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On wavelet multiplier and Landau–Pollak–Slepian operators on $L^2(G, \mathbb{H})$

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Abstract
In this paper, we define the wavelet multiplier and Landau–Pollak–Slepian (L.P.S) operators on the Hilbert space $L^2(G^2, \mathbb{H})$, where G is a locally compact abelian topological group, and $\mathbb{H}$ is the quaternion algebra; Also, we will investigate some of their properties. In particular, we show that they are bounded linear operators, as well in Schatten p-class spaces, $1 \leq p \leq \infty$, and we determine their trace class.

Keywords Locally compact abelian group · Dual group · Wavelet multiplier operator · Landau–Pollak–Slepian operator · Admissible wavelets · Unitary representation · Quaternion algebra

Mathematics Subject Classification Primary 43A15; Secondary 43A25 · 42C15

1 Introduction

The representation of a function by simultaneous bandlimiting and timelimiting has been the concern of many researchers in their works, until to the 1960s, when the problem was solved by works of Henry Landau, Henry Pollack and David Slepian [12,13,16], they introduced the self-adjoint Landau–Pollak–Slepian (L.P.S.) operator on $L^2(\mathbb{R}^n)$. In 1999, He and Wong, introduced Wavelet multipliers operator on $L^2(\mathbb{R}^n)$ [11], which was generalized from Landau–Pollak–Slepian operator, they showed that the L.P.S. operator is in fact a wavelet multiplier operator; for more details see [17].

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The authors in [1], generalized L.P.S. and Wavelet multipliers operator on a locally compact abelian topological group, and investigated some of their properties.

Now, we recall some definitions, and fix some notations, which will be used in the sequel. The quaternion is a four- dimensional non-commutative algebra $\mathbb{H}$ [6,8,10,15] which is defined over the field of real numbers $\mathbb{R}$ with the three imaginary units. More precisely,

$$\mathbb{H} = \{q_0 + q_1 i + q_2 j + q_3 k, q_0, q_1, q_2, q_3 \in \mathbb{R}\},$$

where the elements i, j, k satisfy the following

$$i.j = -j.i = k, \quad j.k = -k.j = i, \quad k.i = -i.k = j, \quad i^2 = j^2 = k^2 = -1.$$

Every quaternion element q can be written also as a sum of two parts, Scalar part $Sc[q] = q_0 \in \mathbb{R}$ and Vector part, $Vec[q] = q_1 i + q_2 j + q_3 k$, $Vec[q]$ is often called pure quaternion, hence $q = Sc[q] + Vec[q]$. Now we will briefly review some basic facts on quaternions which are almost the same as in [8]. For any $p, q \in \mathbb{H}$, such that $p = p_0 + p_1 i + p_2 j + p_3 k,\ q = q_0 + q_1 i + q_2 j + q_3 k$, the addition and multiplication on $\mathbb{H}$ are

$$p + q = (p_0 + q_0) + (p_1 + q_1)i + (p_2 + q_2)j + (p_3 + q_3)k,$$

and

$$\begin{aligned} p.q &= (p_0 q_0 - p_1 q_1 - p_2 q_2 - p_3 q_3) + (p_1 q_0 + p_0 q_1 + p_2 q_3 - p_3 q_2)i \\ &\quad + (p_2 q_0 + p_0 q_2 + p_3 q_1 - p_1 q_3)j + (p_3 q_0 + p_0 q_3 + p_1 q_2 - p_2 q_1)k. \end{aligned}$$

The quaternion conjugate of q is defined by $\overline{p} = p_0 - p_1 i - p_2 j - p_3 k$; clearly it is an anti-involution, this means $\overline{(p.q)} = \overline{q}.\overline{p}$. For any two quaternions q, p we define right C_r and left C_l carrier operators as $C_r(p)q = qp$ and $qC_l(p) = pq$, respectively. Also For any $q \in \mathbb{H}$, we have $\overline{C_r(q)} = C_l(\overline{q})$ and vice versa. Although, multiplication on quaternions are non-commutative, it is clear that for all $p, q, r \in \mathbb{H}$, we have $Sc[q.p.r] = Sc[p.r.q] = Sc[r.q.p]$, which is called cyclic multiplication identity and it is an important tool in our work.

Note that throughout this paper, G denotes a locally compact abelian topological group with the Haar measure dx and $\hat{G}$, is the dual group of G with the Haar measure $d\xi$ such that $d\xi$ is the dual measure of dx [4,5,7]. The elements of G are denoted by x, y etc., while the elements of $\hat{G}$ are denoted by ξ, ω and so forth. For any continuous characters $\omega_l : G \to \mathbb{T}_l, l = i, j$ the character ω on $G^2 := G \times G$ is $\omega : G^2 \to \mathbb{T}_q$ in which $\omega(x) = \omega_j(x_2)\omega_i(x_1)$, for any $x = (x_1, x_2) \in G^2$, where $\mathbb{T}_i, \mathbb{T}_j$ and $\mathbb{T}_q$ are the unit circles in $\mathbb{C}_i, \mathbb{C}_j$ and $\mathbb{H}$ respectively, with $\mathbb{C}_l = \{a + bl, a, b \in \mathbb{R}\},\ l = i, j$.

Likewise, the classical inner product for any two elements $f, g \in L^2(G^2, \mathbb{H})$ we define the (usual) inner product on $L^2(G^2, \mathbb{H})$ as $(f, g) = \int_{G^2} f(x)\overline{g(x)}dx$. Also, we can define $\langle f, g\rangle = Sc \int_{G^2} f(x)\overline{g(x)}dx$, as another inner product on $L^2(G^2, \mathbb{H})$ which is called scalar inner product, it is clear that $\langle f, g\rangle = Sc(f, g)$ [10].

We recall that $\mathcal{F}_s(f)(\omega) = \hat{f}(\omega) = \int_{G^2} \overline{\omega_i(x_1)} f(x) \overline{\omega_j(x_2)} dx$ is the two sided quaternion Fourier transform (also called sandwich Fourier transform) of $f \in L^1(G^2, \mathbb{H})$ and the inverse of the two sided quaternion Fourier transform is $\mathcal{F}_s^{-1}(f)(x) = \check{f}(x) = \int_{\hat{G}^2} \omega_i(x_1) f(\omega) \omega_j(x_2) d\omega$ for f (if it exists). The two sided quaternion Fourier transform on $L^1(G^2, \mathbb{H}) \cap L^2(G^2, \mathbb{H})$ can be extended with respect to the scalar inner product uniquely to a unitary isomorphism from $L^2(G^2, \mathbb{H})$ to $L^2(\hat{G}^2, \mathbb{H})$ known as the quaternionic Plancherel Theorem. Now let us consider the closed subspace $M_{L^1(G^2,\mathbb{H}) \cap L^2(G^2,\mathbb{H})}$ which is define as

$$M_{L^1(G^2,\mathbb{H}) \cap L^2(G^2,\mathbb{H})} = \{f \in L^1(G^2, \mathbb{H}) \cap L^2(G^2, \mathbb{H}) : f \textit{ is even with respect to the first component}\},$$

for any function $f \in M_{L^1(G^2,\mathbb{H}) \cap L^2(G^2,\mathbb{H})}$ we have $\mathcal{F}_s(f) \in M_{L^2(\hat{G}^2,\mathbb{H})}$, so we can extend the two sided quaternion Fourier transformation on $M_{L^1(G^2,\mathbb{H}) \cap L^2(G^2,\mathbb{H})}$ with respect to the usual inner product by Plancherel theorem from $M_{L^2(G^2,\mathbb{H})}$ to $M_{L^2(\hat{G}^2,\mathbb{H})}$ for more details see [2].

