Mediterr. J. Math. (2021) 18:27
https://doi.org/10.1007/s00009-020-01677-2
1660-5446/21/010001-12
published online January 12, 2021

Mediterranean Journal of Mathematics

The Structure of Finitely Generated Shift-Invariant Subspaces on Locally Compact Abelian Groups

Seyyed Mohammad Tabatabaie, Kazaros Kazarian, Rajab Ali Kamyabi Gol and Soheila Jokar

Abstract. In this paper, we characterize finitely generated shift-invariant subspaces of $L^2(G)$, where G is a locally compact abelian group. In particular, we give a formula for the coefficients in the known representation of the Fourier transform of the elements of finitely generated shift-invariant subspaces. Also, certain orthogonalization procedure for generators which is reminiscent of the Gram–Schmidt orthogonalization process is given.

Mathematics Subject Classification. Primary 43A15; Secondary 43A25, 46E30.

Keywords. Locally compact groups, Fourier transform, shift-invariant subspaces, summation basis.

1. Introduction and Notations

A discrete subgroup L of a locally compact abelian (LCA) group G is called a (uniform) lattice if the quotient space G/L is compact. For each $y \in G$ and $f : G \to \mathbb{C}$, define the translation $T_y : G \to \mathbb{C}$ by $T_y f(x) := f(y^{-1}x)$. Then, a closed subspace V of $L^2(G)$ is called shift invariant (SI) with respect to a given lattice L if $T_y f \in V$ whenever $f \in V$ and $y \in L$. For each $\Omega \subseteq L^2(G)$, define:

$$V(\Omega) := \overline{\operatorname{span}}\{T_y\varphi : y \in L \text{ and } \varphi \in \Omega\},$$

which is the smallest SI space containing Ω. If Ω is a finite subset of $L^2(G)$, then $V(\Omega)$ is called a finitely generated shift-invariant (FSI) subspace. These subspaces have attracted a lot of attention and have been studied in many papers. For instance, FSI subspaces of $L^2(\mathbb{R}^n)$ were characterized by De Boor et al. [2], and they were studied in the context of locally compact groups and hypergroups in [12] and [16] (see also [3–5,8,9]). Inspired by [14][1], and based on our previous works [12,16], in this paper, we will study FSI subspaces generated by a minimal generating subset of $L^2(G)$. More precisely, we give a formula for determining the coefficients in the known representation of the Fourier transform of the members of FSI subspaces as a linear combination of the Fourier transform of the generators. Also, a sort of Gram–Schmidt orthogonalization process is introduced. The bracket product on $L^2(\widehat{G})$, which was originally introduced in [11] and extended in [15] (see also [13]), plays an important role in this paper. These products are also applicable to extend many ideas and facts from the theory of shift-invariant subspaces, factorable operators, and Weyl–Heisenberg frames on $\mathbb{R}^n$, to the setting of LCA groups in a different way.

2. Main Results

In this section, by bracket product of elements of $L^2(\widehat{G})$, we give a formula for functions in FSI subspaces of $L^2(G)$ in terms of the generating set. We first recall the Weil's formula which plays a key role in the given proofs. Let G be a locally compact abelian group with a lattice L and let $\mathrm{d}\xi$ be a Haar measure on the dual group $\widehat{G}$. It is well known that G is discrete if and only if $\widehat{G}$ is compact. Also, $\widehat{(G/L)} \cong L^\perp$ and $\widehat{G}/L^\perp \cong \widehat{L}$, where $L^\perp := \{\xi \in \widehat{G} : \xi(L) = \{1\}\}$ [7]. Easily, one can see that if L is a lattice in G, then $L^\perp$ is also a lattice for $\widehat{G}$. There is a suitable $\widehat{G}$-invariant measure $\mathrm{d}\dot{\xi}$ on $\widehat{G}/L^\perp$, where $\dot{\xi} := \xi L^\perp$, such that the following identity (called Weil's formula) holds:

$$\int_{\widehat{G}} f(\xi)\,\mathrm{d}\xi = \int_{\widehat{G}/L^\perp} \sum_{\eta \in L^\perp} f(\xi\eta)\,\mathrm{d}\dot{\xi}, \quad (f \in L^1(\widehat{G})).$$

Definition 2.1. Let G be a locally compact abelian group and L be a lattice in G. The bracket product of each $f, g \in L^2(\widehat{G})$ is defined by:

$$[f, g](\xi) := \sum_{\eta \in L^\perp} f(\xi\eta)\overline{g(\xi\eta)}, \qquad (\xi \in \widehat{G}).$$

Remark 2.2. Since the mapping $\Psi : \widehat{G}/L^\perp \to \widehat{L}$ defined by $\Psi(\xi L^\perp) := \xi|_L$ for all $\xi \in \widehat{G}$, is an isomorphism of topological groups [7, Theorem 4.39], every element of $\widehat{L}$ is the restriction of an (not necessarily unique) element of $\widehat{G}$ to L. If $\xi_1, \xi_2 \in \widehat{G}$ and $\xi_1 = \xi_2$ on L, then we have $\xi_1 L^\perp = \xi_2 L^\perp$. This easily implies that $[f, g](\xi_1) = [f, g](\xi_2)$ for all $f, g \in L^2(\widehat{G})$. In other words, brackets can admit inputs from $\widehat{L}$ by $[f, g](\xi|_L) := [f, g](\xi)$ for all $\xi \in \widehat{G}$. On the other hand, for each $\xi \in \widehat{G}$ and $\eta \in L^\perp$, we have $[f, g](\xi\eta) = [f, g](\xi)$. Thus, $[f, g]$ is constant on $L^\perp$-cosets, and so brackets can admit inputs from $\widehat{G}/L^\perp$ too by setting:

$$[f, g](\dot{\xi}) := [f, g](\xi), \tag{2.1}$$

[1] A first version of the main results of this paper, restricted to the case of the real line, has been posted on ArXiv [14] by one of the authors.

where $\xi \in \widehat{G}$ and $\dot{\xi} := \xi L^{\perp}$. Therefore, in different situations, we can consider appropriate inputs for brackets.

