able to achieve a reasonable prediction of the failure probability by applying a small number of initial samples.

Although the active learning process is a time-consuming procedure in some of the problems, this significant advantage of the proposed AK-SESC method makes it an advantageous reliability approach in especially practical problems with the need of constructing actual samples. The aim of this study was not to reduce time computational cost but the reduction of the problem function calls. So, then it could be helpful in the industries that need to build initial samples to predict the probability of their own product or project failure or in the problems which required FEM analysis which is too time-consuming. It should also be noted that because of the inherent limitations of Kriging for the prediction of complex problems in high dimensional space, application of this method

for solving problems with many random variables (i.e., problems upper than 20 dimensions) should be considered with reservation.

For further studies and expanding this method, it is suggested to use other active learning approaches and compare the reliability analysis results to find the best integration.

CRediT authorship contribution statement

Ala Ameryan: Conceptualization, Validation, Methodology, Formal analysis, Visualization, Investigation, Writing – original draft, Software. **Mansour Ghalehnovi:** Supervision, Conceptualization, Writing – review & editing, Resources, Project administration, Validation. **Mohsen Rashki:** Supervision, Conceptualization, Software, Writing – review & editing, Resources.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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