The translation and modulation operators L_y, M_ξ are defined by $L_y f(x) = f(y^{-1}x)$ and $M_\xi f(x) = \xi_j(x_2) f(x) \xi_i(x_1)$, respectively. Note that the C_r-modulation operator $M_\xi^{C_r}$ is defined as $M_\xi^{C_r} f(x) = \xi_j(x_2) f(x) C_r \xi_i(x_1)$, for any quaternionic function f. The space $L^\infty(G^2, \mathbb{H})$ is defined as follows

$$L^\infty(G^2, \mathbb{H}) = \{f : G^2 \longrightarrow \mathbb{H} : f \textit{ is measurable and } \|f\|_\infty < \infty\},$$

where

$$\|f\|_\infty = inf \{a \geq 0 : \mu(\{x : |f(x)| > a\}) = 0\}.$$

Our aim in this paper is to give a generalization of wavelet multiplier and L.P.S. operators on $L^2(G^2, \mathbb{H})$ where G is a locally compact abelian topological group and $\mathbb{H}$ is the quaternion algebra. For this, we will define a unitary representation [4,5,7,9] on the Hilbert space $L^2(G^2, \mathbb{H})$ by using properties of dual groups [4,5,7], and we find, among other things, the set of all admissible wavelets [14,17] for this unitary representation. Then, we investigate boundedness [3] of wavelet multiplier operator and show it is in Schatten p-class spaces,$1 \leq p \leq \infty$, and also we will determine their trace class. At last, we will show that the L.P.S. operator is a special case of wavelet multiplier operator.

More precisely, this paper is organized as follows: Sect. 2, starts with the definition of a unitary representation on $L^2(G^2, \mathbb{H})$. Then, we calculate the admissible wavelet for this unitary representation, then, we show the operator $P_{\sigma,\varphi} : M_{L^2(G^2,\mathbb{H})} \rightarrow M_{L^2(G^2,\mathbb{H})}$, is unitarily equivalent to the wavelet operator $\varphi T_\sigma \overline{\varphi} : M_{L^2(G^2,\mathbb{H})} \rightarrow M_{L^2(G^2,\mathbb{H})}$, and state some preliminaries and related notations of these operators. Also we will discuss the boundedness of wavelet multipliers operator at two stages, first for $\sigma \in L^1(\hat{G}^2, \mathbb{H})$, and second for $\sigma \in M_{L^p(\hat{G}^2,\mathbb{H})}$, $1 < p \leq \infty$ by using The Riesz–Thorin Theorem [17]. In Sect. 3, we will explain that the wavelet multiplier operators are in the Schatten p-class spaces [17,18] and then we will determine the trace of these operators. In the end, in Sect. 4, we will give the definition of the L.P.S. operator $Q_C P_\Omega Q_C : M_{L^2(G^2,\mathbb{H})} \to M_{L^2(G^2,\mathbb{H})}$, investigate some of its properties including the relationship between wavelet multiplier and L.P.S. operators in a special case, and finally evaluate the trace of this operator.

2 Wavelet multipliers operator on $L^2(G^2, \mathbb{H})$

In this section we introduce the wavelet multipliers operator with respect to the various kinds of inner product on $L^2(G^2, \mathbb{H})$ in two ways, first according to the usual inner product $(f, g) = \int_{G^2} f(x)\overline{g(x)}dx$ for any $f, g \in L^2(G^2, \mathbb{H})$, and the second with respect to the scalar inner product $\langle f, g\rangle = Sc(f, g)$ for all $f, g \in L^2(G^2, \mathbb{H})$.

Also to show the boundedness of the wavelet multipliers operator, we use the Riesz–Thorin Theorem, which will be included for the readers' convenience [17].

Let (X, μ) be a measure space and (Y, ν) be a σ-finite measure space. Let T be a linear transformation with domain $\mathcal{D}$ consisting of all simple functions f on X such that

$$\mu(\{s \in X : f(s) \neq 0\}) < \infty,$$

and such that the range of T is contained in the set of all measurable functions on Y. Suppose that $\alpha_1, \alpha_2, \beta_1$ and β_2 are numbers in the interval [0, 1] and there exist positive constants M_1 and M_2 such that

$$\|Tf\|_{L^{\frac{1}{\beta_j}}(Y)} \leq M_j \|f\|_{L^{\frac{1}{\alpha_j}}(Y)}, \quad f \in \mathcal{D}, \quad j = 1, 2.$$

Then there exist a $0 < \theta < 1$, such that $\alpha = (1-\theta)\alpha_1 + \theta\alpha_2$, $\beta = (1-\theta)\beta_1 + \theta\beta_2$, and

$$\|Tf\|_{L^{\frac{1}{\beta}}(Y)} \leq M_1^{1-\theta} M_2^{\theta} \|f\|_{L^{\frac{1}{\alpha}}(Y)}, \quad f \in \mathcal{D}.$$

The group of all unitary operators on $L^2(G^2, \mathbb{H})$ with respect to the usual composition of mappings denoted by $U(L^2(G^2, \mathbb{H}))$, a group homomorphism $\pi_q : \hat{G}^2 \to U(L^2(G^2, \mathbb{H}))$ is said to be a quaternion unitary representation of the group $\hat{G}^2$ on the Hilbert space $L^2(G^2, \mathbb{H})$, which is denoted by $\{\pi_q, L^2(G^2, \mathbb{H})\}$ and defined as follows

$$(\pi_q(\xi)u)(x) = \xi_j(x_2)u(x)C_r(\xi_i(x_1)) = M_\xi^{C_r} u(x), \quad \xi \in \hat{G}^2, x \in G^2.$$

The nonzero element $\varphi \in L^2(G^2, \mathbb{H})$ is called an admissible wavelet for the quaternion unitary representation $\{\pi_q, L^2(G^2, \mathbb{H})\}$ if

$$\int_{\hat{G}^2} |\langle \varphi, \pi_q(\omega)\varphi\rangle|^2 d\omega < \infty.$$

In this case, the value of the above integral is called the wavelet constant associated with the admissible φ and denoted by c_φ , and $\{\pi_q, L^2(G^2, \mathbb{H})\}$ is called square integrable representation [11,14,17].

2.1 Wavelet multiplier operator according to the usual inner product

In this section we introduce the wavelet multiplier operator $\varphi T_\sigma \overline{\varphi}$ where $T_\sigma \in B(L^2(G^2, \mathbb{H}))$ which is defined by $T_\sigma = \mathcal{F}^{-1}\sigma\mathcal{F}$, $\sigma \in M_{L^p(\hat{G}^2,\mathbb{H})}$, $1 \leq p \leq \infty$ and $\varphi \in M_{L^2(G^2,\mathbb{H})\bigcap L^\infty(G^2,\mathbb{H})}$, we will also establish some of its properties. The following fact will be used frequently.

$$(u, \pi_q(\xi)\varphi) = (u.\overline{\varphi})^\wedge(\xi), \tag{2.1}$$

for $u, \varphi \in L^2(G^2, \mathbb{H})$ and $\xi \in \hat{G}^2$. Clearly from (2.1), we have $(\pi_q(\xi)\varphi, \nu) = \overline{(\nu.\overline{\varphi})^\wedge(\xi)}$.

The following proposition, characterizes the admissible vectors for the quaternion unitary representation $\{\pi_q, L^2(G^2, \mathbb{H})\}$.

Proposition 2.1 *The admissible wavelet for the unitary representation* $\{\pi_q, L^2(G^2, \mathbb{H})\}$ *defined on* $\hat{G}^2$ *consists of all functions* $\varphi \in L^2(G^2, \mathbb{H}) \cap L^4(G^2, \mathbb{H})$ *for which* $\|\varphi\|_2 = 1$.