For each $f, g \in L^2(\widehat{G})$, we have $f\bar{g} \in L^1(\widehat{G})$ and:

$$|[f,g](\xi)|^2 \leq [f,f](\xi)\,[g,g](\xi) \quad \text{a.e. on } \widehat{G}. \tag{2.2}$$

Since

$$\|[f,g]\|_{L^1(\widehat{G}/L^{\perp})} = \int_{\widehat{G}/L^{\perp}} \left| \sum_{\eta \in L^{\perp}} f(\xi\eta)\overline{g(\xi\eta)} \right| \mathrm{d}\dot{\xi} \leq \int_{\widehat{G}} |f(\xi)\overline{g(\xi)}| \mathrm{d}\xi < \infty,$$

the bracket product $[\cdot,\cdot] : L^2(\widehat{G}) \times L^2(\widehat{G}) \to L^1(\widehat{G}/L^{\perp})$ defined by (2.1) is well defined.

Definition 2.3. For each $\varphi \in L^2(G)$, we define:

$$w_\varphi(\xi) := [\hat{\varphi}, \hat{\varphi}](\xi) = \sum_{\eta \in L^{\perp}} |\hat{\varphi}(\xi\eta)|^2, \qquad (\xi \in \widehat{G}),$$

where by $\hat{\varphi}$, we denote the Fourier transform of the function φ (see [7]). The space of all functions $r : \widehat{L} \to \mathbb{C}$ satisfying:

$$\int_{\widehat{L}} |r(\xi)|^2\, w_\varphi(\xi)\, \mathrm{d}\xi < \infty,$$

is denoted by $L^2(\widehat{L}, w_\varphi)$, where $\mathrm{d}\xi$ is the Plancherel measure on $\widehat{L}$. For each $r \in L^2(\widehat{L}, w_\varphi)$, we define the following norm:

$$\|r\|_{L^2(\widehat{L}, w_\varphi)} := \left(\int_{\widehat{L}} |r(\xi)|^2\, w_\varphi(\xi)\, \mathrm{d}\xi \right)^{\frac{1}{2}}.$$

Definition 2.4. Let $\Omega := \{\varphi_i\}_{i=1}^N \subseteq L^2(G)$ be a finite subset of non-zero functions. For each $1 \leq i \leq N$, we denote $\Omega^{(i)} := \Omega \backslash \{\varphi_i\}$. The set Ω is called a *minimal generating set* for $V(\Omega)$ if, for each $1 \leq i \leq N$, $\varphi_i \notin V(\Omega^{(i)})$. Also, Ω is called *B-orthogonal set* (with respect to a given lattice L) if, for each distinct $1 \leq i, j \leq N$, $[\widehat{\varphi_i}, \widehat{\varphi_j}](\xi) = 0$ a.e. on $\widehat{L}$.

We recall the following lemma from [12].

Lemma 2.5. *Let $\varphi \in L^2(G)$. Then, $f \in V_\varphi$ if and only if there exists a function $r \in L^2(\widehat{L}, w_\varphi)$, such that $\hat{f}(\xi) = r(\dot{\xi})\hat{\varphi}(\xi)$ $(\xi \in \widehat{G})$, and $\|f\|_{L^2(G)} = \|r\|_{L^2(\widehat{L}, w_\varphi)}$.*

The next theorem generalizes the above lemma for orthogonal finite subsets of $L^2(G)$, and can be considered also as a consequence of the characterization of Riesz basis property for SI spaces in [3,5].

Theorem 2.6. *Let $\Omega := \{\varphi_i\}_{i=1}^N$ be a B-orthogonal subset of $L^2(G)$. For each $f \in L^2(G)$ and $i = 1, \ldots, N$, we put:*

$$m_i(f) := \frac{[\hat{f}, \widehat{\varphi_i}]}{[\widehat{\varphi_i}, \widehat{\varphi_i}]},$$

where $m_i(f)(\xi) := 0$ *whenever* $[\widehat{\varphi}_i, \widehat{\varphi}_i](\xi) = 0$*. Then,* $m_i(f) \in L^2(\widehat{L}, w_{\varphi_i})$*, and* $f \in V(\Omega)$ *if and only if:*

$$\hat{f} = \sum_{i=1}^{N} m_i(f)\widehat{\varphi}_i. \tag{2.3}$$

In this case:

$$\|f\|_{L^2(G)} = \left(\sum_{i=1}^{N} \|m_i(f)\|^2_{L^2(\widehat{L}, w_{\varphi_i})}\right)^{\frac{1}{2}}. \tag{2.4}$$

The following result which can be concluded from the Weil's formula would be helpful in the proof of Theorem 2.6.