Proof Using Plancherel Theorem and (2.1), we have

$$\begin{aligned} c_\varphi &= \int_{\hat{G}^2} |(\varphi, \pi_q(\xi)\varphi)|^2 d\xi \\ &= \int_{\hat{G}^2} |(\varphi\bar{\varphi})^\wedge(\xi)|^2 d\xi \\ &= \|(\varphi\bar{\varphi})^\wedge\|_2^2 \\ &= \|\varphi\bar{\varphi}\|_2^2 \\ &= \|\varphi\|_4^4. \end{aligned}$$

□

Now by using (2.1), we can prove the following proposition.

Proposition 2.2 *Let* $\varphi \in M_{L^2(G^2,\mathbb{H})\cap L^\infty(G^2,\mathbb{H})}$, *then for any* $u, v \in M_{L^2(G^2,\mathbb{H})}$,

$$\int_{\hat{G}^2} (u, \pi_q(\xi)\varphi)(\pi_q(\xi)\varphi, \nu) d\xi = (u\overline{\varphi}, \nu\overline{\varphi}).$$

Proof By (2.1) and Plancherel Theorem, we get

$$\int_{\hat{G}^2} (u, \pi_q(\xi)\varphi)(\pi_q(\xi)\varphi, \nu) d\xi$$

$$\begin{aligned}&= \int_{\hat{G}^2} (u\overline{\varphi})^\wedge(\xi).\overline{(v\overline{\varphi})^\wedge(\xi)} d\xi \\ &= ((u\overline{\varphi})^\wedge, (v\overline{\varphi})^\wedge) \\ &= (u\overline{\varphi}, v\overline{\varphi}).\end{aligned}$$

□

Now, for $\sigma \in L^\infty(\hat{G}^2, \mathbb{H})$ and $\varphi \in L^2(G^2, \mathbb{H}) \cap L^\infty(G^2, \mathbb{H})$, we define $P_{\sigma,\varphi} : L^2(G^2, \mathbb{H}) \to L^2(G^2, \mathbb{H})$ for any $u, v \in L^2(G^2, \mathbb{H})$ by

$$(P_{\sigma,\varphi} u, v) = \int_{\hat{G}^2} \sigma(\xi)(u, \pi_q(\xi)\varphi)(\pi_q(\xi)\varphi, v) d\xi. \tag{2.2}$$

At this point, we aim to show that the linear operators $P_{\sigma,\varphi} : L^2(G^2, \mathbb{H}) \to L^2(G^2, \mathbb{H})$ for $\sigma \in L^p(\hat{G}^2, \mathbb{H})$, $1 \le p \le \infty$ are bounded linear operators [3,18]. For the case $\sigma \in L^1(\hat{G}^2, \mathbb{H})$, this is shown in the following proposition.

Proposition 2.3 *Let $\sigma \in L^1(\hat{G}^2, \mathbb{H})$ and let $\varphi \in L^2(G^2, \mathbb{H}) \cap L^\infty(G^2, \mathbb{H})$ such that $\|\varphi\|_2 = 1$. Then $P_{\sigma,\varphi} : L^2(G^2, \mathbb{H}) \to L^2(G^2, \mathbb{H})$ is a bounded linear operator and $\|P_{\sigma,\varphi}\|_{B(L^2(G^2,\mathbb{H}))} \le \|\sigma\|_1$.*

Proof Let $\sigma \in L^1(\hat{G}^2, \mathbb{H})$, $\varphi \in L^2(G^2, \mathbb{H}) \cap L^\infty(G^2, \mathbb{H})$ with $\|\varphi\|_2 = 1$; then

$$\begin{aligned}|(P_{\sigma,\varphi} u, v)| &= \Big| \int_{\hat{G}^2} \sigma\big(\xi\big)(u, \pi_q(\xi)\varphi)(\pi_q(\xi)\varphi, v) d\xi \Big| \\ &\le \int_{\hat{G}^2} \big|\sigma\big(\xi\big)\big|\big|(u, \pi_q(\xi)\varphi)\big|\big|(\pi_q(\xi)\varphi, v)\big| d\xi \\ &\le \int_{\hat{G}^2} \big|\sigma\big(\xi\big)\big| \|u\|_2 \|\pi_q(\xi)\varphi\|_2^2 \|v\|_2 d\xi \\ &= \int_{\hat{G}} \big|\sigma\big(\xi\big)\big| \|u\|_2 \|\varphi\|_2^2 \|v\|_2 d\xi \\ &= \|u\|_2 \|v\|_2 \int_{\hat{G}^2} \big|\sigma(\xi)\big| d\xi \\ &= \|u\|_2 \|v\|_2 \|\sigma\|_1.\end{aligned}$$

So that $\|P_{\sigma,\varphi}\|_{B(L^2(G^2,\mathbb{H}))} \le \|\sigma\|_1$. □

Theorem 2.4 *Let $\sigma \in L^p(\hat{G}^2, \mathbb{H})$, $1 < p \le \infty$ and let $\varphi \in L^2(G^2, \mathbb{H}) \cap L^\infty(G^2, \mathbb{H})$ be such that $\|\varphi\|_2 = 1$. Then there exists a unique bounded linear operator $P_{\sigma,\varphi} : L^2(G^2, \mathbb{H}) \to L^2(G^2, \mathbb{H})$ such that $\|P_{\sigma,\varphi}\|_{B(L^2(G^2,\mathbb{H}))} \le \|\varphi\|_{L^\infty(G^2,\mathbb{H})}^{\frac{2}{q}} \|\sigma\|_{L^p(\hat{G}^2,\mathbb{H})}$ and for all $u, v \in L^2(G^2, \mathbb{H})$, $(P_{\sigma,\varphi} u, v)$ is given in (2.2) for all simple functions σ on $\hat{G}^2$ for which the Haar measure of the set $\{\xi \in \hat{G}^2 : \sigma(\xi) \ne 0\}$ is finite.*

Proof For $\sigma \in L^{\infty}(\hat{G}^2, \mathbb{H})$, we get

$$\begin{aligned}
\left|(P_{\sigma,\varphi}u, v)\right| &= \Big| \int_{\hat{G}^2} \sigma\big(\xi\big)(u, \pi_q(\xi)\varphi)(\pi_q(\xi)\varphi, v)d\xi \Big| \\
&\leq \int_{\hat{G}^2} \left|\sigma\big(\xi\big)\right|\left|(u, \pi_q(\xi)\varphi)\right|\left|(\pi_q(\xi)\varphi, v)\right|d\xi \\
&\leq \|\sigma\|_{L^{\infty}(\hat{G}^2,\mathbb{H})}\Big[\int_{\hat{G}^2} \left|(u, \pi_q(\xi)\varphi)\right|^2 d\xi\Big]^{\frac{1}{2}}\Big[\int_{\hat{G}^2} \left|(\pi_q(\xi)\varphi, v)\right|^2 d\xi\Big]^{\frac{1}{2}} \\
&= \|\sigma\|_{L^{\infty}(\hat{G}^2,\mathbb{H})}\Big[\int_{\hat{G}^2} \left|(u\overline{\varphi})^{\wedge}(\xi)\right|^2 d\xi\Big]^{\frac{1}{2}}\Big[\int_{\hat{G}^2} \left|(v\overline{\varphi})^{\wedge}(\xi)\right|^2 d\xi\Big]^{\frac{1}{2}} \\
&= \|\sigma\|_{L^{\infty}(\hat{G}^2,\mathbb{H})}\|(u\overline{\varphi})^{\wedge}\|_2\|(v\overline{\varphi})^{\wedge}\|_2 \\
&= \|\sigma\|_{L^{\infty}(\hat{G}^2,\mathbb{H})}\|u\overline{\varphi}\|_2\|v\overline{\varphi}\|_2 \\
&= \|\sigma\|_{L^{\infty}(\hat{G}^2,\mathbb{H})}\|\varphi\|^2_{L^{\infty}(G^2,\mathbb{H})}\|u\|_2\|v\|_2,
\end{aligned}$$

thus

$$\|P_{\sigma,\varphi}\|_{B(L^2(G)} \leq \|\sigma\|_{L^{\infty}(\hat{G}^2,\mathbb{H})}\|\varphi\|^2_{L^{\infty}(G^2,\mathbb{H})}.$$

For $1 < p < \infty$, the Riesz–Thorin Theorem completes the proof. □

Now Proposition 2.3 and Theorem 2.4 allow us to define the wavelet multiplier operator $\varphi T_\sigma \bar{\varphi} : M_{L^2(G^2,\mathbb{H})} \to M_{L^2(G^2,\mathbb{H})}$ for all $\sigma \in M_{L^p(\hat{G}^2,\mathbb{H})}$, $1 \leq p \leq \infty$ and all $\varphi \in M_{L^2(G^2,\mathbb{R})\cap L^{\infty}(G^2,\mathbb{R})}$ with $\|\varphi\|_2 = 1$ which is the same as the bounded linear operator $P_{\sigma,\varphi} : M_{L^2(G^2,\mathbb{H})} \to M_{L^2(G^2,\mathbb{H})}$. In other words, for any $\sigma \in M_{L^p(\hat{G}^2,\mathbb{H})}$, we have $(P_{\sigma,\varphi}u, v) = (\varphi T_\sigma \bar{\varphi}u, v)$, for all $u, v \in M^{\varphi}_{L^2(G^2,\mathbb{H})}$. Indeed

$$\begin{aligned}
(P_{\sigma,\varphi}u, v) &= \int_{\hat{G}^2} \sigma(\xi)(u, \pi_q(\xi)\varphi)(\pi_q(\xi)\varphi, v)d\xi \\
&= \int_{\hat{G}^2} \sigma(\xi)(u\overline{\varphi})^{\wedge}(\xi)\overline{(v\overline{\varphi})^{\wedge}(\xi)}d\xi \\
&= \int_{\hat{G}^2} \sigma(u\overline{\varphi})^{\wedge}(\xi).\overline{(v\overline{\varphi})^{\wedge}(\xi)}d\xi \\
&= (\sigma(u\overline{\varphi})^{\wedge}, (v\overline{\varphi})^{\wedge}) \\
&= ((\sigma(u\overline{\varphi})^{\wedge})^{\vee}, v\overline{\varphi}) \\
&= (\varphi T_\sigma \bar{\varphi}\, u, v).
\end{aligned}$$

Remark 2.5 Let φ be an admissible wavelet for the square integralable representation $\{\pi_q, L^2(G^2, \mathbb{H})\}$, then the linear operator $L_{\sigma,\varphi} : L^2(G^2, \mathbb{H}) \to L^2(G^2, \mathbb{H})$ which is defined as $(L_{\sigma,\varphi}u, v) = \dfrac{1}{c_\varphi} \int_{\hat{G}^2} \sigma(\xi)(u, \pi_q(\xi)\varphi)(\pi_q(\xi)\varphi, v)d\xi$ is called the localization operator associated with the symbol σ and admissible wavelet φ , hence from Proposition 2.1, we have $c_\varphi = \|\varphi\|_4^4$ and from (2.2) we get that $P_{\sigma,\varphi} = c_\varphi L_{\sigma,\varphi}$ also $L_{\sigma,\varphi} \in S_1$ with $\|L_{\sigma,\varphi}\|_{S_1} \leq \frac{1}{c_\varphi}\|\sigma\|_{L^1(\hat{G}^2)}$ for more details see [14,17].

2.2 Wavelet multiplier operator according to scalar inner product

In this section we introduce the wavelet multiplier operator $\varphi T_\sigma \overline{\varphi}$ with respect to scalar inner product where $T_\sigma \in B(L^2(G^2, \mathbb{H}))$ is defined by $T_\sigma = \mathcal{F}^{-1}\sigma Sc\mathcal{F}$ and $\varphi \in L^p(G^2, \mathbb{H})$, $1 \leq p \leq \infty$ and we also establish some of its properties. From (2.1) we have the following formula which will be used frequently.

$$\langle u, \pi_q(\xi)\varphi\rangle = Sc(u\overline{\varphi})^\wedge(\xi). \tag{2.3}$$

For any $u, \varphi \in L^2(G^2, \mathbb{H})$, note that from the definition of scalar inner product, we have $\langle u, \pi_q(\xi)\varphi\rangle = \langle \pi_q(\xi)\varphi, u\rangle$.

The following proposition characterizes the admissible vectors for the quaternion unitary representation $\{\pi_q, L^2(G^2, \mathbb{H})\}$.

Proposition 2.6 *The admissible wavelet for the unitary representation $\{\pi_q, L^2(G^2, \mathbb{H})\}$ defined on $\hat{G}^2$ consists of all functions $\varphi \in L^2(G^2, \mathbb{H}) \cap L^4(G^2, \mathbb{H})$ for which $\|\varphi\|_2 = 1$.*

Proof Using Plancherel Theorem and (2.3), we have

$$\begin{aligned} c_\varphi &= \int_{\hat{G}^2} |\langle \varphi, \pi_q(\xi)\varphi\rangle|^2 d\xi \\ &= \int_{\hat{G}^2} |Sc(\varphi\bar{\varphi})^\wedge(\xi)|^2 d\xi \\ &\leq \int_{\hat{G}^2} |(\varphi\bar{\varphi})^\wedge(\xi)|^2 d\xi \\ &= \|(\varphi\bar{\varphi})^\wedge\|_2^2 \\ &= \|\varphi\bar{\varphi}\|_2^2 \\ &= \||\varphi|^2\|_2^2 \\ &= \|\varphi\|_4^4. \end{aligned}$$

□

Now by using (2.3), we can prove the following proposition.

Proposition 2.7 *Let $\varphi \in L^2(G^2, \mathbb{H}) \cap L^\infty(G^2, \mathbb{H})$, then for any $u, v \in L^2(G^2, \mathbb{H})$,*

$$\int_{\hat{G}^2} \langle u, \pi_q(\xi)\varphi\rangle\langle \pi_q(\xi)\varphi, v\rangle d\xi = \langle Sc(u\overline{\varphi})^\wedge, (v\overline{\varphi})^\wedge\rangle.$$

Proof By (2.3), we get

$$\begin{aligned} &\int_{\hat{G}^2} \langle u, \pi_q(\xi)\varphi\rangle\langle \pi_q(\xi)\varphi, v\rangle \, d\xi \\ &\qquad = \int_{\hat{G}^2} Sc(u\bar{\varphi})^\wedge(\xi).Sc(v\bar{\varphi})^\wedge(\xi) d\xi \end{aligned}$$

$$= \int_{\hat{G}^2} Sc[Sc(u\bar{\varphi})^\wedge(\xi).(v\bar{\varphi})^\wedge(\xi)]d\xi$$
$$= Sc \int_{\hat{G}^2} Sc(u\bar{\varphi})^\wedge(\xi).(v\bar{\varphi})^\wedge(\xi)d\xi$$
$$= Sc \int_{\hat{G}^2} Sc(u\bar{\varphi})^\wedge(\xi).\overline{(v\bar{\varphi})^\wedge(\xi)}d\xi$$
$$= \langle Sc(u\bar{\varphi})^\wedge, (v\bar{\varphi})^\wedge \rangle.$$

□

Now for $\sigma \in L^p(\hat{G}^2, \mathbb{R})$, $1 \leq p \leq \infty$ and $\varphi \in L^2(G^2, \mathbb{H}) \cap L^\infty(G^2, \mathbb{H})$, we define $P_{\sigma,\varphi} : L^2(G^2, \mathbb{H}) \to L^2(G^2, \mathbb{H})$ for any $u, v \in L^2(G^2, \mathbb{H})$ by

$$\langle P_{\sigma,\varphi}u, v\rangle = \int_{\hat{G}^2} \sigma(\xi)\langle u, \pi_q(\xi)\varphi\rangle\langle \pi_q(\xi)\varphi, v\rangle d\xi. \tag{2.4}$$

Now, we aim to show that the linear operators $P_{\sigma,\varphi} : L^2(G^2, \mathbb{H}) \to L^2(G^2, \mathbb{H})$ for $\sigma \in L^p(\hat{G}^2, \mathbb{R})$, $1 \leq p \leq \infty$ are bounded linear operators. For the case $\sigma \in L^1(\hat{G}^2, \mathbb{R})$, this is shown in the following proposition.