Proposition 2.7. *If* G *is an LCA group with a lattice* L*, then for each* $f, g \in L^2(G)$*:*

$$\langle f, g\rangle = \int_{\widehat{G}/L^{\perp}} [\hat{f}, \hat{g}]\, \mathrm{d}\dot{\xi}, \tag{2.5}$$

where $\dot{\xi} := \xi L^{\perp}$ *for all* $\xi \in \widehat{G}$*.*

Proof of Theorem 2.6. Let $f \in L^2(G)$ and $i = 1, 2, \ldots, N$. We have $m_i(f) \in L^2(\widehat{L}, w_{\varphi_i})$, since by (2.2) and (2.5):

$$\begin{aligned}
\|m_i(f)\|^2_{L^2(\widehat{L}, w_{\varphi_i})} &= \int_{\widehat{L}} \left|\frac{[\hat{f}, \widehat{\varphi}_i]}{[\widehat{\varphi}_i, \widehat{\varphi}_i]}\right|^2 (\xi)\, w_{\varphi_i}(\xi)\, \mathrm{d}\xi \\
&\le \int_{\widehat{L}} \frac{[\hat{f}, \hat{f}](\xi)[\widehat{\varphi}_i, \widehat{\varphi}_i](\xi)}{[\widehat{\varphi}_i, \widehat{\varphi}_i]^2(\xi)}\, [\widehat{\varphi}_i, \widehat{\varphi}_i](\xi)\, \mathrm{d}\xi \\
&= \int_{\widehat{G}/L^{\perp}} [\hat{f}, \hat{f}](\xi)\, \mathrm{d}\dot{\xi} \\
&= \langle f, f\rangle = \|f\|_2^2 < \infty.
\end{aligned}$$

If $N = 1$, for a function $\varphi \in L^2(G)$, we have $V(\Omega) = V_\varphi$. By Lemma 2.5, $f \in V_\varphi$ if and only if for some $r \in L^2(\widehat{L}, w_\varphi)$, $\hat{f} = r\,\hat{\varphi}$ and $\|f\|_{L^2(G)} = \|r\|_{L^2(\widehat{L}, w_\varphi)}$. By [12, Proposition 2.2], if $f \in A_\varphi := \mathrm{span}\{T_y\varphi : y \in L\}$, then $\hat{f} = r\,\hat{\varphi}$, where $r(\xi) = r(\dot{\xi}) = \sum_{i=1}^{n} a_i \overline{\xi(y_i)}$ for some $a_1, \ldots, a_n \in \mathbb{C}$ and $y_1, \ldots, y_n \in L$. Therefore:

$$\begin{aligned}
[\hat{f}, \hat{\varphi}](\xi) &= \sum_{\eta \in L^{\perp}} \hat{f}(\xi\eta)\overline{\hat{\varphi}(\xi\eta)} \\
&= \sum_{i=1}^{n} \sum_{\eta \in L^{\perp}} a_i \cdot (\overline{\xi\eta})(y_i)\hat{\varphi}(\xi\eta)\overline{\hat{\varphi}(\xi\eta)} \\
&= \sum_{i=1}^{n} a_i \bar{\xi}(y_i) \sum_{\eta \in L^{\perp}} |\hat{\varphi}(\xi\eta)|^2 \\
&= r(\xi)\, [\hat{\varphi}, \hat{\varphi}](\xi),
\end{aligned}$$

and so, $\hat{f} = \frac{[\hat{f},\hat{\varphi}]}{[\hat{\varphi},\hat{\varphi}]}\hat{\varphi}$. Now, let $f \in V_\varphi \backslash A_\varphi$. Then, there is a sequence (f_n) in A_φ, such that:

$$\lim_{n\to\infty} \|\widehat{f_n} - \hat{f}\|_{L^2(\widehat{G})} = \lim_{n\to\infty} \|f_n - f\|_{L^2(G)} = 0.$$

By the above argument, there is a sequence (r_n) in $L^2(\widehat{L}, w_\varphi)$, such that for all $n \in \mathbb{N}$, $\widehat{f_n} = r_n\hat{\varphi}$ and $r_n = \frac{[\widehat{f_n},\hat{\varphi}]}{[\hat{\varphi},\hat{\varphi}]}$. For each m and n, we have:

$$\|r_n - r_m\|_{L^2(\widehat{L},w_\varphi)} = \|f_n - f_m\|_{L^2(G)}.$$

Therefore, (r_n) is a Cauchy sequence. This implies that for some $r \in L^2(\widehat{L}, w_\varphi)$:

$$\lim_{n\to\infty} \|r_n - r\|_{L^2(\widehat{L},w_\varphi)} = 0,$$

and by

$$\|\widehat{f_n} - r\hat{\varphi}\|_{L^2(\widehat{G})} = \|r_n\hat{\varphi} - r\hat{\varphi}\|_{L^2(\widehat{G})} = \|(r_n - r)\hat{\varphi}\|_{L^2(\widehat{G})} = \|r_n - r\|_{L^2(\widehat{L},w_\varphi)},$$

we have $\hat{f} = r\hat{\varphi}$. However:

$$r = \lim_{n\to\infty} r_n = \lim_{n\to\infty} \frac{[\widehat{f_n},\hat{\varphi}]}{[\hat{\varphi},\hat{\varphi}]} = \frac{[\widehat{f},\hat{\varphi}]}{[\hat{\varphi},\hat{\varphi}]},$$

since by the relation (2.5) and the inequality (2.2):

$$\left\| \frac{[\widehat{f_n},\hat{\varphi}] - [\widehat{f},\hat{\varphi}]}{[\hat{\varphi},\hat{\varphi}]} \right\|_{L^2(\widehat{L},w_\varphi)} = \left\| \frac{[\widehat{f_n} - \hat{f},\hat{\varphi}]}{[\hat{\varphi},\hat{\varphi}]} \right\|_{L^2(\widehat{L},w_\varphi)} \leq \|f_n - f\|_{L^2(G)}.$$

This completes the proof for $N = 1$.

Now, let $N > 1$. If

$$f \in A(\Omega) := \operatorname{span}\{T_y\varphi_i : y \in L, i = 1, \ldots, N\},$$

then $f = \sum_{i=1}^N f_i$, where for any $1 \leq j \leq N$, $f_i \in A_{\varphi_i}$. By the above argument, for each $i = 1, \ldots, N$ we have $\widehat{f_i} = r_i\widehat{\varphi_i}$, where $r_i = m_i(f_i)$. However, by B-orthogonality of Ω and the relation (2.5), for each $1 \leq l, j \leq N$ with $l \neq j$, we have $V_{\varphi_l} \perp V_{\varphi_j}$ in $L^2(G)$. This implies that $m_i(f) = m_i(f_i)$ for all $i = 1, \ldots, N$. Therefore, the relation (2.3) holds. Also, by orthogonality, we have:

$$\|f\|^2_{L^2(G)} = \sum_{i=1}^N \|f_i\|^2_{L^2(G)} = \sum_{i=1}^N \|m_i(f)\|^2_{L^2(\widehat{L},w_{\varphi_i})}.$$

If $f \in V(\Omega)\backslash A(\Omega)$, similar to the case $N = 1$, one can see that the relations (2.3) and (2.4) hold and the proof of necessity is completed.

Conversely, let:

$$\hat{f} = \sum_{i=1}^N m_i(f)\widehat{\varphi_i}, \qquad \text{where} \qquad m_i(f) = \frac{[\hat{f},\widehat{\varphi_i}]}{[\widehat{\varphi_i},\widehat{\varphi_i}]} \in L^2(\widehat{L}, w_{\varphi_i}).$$

If, for any $1 \leq i \leq N$, we put $\hat{f_i} := m_i(f)\widehat{\varphi_i}$, then by Lemma 2.5, $f_i \in V_{\varphi_i}$. Now, since $\widehat{f} = \sum_{i=1}^N m_i(f)\widehat{\varphi_i}$, we have $f \in V(\Omega)$. □

Remark 2.8. The "only if" part of Theorem 2.6 holds more generally for hypergroups. In fact, if K is a commutative Pontryagin hypergroup with a lattice L satisfying the Weil's formula, and $\Omega := \{\varphi_i\}_{i=1}^N$ is a B-orthogonal subset of non-zero elements of $L^2(K)$, then for each $f \in V(\Omega)$, we have:

$$\hat{f} = \sum_{i=1}^{N} m_i(f)\widehat{\varphi}_i,$$

on the center of $\widehat{K}$, where $m_i(f) := \frac{[\hat{f},\widehat{\varphi_i}]}{[\widehat{\varphi_i},\widehat{\varphi_i}]}$. This statement would be a generalization of one of the main results of [16]. We refer to the monograph [1] and the paper [10] (in which hypergroups are called *convo*) for examples, basic definitions and properties related to hypergroups which are extensions of locally compact groups; see also [16].

Example 2.9. Let $G := (\mathbb{R}, +)$, and so $\widehat{G} = \mathbb{R}$. For the lattice $L := \mathbb{Z}$ in $\mathbb{R}$, we have $\widehat{L} = \mathbb{T}$ and $L^\perp = \mathbb{Z}$. Let $f \in L^2(\mathbb{R})$ and $\Omega := \{\varphi_k\}_{k=1}^N$ be a B-orthogonal subset of $L^2(\mathbb{R})$. Then:

$$[\hat{f},\widehat{\varphi_k}](\xi) = \sum_{\eta\in\mathbb{Z}} \hat{f}(\xi+\eta)\overline{\widehat{\varphi_k}(\xi+\eta)} \quad \text{and} \quad w_{\varphi_k}(\xi) := \sum_{\eta\in\mathbb{Z}} |\widehat{\varphi_k}(\xi+\eta)|^2 \quad (\xi \in \mathbb{R}).$$

By Theorem 2.6, $f \in V(\Omega)$ if and only if:

$$\hat{f} = \sum_{k=1}^{N} m_k(f)\widehat{\varphi_k}, \quad \text{where} \quad m_k(f) := \frac{[\hat{f},\widehat{\varphi_k}]}{[\widehat{\varphi_k},\widehat{\varphi_k}]} \in L^2(\mathbb{T}, w_{\varphi_k}).$$

Also, $\|f\|_{L^2(\mathbb{R})} = \left(\sum_{k=1}^N \|m_k(f)\|^2_{L^2(\mathbb{T},w_{\varphi_k})}\right)^{\frac{1}{2}}$.

In particular, since the Fourier transform is a unitary isomorphism from $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$ for a given N and each $1 \le k \le N$, there are $\varphi_k \in L^2(\mathbb{R})$, such that $\widehat{\varphi_k} = \chi_{[k,k+1)}$. Hence, $\{\varphi_k\}_{k=1}^N$ is B-orthogonal. In this case, for each $f \in L^2(\mathbb{R})$ and $\xi \in \mathbb{T}$,

$$[\hat{f},\widehat{\varphi_k}](\xi) = \hat{f}(t(\xi)+k),$$

where $t(\xi) := \xi \bmod 1$ in $[0,1)$, and $w_{\varphi_k}(\xi) \equiv 1$. Hence, $m_k(f)(\xi) = \hat{f}(t(\xi)+k)$.

Also, $f \in V(\{\varphi_k\}_{k=1}^N)$ if and only if $\hat{f} \in L^2(\mathbb{R})$ and $\hat{f}(t) = 0$ a.e. on $\mathbb{R}\backslash[1, N+1)$.