Proposition 2.8 *Let $\sigma \in L^1(\hat{G}^2, \mathbb{R})$ and let $\varphi \in L^2(G^2, \mathbb{H}) \cap L^\infty(G^2, \mathbb{H})$ be such that $\|\varphi\|_2 = 1$. Then $P_{\sigma,\varphi} : L^2(G^2, \mathbb{H}) \to L^2(G^2, \mathbb{H})$ is a bounded linear operator and $\|P_{\sigma,\varphi}\|_{B(L^2(G^2,\mathbb{H}))} \leq \|\sigma\|_1$.*

Proof Let $\sigma \in L^1(\hat{G}^2, \mathbb{R})$, $\varphi \in L^2(G^2, \mathbb{H}) \cap L^\infty(G^2, \mathbb{H})$ with $\|\varphi\|_2 = 1$; Then

$$\begin{aligned} |\langle P_{\sigma,\varphi}u, v\rangle| &= \Big| \int_{\hat{G}^2} \sigma\big(\xi\big)\langle u, \pi_q(\xi)\varphi\rangle\langle \pi_q(\xi)\varphi, v\rangle d\xi \Big| \\ &\leq \int_{\hat{G}^2} \big|\sigma\big(\xi\big)\big|\big|\langle u, \pi_q(\xi)\varphi\rangle\big|\big|\langle \pi_q(\xi)\varphi, v\rangle\big|d\xi \\ &\leq \int_{\hat{G}^2} \big|\sigma\big(\xi\big)\big|\|u\|_2\|\pi_q(\xi)\varphi\|_2^2\|v\|_2 d\xi \\ &= \int_{\hat{G}} \big|\sigma\big(\xi\big)\big|\|u\|_2\|\varphi\|_2^2\|v\|_2 d\xi \\ &= \|u\|_2\|v\|_2 \int_{\hat{G}^2} \big|\sigma(\xi)\big|d\xi \\ &= \|u\|_2\|v\|_2\|\sigma\|_1. \end{aligned}$$

So that $\|P_{\sigma,\varphi}\|_{B(L^2(G^2,\mathbb{H}))} \leq \|\sigma\|_1$. □

Theorem 2.9 *Let $\sigma \in L^p(\hat{G}^2, \mathbb{R})$, $1 < p \leq \infty$ and let $\varphi \in L^2(G^2, \mathbb{H}) \cap L^\infty(G^2, \mathbb{H})$ be such that $\|\varphi\|_2 = 1$. Then there exists a unique bounded linear operator $P_{\sigma,\varphi}$: $L^2(G^2, \mathbb{H}) \to L^2(G^2, \mathbb{H})$ such that $\|P_{\sigma,\varphi}\|_{B(L^2(G^2,\mathbb{H}))} \leq \|\varphi\|^{\frac{2}{q}}_{L^\infty(G^2,\mathbb{H})}\|\sigma\|_{L^p(\hat{G}^2,\mathbb{H})}$ and for all $u, v \in L^2(G^2, \mathbb{H})$, $\langle P_{\sigma,\varphi}u, v\rangle$ is given in (2.4) for all simple functions σ on $\hat{G}^2$ for which the Haar measure of the set $\{\xi \in \hat{G}^2 : \sigma(\xi) \neq 0\}$ is finite.*

Proof For $\sigma \in L^{\infty}(\hat{G}^2, \mathbb{R})$, we get

$$\begin{aligned}
\big|\langle P_{\sigma,\varphi}u, v\rangle\big| &= \Big| \int_{\hat{G}^2} \sigma\big(\xi\big)\langle u, \pi_q(\xi)\varphi\rangle\langle \pi_q(\xi)\varphi, v\rangle d\xi \Big| \\
&\le \int_{\hat{G}^2} \big|\sigma\big(\xi\big)\big|\big|\langle u, \pi_q(\xi)\varphi\rangle\big|\big|\langle \pi_q(\xi)\varphi, v\rangle\big| d\xi \\
&\le \|\sigma\|_\infty \Big[\int_{\hat{G}^2} \big|\langle u, \pi_q(\xi)\varphi\rangle\big|^2 d\xi\Big]^{\frac{1}{2}} \Big[\int_{\hat{G}^2} \big|\langle \pi_q(\xi)\varphi, v\rangle\big|^2 d\xi\Big]^{\frac{1}{2}} \\
&= \|\sigma\|_\infty \Big[\int_{\hat{G}^2} \big|Sc(u\bar{\varphi})^\wedge(\xi)\big|^2 d\xi\Big]^{\frac{1}{2}} \Big[\int_{\hat{G}^2} \big|Sc(v\bar{\varphi})^\wedge(\xi)\big|^2 d\xi\Big]^{\frac{1}{2}} \\
&= \|\sigma\|_\infty \|Sc(u\bar{\varphi})^\wedge\|_2 \|Sc(v\bar{\varphi})^\wedge\|_2 \\
&\le \|\sigma\|_\infty \|(u\bar{\varphi})^\wedge\|_2 \|(v\bar{\varphi})^\wedge\|_2 \\
&= \|\sigma\|_\infty \|u\bar{\varphi}\|_2 \|v\bar{\varphi}\|_2 \\
&= \|\sigma\|_\infty \|\varphi\|^2_{L^\infty(G^2,\mathbb{H})} \|u\|_2 \|v\|_2,
\end{aligned}$$

thus

$$\|P_{\sigma,\varphi}\|_{B(L^2(G)} \le \|\sigma\|_{L^\infty(\hat{G}^2,\mathbb{H})} \|\varphi\|^2_{L^\infty(G^2,\mathbb{H})}.$$

For $1 < p < \infty$, the Riesz–Thorin Theorem completes the proof. □

Now Proposition 2.8 and Theorem 2.9 allow us to define the wavelet multiplier operator $\varphi T_\sigma \bar{\varphi} : L^2(G^2, \mathbb{H}) \to L^2(G^2, \mathbb{H})$ for all $\sigma \in L^p(\hat{G}^2, \mathbb{R})$, $1 \le p \le \infty$ and all $\varphi \in L^2(G^2, \mathbb{R}) \cap L^\infty(G^2, \mathbb{R})$ with $\|\varphi\|_2 = 1$ which is the same as the bounded linear operator $P_{\sigma,\varphi} : L^2(G^2, \mathbb{H}) \to L^2(G^2, \mathbb{H})$. In other words, for any $\sigma \in L^p(\hat{G}^2, \mathbb{R})$, $1 \le p \le \infty$ we have $\langle P_{\sigma,\varphi}u, v\rangle = \langle \varphi T_\sigma \bar{\varphi} u, v\rangle$, for all $u, v \in L^2(G^2, \mathbb{H})$. Indeed