Example 2.10. Let G be the unit circle $\mathbb{T} \cong [0,1)$, and so $\widehat{G} = \mathbb{Z}$. The sets of the form $L = \frac{1}{n}\mathbb{Z}_n = \{0, \frac{1}{n}, \dots, \frac{n-1}{n}\}$ $(n \in \mathbb{N})$, are the only lattices in G [6, page 525]. The dual group of L is $\widehat{L} = \mathbb{Z}_n$ and its annihilator is $L^\perp = n\mathbb{Z}$. Let $\varphi \in L^2(\mathbb{T})$ and $n \in \mathbb{N}$. By Lemma 2.5, $f \in V_\varphi$ if and only if there exists a function $r \in L^2(\mathbb{Z}_n, w_\varphi)$, such that $\hat{f}(\xi) = r(\xi)\hat{\varphi}(\xi)$ $(\xi \in \mathbb{Z}_n)$, and $\|f\|_{L^2(\mathbb{T})} = \|r\|_{L^2(\mathbb{Z}_n,w_\varphi)}$, where $w_\varphi(\xi) := \sum_{\eta\in n\mathbb{Z}} |\hat{\varphi}(\xi\eta)|^2 \quad (\xi \in \mathbb{Z}_n)$. If $\Omega := \{\varphi_k\}_{k=1}^N$ is a B-orthogonal subset of $L^2(\mathbb{T})$, then:

$$[\hat{f},\widehat{\varphi_k}](\xi) = \sum_{\eta\in n\mathbb{Z}} \hat{f}(\xi\eta)\overline{\widehat{\varphi_k}(\xi\eta)} \quad \text{and} \quad w_{\varphi_k}(\xi) := \sum_{\eta\in n\mathbb{Z}} |\widehat{\varphi_k}(\xi\eta)|^2 \quad (\xi \in \mathbb{Z}_n).$$

By Theorem 2.6, $f \in V(\Omega)$ if and only if:

$$\hat{f} = \sum_{k=1}^{N} m_k(f)\widehat{\varphi_k} \quad \text{and} \quad \|f\|_{L^2(\mathbb{T})} = \left(\sum_{k=1}^{N} \|m_k(f)\|^2_{L^2(\mathbb{Z}_n, w_{\varphi_k})}\right)^{\frac{1}{2}},$$

where $m_k(f) := \frac{[\hat{f}, \widehat{\varphi_k}]}{[\widehat{\varphi_k}, \widehat{\varphi_k}]} \in L^2(\mathbb{Z}_n, w_{\varphi_k})$.

The following theorem can be considered as a Gram–Schmidt orthogonalization algorithm on fibers.

Theorem 2.11. *Suppose that $\Omega := \{\varphi_i\}_{i=1}^N \subseteq L^2(G)$ is a minimal generating set for $V(\Omega)$, and the functions $g_1, \ldots, g_N$ are defined by the relations $g_1 := \varphi_1$ and:*

$$\widehat{g_i} = \widehat{\varphi_i} - \sum_{j=1}^{i-1} b_j^{(i)} \widehat{g_j} \qquad (2 \le i \le N), \tag{2.6}$$

where

$$b_j^{(i)} := [\widehat{\varphi_i}, \widehat{g_j}][\widehat{g_j}, \widehat{g_j}]^{-1} \in L^2(\widehat{L}, w_{g_j}) \quad (1 \le j \le N-1), \tag{2.7}$$

and we define $b_j^{(i)}(\xi) := 0$ if $[\widehat{g_j}, \widehat{g_j}](\xi) = 0$. Then, for any $i, 1 \le i \le N$, we have $g_i \in V(\Omega_i)$, where $\Omega_i := \{\varphi_j\}_{j=1}^i$, and for any distinct $1 \le i, j \le N$, $[\widehat{g_i}, \widehat{g_j}](\xi) = 0 \quad$ a.e. on $\widehat{L}$.

Proof. First, it is shown by induction that for each $1 \le i \le N$, and $1 \le j \le i$, $b_j^{(i)} \in L^2(\widehat{L}, w_{g_j})$ and $\widehat{g_i}$ belongs to $L^2(\widehat{G})$. Trivially, $\widehat{g_1} = \widehat{\varphi_1} \in L^2(\widehat{G})$, and $b_1^{(1)} = 1 \in L^2(\widehat{L}, w_{g_1})$, since:

$$\|1\|_{L^2(\widehat{L}, w_{g_1})} = \int_{\widehat{L}} [\widehat{\varphi_1}, \widehat{\varphi_1}](\xi)\, \mathrm{d}\xi = \|\widehat{\varphi_1}\|_2^2 < \infty.$$

Let $g_1, \ldots, g_i \in L^2(G)$. Then, $b_j^{(i)} \in L^2(\widehat{L}, w_{g_j})$, since by the relations (2.2) and (2.5):

$$\begin{aligned}
\|b_j^{(i)}\|^2_{L^2(\widehat{L}, w_{g_j})} &= \int_{\widehat{G}/L^\perp} \left| \frac{[\widehat{\varphi_i}, \widehat{g_j}](\xi)}{[\widehat{g_j}, \widehat{g_j}](\xi)} \right|^2 w_{g_j}(\xi)\, \mathrm{d}\dot{\xi} \\
&\le \int_{\widehat{G}/L^\perp} \frac{[\widehat{\varphi_i}, \widehat{\varphi_i}](\xi)[\widehat{g_j}, \widehat{g_j}](\xi)}{[\widehat{g_j}, \widehat{g_j}]^2(\xi)} [\widehat{g_j}, \widehat{g_j}](\xi)\, \mathrm{d}\dot{\xi} \\
&= \langle \varphi_i, \varphi_i \rangle = \|\varphi_i\|^2_{L^2(G)}.
\end{aligned}$$

Also, by Lemma 2.5 immediately, we have $g_{i+1} \in L^2(G)$.

Now, by induction for $N \ge 2$, we prove that for any distinct $1 \le i, j \le N$, $[\widehat{g_i}, \widehat{g_j}](\xi) = 0 \quad$ a.e. on $\widehat{L}$.

If $N = 2$, we have:

$$\begin{aligned}
[\widehat{g_1}, \widehat{g_2}](\xi) &= \left[\widehat{\varphi_1}, \widehat{\varphi_2} - \frac{[\widehat{\varphi_2}, \widehat{\varphi_1}]}{[\widehat{\varphi_1}, \widehat{\varphi_1}]} \cdot \widehat{\varphi_1} \right](\xi) \\
&= \sum_{\eta \in L^\perp} \widehat{\varphi_1}(\xi\eta) \overline{\left(\widehat{\varphi_2} - \frac{[\widehat{\varphi_2}, \widehat{\varphi_1}]}{[\widehat{\varphi_1}, \widehat{\varphi_1}]} \cdot \widehat{\varphi_1} \right)}(\xi\eta)
\end{aligned}$$

$$= [\widehat{\varphi_1}, \widehat{\varphi_2}](\xi) - \sum_{\eta \in L^\perp} \frac{\overline{[\widehat{\varphi_2}, \widehat{\varphi_1}](\xi\eta)}}{[\widehat{\varphi_1}, \widehat{\varphi_1}](\xi\eta)} \cdot \widehat{\varphi_1}(\xi\eta)\overline{\widehat{\varphi_1}(\xi\eta)}$$
$$= [\widehat{\varphi_1}, \widehat{\varphi_2}](\xi) - \frac{\overline{[\widehat{\varphi_2}, \widehat{\varphi_1}](\xi)}}{[\widehat{\varphi_1}, \widehat{\varphi_1}](\xi)} \cdot \sum_{\eta \in L^\perp} \widehat{\varphi_1}(\xi\eta)\overline{\widehat{\varphi_1}(\xi\eta)}$$
$$= [\widehat{\varphi_1}, \widehat{\varphi_2}](\xi) - [\widehat{\varphi_1}, \widehat{\varphi_2}](\xi) = 0.$$

Now, suppose that $[\widehat{g}_i, \widehat{g}_j](\xi) = 0 \quad \text{a.e. on } \widehat{L}$ for all $2 \le m \le N-1$ and all distinct $1 \le i, j \le m-1$. Put $b_j^{(m)} := [\widehat{\varphi_m}, \widehat{g}_j]\,[\widehat{g}_j, \widehat{g}_j]^{-1}$, and let $\widehat{g_m} = \widehat{\varphi_m} - \sum_{j=1}^{m-1} b_j^{(m)} \widehat{g}_j$. Then, by the assumption of induction, for any $1 \le l \le m-1$, we have:

$$[\widehat{g_m}, \widehat{g}_l] = [\widehat{\varphi_m}, \widehat{g}_l] - b_l^{(m)}[\widehat{g}_l, \widehat{g}_l] = [\widehat{\varphi_m}, \widehat{g}_l] - [\widehat{\varphi_m}, \widehat{g}_l] = 0, \text{ a.e.}$$

Finally, we prove that for any $i, 1 \le i \le N$, $g_i \in V(\Omega_i)$. If we put $\widehat{f} = \frac{[\widehat{\varphi_2}, \widehat{\varphi_1}]}{[\widehat{\varphi_1}, \widehat{\varphi_1}]} \cdot \widehat{\varphi_1}$, then by Theorem 2.6, $f \in V_{\varphi_1}$, and clearly, $\varphi_2 \in V_{\varphi_2}$. Therefore, by relation (2.6), $g_2 \in V(\Omega_2)$. The proof of the general case is similar and we skip it. □

Proposition 2.12. *Under assumptions of Theorem 2.11, we have:*

$$V(\Omega) = \bigoplus_{i=1}^{N} V_{g_i}.$$

Proof. For the orthogonality, we note that by Theorem 2.11, for each distinct i, j, we have $[\widehat{g}_i, \widehat{g}_j](\xi) = 0 \quad \text{a.e. on } \widehat{L}$, and by the relation (2.5):

$$\langle T_x g_i, T_y g_j \rangle = \int_{\widehat{G}/L^\perp} [\widehat{T_x g_i}, \widehat{T_y g_j}](\xi)\, \mathrm{d}\dot{\xi}$$
$$= \int_{\widehat{G}/L^\perp} \xi(x^{-1}y)[\widehat{g}_i, \widehat{g}_j](\xi)\, \mathrm{d}\dot{\xi} = 0$$

for all $x, y \in L$.

Let $f \in \bigoplus_{i=1}^{N} V_{g_i}$. Then, $f = \sum_{i=1}^{N} f_i$, where for each $1 \le i \le N$, $f_i \in V_{g_i}$. By Theorem 2.6, for each $1 \le i \le N$, there is $F_i \in L^2(\widehat{L}, w_{g_i})$, such that $\widehat{f_i} = F_i \cdot \widehat{g}_i$. Hence, $\hat{f} = \sum_{i=1}^{N} F_i \cdot \widehat{g}_i$, and by Theorem 2.6, we have $f \in V(\{g_i\}_{i=1}^{N})$. On the other hand, by Proposition 2.11, for each $1 \le i \le N$, we have $g_i \in V(\Omega_i)$, and so $f \in V(\Omega)$.

Conversely, suppose that $f \in V(\Omega)$, $\Omega := \{\varphi_i\}_{i=1}^{N}$. By relations (2.6) and (2.7) and by B-orthogonality of the set $\{g_i\}_{i=1}^{N}$, for each $1 \le m \le N$, we have:

$$b_m^{(m)} = \frac{[\widehat{\varphi_m}, \widehat{g_m}]}{[\widehat{g_m}, \widehat{g_m}]}$$
$$= \frac{[\widehat{g_m} + \sum_{j=1}^{m-1} b_j^{(m)} \widehat{g}_j, \widehat{g_m}]}{[\widehat{g_m}, \widehat{g_m}]}$$
$$= \frac{[\widehat{g_m}, \widehat{g_m}] + b_j^{(m)} \sum_{j=1}^{m-1} [\widehat{g}_j, \widehat{g_m}]}{[\widehat{g_m}, \widehat{g_m}]} = 1,$$

where $b_m^{(m)}$ for $m = N$ is similarly defined by $b_N^{(N)} := [\widehat{\varphi_N}, \widehat{g_N}][\widehat{g_N}, \widehat{g_N}]^{-1}$. Therefore:

$$\widehat{\varphi_m} = \widehat{g_m} + \sum_{j=1}^{m-1} b_j^{(m)} \widehat{g_j} = \sum_{j=1}^{m} b_j^{(m)} \widehat{g_j} = \sum_{j=1}^{m} \frac{[\widehat{\varphi_m}, \widehat{g_j}]}{[\widehat{g_j}, \widehat{g_j}]} \widehat{g_j}.$$