$$\begin{aligned}
\langle P_{\sigma,\varphi}u, v\rangle &= \int_{\hat{G}^2} \sigma(\xi)\langle u, \pi_q(\xi)\varphi\rangle\langle \pi_q(\xi)\varphi, v\rangle d\xi \\
&= \int_{\hat{G}^2} \sigma(\xi) Sc(u\bar{\varphi})^\wedge(\xi) Sc(v\bar{\varphi})^\wedge(\xi) d\xi \\
&= \int_{\hat{G}^2} \sigma(\xi) Sc[Sc(u\bar{\varphi})^\wedge(\xi).(v\bar{\varphi})^\wedge(\xi)] d\xi \\
&= Sc \int_{\hat{G}^2} \sigma(\xi) Sc(u\bar{\varphi})^\wedge(\xi).(v\bar{\varphi})^\wedge(\xi) d\xi \\
&= Sc \int_{\hat{G}^2} (\sigma Sc(u\bar{\varphi})^\wedge)(\xi).\overline{(v\bar{\varphi})^\wedge(\xi)} d\xi \\
&= \langle \sigma Sc(u\bar{\varphi})^\wedge, (v\bar{\varphi})^\wedge\rangle \\
&= \langle (\sigma Sc(u\bar{\varphi})^\wedge)^\vee, v\bar{\varphi}\rangle \\
&= \langle \varphi T_\sigma \bar{\varphi} u, v\rangle.
\end{aligned}$$

Remark 2.10 Let φ be an admissible wavelet for the square integralable representation $\{\pi_q, L^2(G^2, \mathbb{H})\}$, then the linear operator $L_{\sigma,\varphi} : L^2(G^2, \mathbb{H}) \to L^2(G^2, \mathbb{H})$ which is defined as $\langle L_{\sigma,\varphi}u, v\rangle = \dfrac{1}{c_\varphi} \int_{\hat{G}^2} \sigma(\xi)\langle u, \pi(\xi)\varphi\rangle\langle \pi_q(\xi)\varphi, v\rangle d\xi$ is called the localization operator associated with the symbol σ and admissible wavelet φ , hence from Proposition 2.6, we have $c_\varphi \leq \|\varphi\|_4^4$ and from (2.4) we get $P_{\sigma,\varphi} = c_\varphi L_{\sigma,\varphi}$ also $L_{\sigma,\varphi} \in S_1$ with $\|L_{\sigma,\varphi}\|_{S_1} \leq \frac{1}{c_\varphi}\|\sigma\|_{L^1(\hat{G}^2)}$.

3 The Schatten–von Neumann property

We recall that an operator T on a Hilbert space $\mathcal{H}$ is called a compact operator [3,18] (or completely continuous operator) if, for every bounded sequence $\{x_n\}$ in $\mathcal{H}$, the sequence $\{Tx_n\}$ contains a convergent subsequence. Now if T is a compact operator on a separable Hilbert space $\mathcal{H}$, then there exist orthonormal sets $\{e_n\}$ and $\{\sigma_n\}$ in $\mathcal{H}$ such that

$$T(x) = \sum_n \lambda_n \langle x, e_n\rangle \sigma_n, \quad x \in \mathcal{H},$$

where λ_n is the n-th singular value of T [3,18]. Given $0 < p < \infty$, we define the Schatten p-class of $\mathcal{H}$, denoted by $\mathcal{S}_p(\mathcal{H})$ or simply $\mathcal{S}_p$, to be the space of all compact operators T on $\mathcal{H}$ such that its singular value sequence $\{\lambda_n\}$ belongs to ℓ_p (the p-summable sequence space) [18]. We will be mainly concerned with the range $1 \leq p < \infty$. In this case, $\mathcal{S}_p$ is a Banach space with the norm $\|T\|_p$ defined by

$$\|T\|_p = \Big[\sum_n |\lambda_n|^p\Big]^{\frac{1}{p}},$$

$\mathcal{S}_1$ is also called the trace class, and $\mathcal{S}_2$ is usually called the Hilbert- Schmidt class. The following theorem contains sufficient conditions for the wavelet multipliers operator in trace class.

Theorem 3.1 *Let $\sigma \in M_{L^1(\hat{G}^2,\mathbb{H})}$ (or $\in L^1(\hat{G}^2, \mathbb{R})$ in the case of scalar inner product) and $\varphi \in M_{L^2(G^2,\mathbb{R})\cap L^4(G^2,\mathbb{R})\cap L^\infty(G^2,\mathbb{R})}$ (or $\varphi \in L^2(G^2, \mathbb{R}) \cap L^4(G^2, \mathbb{R}) \cap L^\infty(G^2, \mathbb{R})$ in the case of scalar inner product) such that $\|\varphi\|_2 = 1$. Then the wavelet multiplier operator $\varphi T_\sigma \bar{\varphi}$ is in $\mathcal{S}_1$ and $\|\varphi T_\sigma \bar{\varphi}\|_{\mathcal{S}_1} \leq \|\sigma\|_1$.*

Proof By Remark 2.5 (Remark 2.10) the proof is clear. □

Now we are going to show that the wavelets multipliers operators $\varphi T_\sigma \bar{\varphi}$ is in $\mathcal{S}_p$ for $1 \leq p \leq \infty$, where $\sigma \in M_{L^p(\hat{G}^2,\mathbb{H})}(\in L^p(\hat{G}^2, \mathbb{R}))$. To do this, we need to recall some notations and terminologies.

Let B_0 and B_1 be two complex Banach spaces, we called B_0 and B_1 compatible if we have $B_k \subseteq V$, $k = 0, 1$ for some complex vector space V. Suppose that $\mathcal{S} = \{z \in \mathbb{C} : 0 \leq Re(z) \leq 1\}$ and let B be any complex Banach space, a function $f : \mathcal{S} \to B$

is called analytic on $\mathcal{S}$ if for every g (bounded linear functional on B) we have the composition $g \circ f : \mathcal{S} \to \mathbb{C}$ is analytic on $\mathcal{S}$. Now, let $\mathcal{F}(B_0, B_1)$ (B_0 and B_1 are compatible Banach spaces), be the set of all bounded and continuous functions f from $\overline{\mathcal{S}}$ into $B_0 + B_1$ such that f is analytic on $\mathcal{S}$ and the mappings

$$\mathbb{R} \ni y \to f(k + iy) \in B_k,\ k = 0, 1,$$

are continuous from $\mathbb{R}$ into B_k, $k = 0, 1$. Now one can show that $\mathcal{F}(B_0, B_1)$ is a complex Banach space with the norm $\| \ \|_{\mathcal{F}}$ defined as

$$\|f\|_{\mathcal{F}} = \max_{k=0,1} \sup_{y\in\mathbb{R}} \|f(k + iy)\|_{B_{k'}}, \quad f \in \mathcal{F}(B_0, B_1).$$

For any θ in the interval [0, 1], B_θ is the subspace of $B_0 + B_1$ consisting of all elements b in $B_0 + B_1$ such that $b = f(\theta)$ for some f in $\mathcal{F}(B_0, B_1)$, then B_θ is a complex Banach space with respect to the norm $\| \ \|_\theta$ defined as

$$\|b\|_\theta = \inf_{b=f(\theta)} \|f\|_{\mathcal{F}}, \quad b \in B_\theta,$$

and the interpolation space between the spaces B_0 and B_1 is B_θ, which denoted by $[B_0, B_1]$.

Suppose that we have two pairs of compatible Banach spaces, like B_0, B_1 and $\tilde{B_0}, \tilde{B_1}$, and let T be any bounded linear operator from $B_0 + B_1$ into $\tilde{B_0} + \tilde{B_1}$, so as, T is a bounded linear operator from B_k into $\tilde{B_k}$ with norm less than or equal to M_k, $k = 0, 1$. Then for any real number θ in the interval (0, 1), T is a bounded linear operator from $[B_0, B_1]_\theta$ into $[\tilde{B_0}, \tilde{B_1}]_\theta$ with a norm not bigger than $M_0^{1-\theta} M_1^\theta$.