By Theorem 2.6, $\varphi_m \in V(\{g_i\}_{i=1}^m)$. Hence, $f \in \bigoplus_{i=1}^N V_{g_i}$. □

The following lemma gives a formula for the orthogonal projection on $V(\Omega)$ for a finite minimal generating subset $\Omega \subseteq L^2(G)$.

Lemma 2.13. *Let $\Omega := \{\varphi_i\}_{i=1}^N \subseteq L^2(G)$ be a finite minimal generating set for $V(\Omega)$. Then, for each $f \in L^2(G)$, the orthogonal projection $P_\Omega(f)$ of f on $V(\Omega)$ is given by:*

$$\widehat{P_\Omega(f)} = \sum_{i=1}^{N} [\hat{f}, \widehat{g_i}][\widehat{g_i}, \widehat{g_i}]^{-1} \widehat{g_i},$$

where $\{g_i\}_{i=1}^N$ are defined by the relation (2.6).

Proof. Let $f = f_1 \oplus f_2$, where $f_1 \in V(\Omega)$ and $f_2 \in [V(\Omega)]^\perp$. Then, $P_\Omega(f) = f_1$. By Proposition 2.12, for all $1 \le i \le N$, there is $F_i \in V_{g_i}$, such that $f_1 = \sum_{i=1}^N F_i$. By Theorem 2.6, for each $1 \le i \le N$, we have $\widehat{F_i} = m(F_i)\widehat{g_i}$, where:

$$m(F_i) := \frac{[\widehat{F_i}, \widehat{g_i}]}{[\widehat{g_i}, \widehat{g_i}]} \in L^2(\widehat{L}, w_{g_i}).$$

Hence:

$$\widehat{P_\Omega(f)} = \sum_{i=1}^{N} \frac{[\widehat{F_i}, \widehat{g_i}]}{[\widehat{g_i}, \widehat{g_i}]} \cdot \widehat{g_i}.$$

Since $f_2 \in [V(\Omega)]^\perp$ and $g_i \in V(\Omega)$ $(1 \le i \le N)$, for each $x \in L$, we have:

$$\begin{aligned}
\mathcal{F}^{-1}[\widehat{f_2}, \widehat{g_i}](x) &= \int_{\widehat{G}/L^\perp} \xi(x)[\widehat{f_2}, \widehat{g_i}](\xi)\, d\dot{\xi} \\
&= \int_{\widehat{G}/L^\perp} \sum_{\eta \in L^\perp} (\xi\eta)(x) \widehat{f_2}(\xi\eta)\overline{\widehat{g_i}(\xi\eta)}\, d\dot{\xi} \\
&= \int_{\widehat{G}} \xi(x) \widehat{f_2}(\xi)\overline{\widehat{g_i}(\xi)}\, d\xi \\
&= \int_{\widehat{G}} \widehat{f_2}(\xi)\overline{\widehat{T_x g_i}(\xi)}\, d\xi \\
&= \langle \widehat{f_2}, \widehat{T_x g_i} \rangle = \langle f_2, T_x g_i \rangle = 0,
\end{aligned}$$

where $\mathcal{F}^{-1}$ is the inverse Fourier transform. Therefore, $[\widehat{f_2}, \widehat{g_i}] = 0$. This implies that $[\widehat{f}, \widehat{g_i}] = [\widehat{f_1}, \widehat{g_i}]$, and by B-orthogonality of $\{g_i\}_{i=1}^N$, we have

$[\hat{f},\widehat{g_i}]=\sum_{j=1}^{N}[\widehat{F_j},\widehat{g_i}]=[\widehat{F_i},\widehat{g_i}]$. Therefore:

$$\widehat{P_\Omega(f)}=\sum_{i=1}^{N}[\hat{f},\widehat{g_i}][\widehat{g_i},\widehat{g_i}]^{-1}\widehat{g_i}.$$

□

Theorem 2.14. *Let $\Omega:=\{\varphi_i\}_{i=1}^N\subseteq L^2(G)$ be a finite minimal generating set for $V(\Omega)$. Then, for each $f\in V(\Omega)$, we have:*

$$\hat{f}=\sum_{i=1}^{N}m_i(f)\widehat{\varphi_i},\qquad where\qquad m_i(f):=\frac{[\hat{f},\widehat{h_i}]}{[\widehat{h_i},\widehat{h_i}]}\in L^2(\widehat{L},w_{h_i}),\tag{2.8}$$

and

$$\sum_{i=1}^{N}\|m_i(f)\|^2_{L^2(\widehat{L},w_{h_i})}\leq\|f\|^2_{L^2(G)},\tag{2.9}$$

where

$$\widehat{h_i}=\widehat{\varphi_i}-\widehat{P_{\Omega^{(i)}}(\varphi_i)},\tag{2.10}$$

and $\Omega^{(i)}:=\Omega\backslash\{\varphi_i\}$.

Proof. By Theorem 2.6, the affirmation is true when $N=1$.

Suppose that the theorem is true for some $N\in\mathbb{N}$. Let $\Gamma:=\{\phi_i\}_{i=1}^{N+1}$ be a minimal generating set for $V(\Gamma)$, and put:

$$\widehat{\varphi_i}=\widehat{\phi_i}-\frac{[\widehat{\phi_i},\widehat{\phi}_{N+1}]}{[\widehat{\phi}_{N+1},\widehat{\phi}_{N+1}]}\widehat{\phi}_{N+1},\qquad(1\leq i\leq N).$$

By Lemma 2.13, it follows that for any $1\leq i\leq N$:

$$\left[\widehat{\varphi_i},\widehat{\phi}_{N+1}\right](\xi)=0\qquad\text{a.e.}$$

Therefore, setting $\Omega:=\{\varphi_i\}_{i=1}^N$, by the relation (2.5), we have $V(\Omega)\perp V_{\phi_{N+1}}$, and so, $V(\Omega)\oplus V_{\phi_{N+1}}=V(\Gamma)$. Hence, by the induction assumption, for any $f\in V(\Gamma)$, we have:

$$\hat{f}=\sum_{i=1}^{N}m_i(f)\widehat{\varphi_i}+\frac{\left[\hat{f},\widehat{\phi}_{N+1}\right]}{\left[\widehat{\phi}_{N+1},\widehat{\phi}_{N+1}\right]}\widehat{\phi}_{N+1},$$

where $m_i(f)$ is defined by (2.8).
For any $1\leq i\leq N$:

$$\widehat{\phi_i}-\widehat{P_{\Gamma^{(i)}}(\phi_i)}=\widehat{\phi_i}-\widehat{P_{\Gamma^{(i)}}}(\varphi_i)-\frac{\left[\widehat{\phi_i},\widehat{\phi}_{N+1}\right]}{\left[\widehat{\phi}_{N+1},\widehat{\phi}_{N+1}\right]}\widehat{\phi}_{N+1}=\widehat{\varphi_i}-\widehat{P_{\Omega^{(i)}}}(\varphi_i)=\widehat{h_i}.$$

Putting $\widehat{h}_{N+1}=\widehat{\phi}_{N+1}-\widehat{P_{\Gamma^{(N+1)}}}(\phi_{N+1})$, easily, one can show $w_{h_{N+1}}(\xi)\leq w_{\phi_{N+1}}(\xi)$ and the proof is completed. □

Acknowledgements

We would like to thank the referee for carefully reading our manuscript and her/his very nice comments.

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

References

[1] Bloom, W.R., Heyer, H.: Harmonic Analysis of Probability Measures on Hypergroups. De Gruyter, Berlin (1995)

[2] De Boor, C., DeVore, R.A., Ron, A.: The structure of finitely generated shift invariant spaces in $L^2(\mathbb{R}^d)$. J. Funct. Anal. **119**, 37–78 (1994)

[3] Bownik, M.: The structure of shift-invariant subspaces of $L^2(\mathbb{R}^n)$. J. Funct. Anal. **177**(2), 282–309 (2000)

[4] Bownik, M., Ross, K.: The structure of translation-invariant spaces on locally compact abelian groups. J. Fourier Anal. Appl. **21**(4), 849–884 (2015)

[5] Cabrelli, C., Paternostro, V.: Shift-invariant spaces on LCA groups. J. Funct. Anal. **258**(6), 2034–2059 (2010)

[6] Christensen, O.: An Introduction to Frames and Riesz Bases, 2nd edn. Springer International Publishing, Cham (2016)

[7] Folland, G.B.: A Course in Abstract Harmonic Analysis. CRC Press, London (1995)

[8] Iverson, J.: Subspaces of $L^2(G)$ invariant under translation by an abelian subgroup. J. Funct. Anal. **269**(3), 865–913 (2015)

[9] Jakobsen, M.S., Lemvig, J.: Reproducing formulas for generalized translation invariant systems on locally compact abelian groups. Trans. Am. Math. Soc. **368**(12), 8447–8480 (2016)

[10] Jewett, R.I.: Spaces with an abstract convolution of measures. Adv. Math. **18**, 1–101 (1975)

[11] Jia, R.Q., Micchelli, C.A.: Using the refinement equation for the construction of pre-wavelets, II. Powers of two. In: Laurent, P.J., Le Méhauté, A., Schumaker, L.L. (eds.) Curves and Surfaces, pp. 209–246. Academic Press, New York (1991)

[12] Kamyabi Gol, R.A., Raisi Tousi, R.: The structure of shift invariant spaces on a locally compact abelian group. J. Math. Anal. Appl. **340**, 219–225 (2008a)

[13] Kamyabi Gol, R.A., Raisi Tousi, R.: Bracket products on locally compact abelian groups. J. Sci. I. R. Iran **19**(2), 153–157 (2008b)

[14] Kazarian, K.S.: On the structure of finitely generated shift-invariant subspaces. arXiv:1605.02456v1 [math.CA] (2016)

[15] Ron, A., Shen, Z.: Frames and stable bases for shift invarianat subspace of $L^2(\mathbb{R}^d)$. Can. J. Math. **47**, 1051–1094 (1995)

[16] Tabatabaie, S.M., Jokar, S.: A characterization of admissible vectors related to representations on hypergroups. Tbilisi Math. J. **10**(4), 143–151 (2017)

Seyyed Mohammad Tabatabaie and Soheila Jokar
Department of Mathematics
University of Qom
Qom
Iran
e-mail: sm.tabatabaie@qom.ac.ir

Soheila Jokar
e-mail: s.jokar@stu.qom.ac.ir

Kazaros Kazarian
Departamento de Matematicas
Universidad Autonoma de Madrid
28049 Madrid
Spain
e-mail: kazaros.kazarian@uam.es

Rajab Ali Kamyabi Gol
Department of Mathematics
Ferdowsi University of Mashhad
Mashhad
Iran
e-mail: kamyabi@ferdowsi.um.ac.ir

Received: October 2, 2019.
Revised: August 4, 2020.
Accepted: December 5, 2020.