In particular for $1 \le p \le \infty$,

$$[L^1(X, \mu), L^\infty(X, \mu)]_{\frac{1}{q}} = L^p(X, \mu),$$

and

$$[\mathcal{S}_1, \mathcal{S}_\infty]_{\frac{1}{q}} = \mathcal{S}_p,$$

where (X, μ) is a measure space and q is the conjugate index of p. See [17,18] for more details.

Theorem 3.2 *Let $\sigma \in M_{L^p(\hat{G}^2,\mathbb{H})}$ (or $\in L^p(\hat{G}^2, \mathbb{R})$ in the case of scalar inner product), $1 \le p \le \infty$ and $\varphi \in M_{L^2(G^2,\mathbb{R})\cap L^4(G^2,\mathbb{R})\cap L^\infty(G^2,\mathbb{R})}$ (or $\in L^2(G^2, \mathbb{R}) \cap L^4(G^2, \mathbb{R}) \cap L^\infty(G^2, \mathbb{R})$ in the case of scalar inner product) with $\|\varphi\|_2 = 1$. Then the wavelet multiplier operator $\varphi T_\sigma \bar{\varphi}$ is in $\mathcal{S}_p$ and $\|\varphi T_\sigma \bar{\varphi}\|_{\mathcal{S}_p} \le \|\varphi\|^{\frac{2}{q}}_{L^\infty(G^2,\mathbb{H})} \|\sigma\|_{L^p(\hat{G}^2,\mathbb{H})}$.*

Proof For $p = 1$ the proof follows from Theorem 3.1; and for $p = \infty$ the proof follows from Theorems 2.4 and 2.9, thus for $1 < p < \infty$ the interpolation Theorem as mentioned above complete the proof. □

In the following theorem, we investigate the trace of the wavelet multiplier operator.

Theorem 3.3 *Let* $\sigma \in M_{L^1(\hat{G}^2,\mathbb{H})}$ *(or* $\in L^1(\hat{G}^2,\mathbb{R})$ *in state scalar inner product) and* $\varphi \in M_{L^2(G^2,\mathbb{R})\cap L^4(G^2,\mathbb{R})\cap L^\infty(G^2,\mathbb{R})}$ *(or* $\in L^2(G^2,\mathbb{R}) \cap L^4(G^2,\mathbb{R}) \cap L^\infty(G^2,\mathbb{R})$ *in state scalar inner product) be such that* $\|\varphi\|_2 = 1$. *Then*

$$\mathrm{tr}(\varphi T_\sigma \bar{\varphi}) = \int_{\hat{G}^2} \sigma(\xi) d\xi.$$

Proof Let $\{\varphi_k\}_{k=1}^\infty$ be an orthonormal basis for $L^2(G^2,\mathbb{R})$. We get

$$\begin{aligned}
\mathrm{tr}(\varphi T_\sigma \bar{\varphi}) = \mathrm{tr}(P_{\sigma,\varphi}) &= \sum_{k=1}^\infty (P_{\sigma,\varphi}\varphi_k, \varphi_k) \\
&= \sum_{k=1}^\infty \int_{\hat{G}^2} \sigma(\xi) |(\varphi_k, \pi_q(\xi)\varphi)|^2 d\xi \\
&= \int_{\hat{G}^2} \sigma(\xi) \sum_{k=1}^\infty |(\varphi_k, \pi_q(\xi)\varphi)|^2 d\xi \\
&= \int_{\hat{G}^2} \|\pi_q(\xi)\varphi\|_2^2 \sigma(\xi) d\xi \\
&= \|\varphi\|_2^2 \int_{\hat{G}^2} \sigma(\xi) d\xi \\
&= \int_{\hat{G}^2} \sigma(\xi) d\xi.
\end{aligned}$$

For $\sigma \in L^1(\hat{G}^2,\mathbb{R})$ with scalar inner product by the same way. □

4 The Landau–Pollak–Slepian opearator

Here we will give the Landau–Pollak–Slepian (L.P.S) operator $Q_C P_\Omega Q_C : M_{L^2(G^2,\mathbb{H})} \to M_{L^2(G^2,\mathbb{H})}$ where C and Ω are compact neighborhoods of identity elements of G^2 and $\hat{G}^2$, respectively, and we will also investigate some properties of the L.P.S. operator and finally we consider the trace of this operator. At first, let us define the linear operators $P_\Omega, Q_C : M_{L^2(G^2,\mathbb{H})} \to M_{L^2(G^2,\mathbb{H})}$ by $(P_\Omega f)^\wedge(\xi) = \chi_\Omega(\xi)\hat{f}(\xi)$ and $(Q_C f)(x) = \chi_C(x) f(x)$, for all $f \in M_{L^2(G^2,\mathbb{H})}$, which are in fact orthogonal projections, as the following proposition shows.

Proposition 4.1 *With the notations as above,* $P_\Omega, Q_C : M_{L^2(G^2,\mathbb{H})} \to M_{L^2(G^2,\mathbb{H})}$ *are orthogonal projections.*

Proof Note that

$$(P_\Omega f, g) = ((P_\Omega f)^\wedge, \hat{g})$$

$$
\begin{aligned}
&= \int_{\hat{G}^2} \chi_\Omega(\xi)\hat{f}(\xi)\overline{\hat{g}(\xi)}d\xi \\
&= \int_{\hat{G}^2} \hat{f}(\xi)\overline{\chi_\Omega(\xi)\hat{g}(\xi)}d\xi \\
&= \int_{\hat{G}^2} \hat{f}(\xi)\overline{(P_\Omega g)^\wedge(\xi)}d\xi \\
&= (\hat{f}, (P_\Omega g)^\wedge) = (f, P_\Omega g).
\end{aligned}
$$

Therefore, $P_\Omega : M_{L^2(G^2,\mathbb{H})} \to M_{L^2(G^2,\mathbb{H})}$ is self-adjoint. Also,

$$
\begin{aligned}
(Q_C f, g) &= \int_{G^2} (Q_C f)(x)\overline{g(x)}dx \\
&= \int_{G^2} \chi_C(x) f(x)\overline{g(x)}dx \\
&= \int_{G^2} f(x)\overline{\chi_C(x) g(x)}dx \\
&= \int_{G^2} f(x)\overline{(Q_C g)(x)}dx \\
&= (f, Q_C g).
\end{aligned}
$$

Therefore, $Q_C : M_{L^2(G^2,\mathbb{H})} \to M_{L^2(G^2,\mathbb{H})}$ is self-adjoint. On the other hand, we have

$$
\begin{aligned}
(P_\Omega^2 f, g) &= (P_\Omega f, P_\Omega g) = ((P_\Omega f)^\wedge, (P_\Omega g)^\wedge) \\
&= \int_{\hat{G}^2} (P_\Omega f)^\wedge(\xi)\overline{(P_\Omega g)^\wedge(\xi)}d\xi \\
&= \int_{\hat{G}^2} \chi_\Omega(\xi)\hat{f}(\xi)\overline{\chi_\Omega(\xi)\hat{g}(\xi)}d\xi \\
&= \int_{\hat{G}^2} \chi_\Omega(\xi)\hat{f}(\xi)\overline{\hat{g}(\xi)}d\xi \\
&= \int_{\hat{G}^2} (P_\Omega f)^\wedge(\xi)\overline{\hat{g}(\xi)}d\xi \\
&= ((P_\Omega f)^\wedge, \hat{g}) \\
&= (P_\Omega f, g).
\end{aligned}
$$

Thus $P_\Omega^2 = P_\Omega$ and hence $P_\Omega : M_{L^2(G^2,\mathbb{H})} \to M_{L^2(G^2,\mathbb{H})}$ is an orthogonal projection. Also,

$$
\begin{aligned}
(Q_C^2 f, g) &= (Q_C f, Q_C g) \\
&= \int_{G^2} (Q_C f)(x)\overline{(Q_C g)(x)}dx \\
&= \int_{G^2} \chi_C(x) f(x)\overline{\chi_C(x) g(x)}dx
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{G^2} \chi_C(x) f(x)\overline{g(x)} dx \\
&= \int_{G^2} (Q_C f)(x)\overline{g(x)} dx \\
&= (Q_C f, g).
\end{aligned}
$$

Thus $Q_C^2 = Q_C$ and hence, $Q_C : M_{L^2(G^2,\mathbb{H})} \to M_{L^2(G^2,\mathbb{H})}$ is an orthogonal projection. □

Using the fact that P_Ω and Q_C are orthogonal projections, we get

$$
\begin{aligned}
&\sup\left\{ \frac{\|P_\Omega Q_C f\|_2^2}{\|f\|_2^2} : f \in M_{L^2(G^2,\mathbb{H})}, \|f\|_2 \neq 0 \right\} \\
&= \sup\left\{ \frac{(P_\Omega Q_C f, P_\Omega Q_C f)}{\|f\|_2^2} : f \in M_{L^2(G^2,\mathbb{H})}, \|f\|_2 \neq 0 \right\} \\
&= \sup\left\{ \frac{(P_\Omega^2 Q_C f, Q_C f)}{\|f\|_2^2} : f \in M_{L^2(G^2,\mathbb{H})}, \|f\|_2 \neq 0 \right\} \\
&= \sup\left\{ \frac{(P_\Omega Q_C f, Q_C f)}{\|f\|_2^2} : f \in M_{L^2(G^2,\mathbb{H})}, \|f\|_2 \neq 0 \right\} \\
&= \sup\left\{ \frac{(Q_C P_\Omega Q_C f, f)}{\|f\|_2^2} : f \in M_{L^2(G^2,\mathbb{H})}, \|f\|_2 \neq 0 \right\} \\
&= \sup\left\{ (Q_C P_\Omega Q_C f, f) : f \in M_{L^2(G^2,\mathbb{H})}, \|f\|_2 = 1 \right\}.
\end{aligned}
$$

Since $Q_C P_\Omega Q_C : M_{L^2(G^2,\mathbb{H})} \to M_{L^2(G^2,\mathbb{H})}$ is self-adjoint, it follows from the above that

$$
\sup\{\frac{\|P_\Omega Q_C f\|_2^2}{\|f\|_2^2} : f \in M_{L^2(G^2,\mathbb{H})}, \|f\|_2 \neq 0\} = \|Q_C P_\Omega Q_C\|_{B(L^2(G^2,\mathbb{H}))}.
$$

Theorem 4.2 *Let φ be the function on G^2 defined by $\varphi(x) = \frac{1}{|C|^{\frac{1}{2}}}\chi_C(x)$, where $|C|$ denotes the Haar measure of C, and let σ be the function on $\hat{G}^2$ defined by $\sigma(\xi) = \chi_\Omega(\xi)$. Then the operator $Q_C P_\Omega Q_C : M_{L^2(G^2,\mathbb{H})} \to M_{L^2(G^2,\mathbb{H})}$ is unitarily equivalent to scalar multiple of the wavelet multiplier $\varphi T_\sigma \varphi : M_{L^2(G^2,\mathbb{H})} \to M_{L^2(G^2,\mathbb{H})}$. In fact $Q_C P_\Omega Q_C = |C|(\varphi T_\sigma \varphi)$.*

Proof From the definition of φ, we get that $\varphi \in M_{L^2(G^2,\mathbb{R})\cap L^\infty(G^2,\mathbb{R})}$ with $\|\varphi\|_2^2 = \int_{G^2} |\varphi(x)|^2 dx = \frac{1}{|C|}\int_C dx = 1$, so for all $u, v \in M_{L^2(G^2,\mathbb{H})}$ we have,

$$
(\varphi T_\sigma \varphi u, v) = \int_{\hat{G}^2} \sigma(\xi)(u, \pi_q(\xi)\varphi)(\pi_q(\xi)\varphi, v) d\xi,
$$

and

$$\begin{aligned}(u, \pi(\xi)\varphi) &= \int_{G^2} u(x)\overline{\pi_q(\xi)\varphi(x)}dx \\ &= \int_{G^2} u(x)\overline{\xi_j(x_2)\varphi(x)C_r(\xi_i(x_1))}dx \\ &= \int_{G^2} u(x)\overline{C_r(\xi_i(x_1))}\varphi(x)\overline{\xi_j(x_2)}dx \\ &= \int_{G^2} u(x)C_l\overline{\xi_i(x_1)}\frac{1}{|C|^{\frac{1}{2}}}\chi_C(x)\overline{\xi_j(x_2)}dx \\ &= \frac{1}{|C|^{\frac{1}{2}}}\int_{G^2} \overline{\xi_i(x_1)}\chi_C(x)u(x)\overline{\xi_j(x_2)}dx \\ &= \frac{1}{|C|^{\frac{1}{2}}}\int_{G^2} \overline{\xi_i(x_1)}(Q_C u)(x)\overline{\xi_j(x_2)}dx \\ &= \frac{1}{|C|^{\frac{1}{2}}}(Q_C u)^\wedge(\xi).\end{aligned}$$

So

$$(u, \pi(\xi)\varphi) = \frac{1}{|C|^{\frac{1}{2}}}(Q_C u)^\wedge(\xi) \text{ and } (\pi(\xi)\varphi, v) = \overline{\frac{1}{|C|^{\frac{1}{2}}}(Q_C v)^\wedge(\xi)}.$$

Now

$$\begin{aligned}(\varphi T_\sigma \varphi u, v) &= \int_{\hat{G}^2} \sigma(\xi)(u, \pi(\xi)\varphi)(\pi(\xi)\varphi, v)d\xi \\ &= \frac{1}{|C|}\int_{\hat{G}^2} \sigma(\xi)(Q_C u)^\wedge(\xi)\overline{(Q_C v)^\wedge(\xi)}d\xi \\ &= \frac{1}{|C|}\int_{\hat{G}^2} \chi_\Omega(\xi)(Q_C u)^\wedge(\xi)\overline{(Q_C v)^\wedge(\xi)}d\xi \\ &= \frac{1}{|C|}\int_{\hat{G}^2} (\chi_\Omega(Q_C u)^\wedge)(\xi)\overline{(Q_C v)^\wedge(\xi)}d\xi \\ &= \frac{1}{|C|}\int_{\hat{G}^2} (P_\Omega(Q_C u))^\wedge(\xi)\overline{(Q_C v)^\wedge(\xi)}d\xi \\ &= \frac{1}{|C|}((P_\Omega(Q_C u))^\wedge, (Q_C v)^\wedge) = \frac{1}{|C|}(P_\Omega Q_C u, Q_C v) \\ &= \frac{1}{|C|}(Q_C P_\Omega Q_C u, v) \text{ for all functions } u, v \in M_{L^2(G^2,\mathbb{H})}.\end{aligned}$$

So

$$Q_C P_\Omega Q_C = |C|(\varphi T_\sigma \varphi).$$

□

Theorem 4.3 *With the above notations* $\operatorname{tr}(Q_C P_\Omega Q_C) = |C||\Omega|$.

Proof Theorem 4.3 is an immediate consequence of Theorems 4.2 and 3.3. □

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