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HIGHER CODERIVATIONS ON COALGEBRAS AND CHARACTERIZATION

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ABSTRACT. In this paper we define higher coderivations on a coalgebra C and then we characterize them in terms of the coderivations on C. Indeed, we show that each higher coderivation is a combination of compositions of coderivations. Finally we prove a one to one correspondence between the set of all higher coderivations on C and all sequences of coderivations on C.

1. INTRODUCTION

A *coalgebra* (C, Δ, ε) over a field κ is a κ-vector space C together with the κ-linear maps $\Delta : C \to C \otimes C$ and $\varepsilon : C \to \kappa$, such that $(I_C \otimes \Delta)\Delta = (\Delta \otimes I_C)\Delta$, (coassociativity) and $(I_C \otimes \varepsilon)\Delta = (\varepsilon \otimes I_C)\Delta$, (counitary). The maps Δ and ε are called, respectively, *coproduct* and *counit* of the coalgebra C. Given an element c of the coalgebra (C, Δ, ε), we know that there exist elements $c_{1,i}$ and $c_{2,i}$ in C such that $\Delta(c) = \sum_i c_{1,i} \otimes c_{2,i}$. In *Sweedlers notation*, this is abbreviated to $\sum c_{(1)} \otimes c_{(2)}$. Here, the subscripts "(1)" and "(2)" indicate the order of the factors in the tensor product. For more about basic definitions in coalgebras notion, you can see [1] and [3].

A κ-linear map $f : C \to C$ on a κ-coalgebra (C, Δ, ε) is called a *coderivation* if $\Delta f = (I_C \otimes f + f \otimes I_C)\Delta$. One can see examples and a general definition of coalgebras and coderivations in the sense of comodules in [2, 4, 6]. In this paper we define higher coderivations on a coalgebra C and then characterize them in terms of the coderivations on C. Indeed, we show that each higher coderivation is a combination of compositions of coderivations. As a corollary we characterize all higher coderivations which are ordinary. We have some nearly same properties for higher derivations, you

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can see in [5] and [7]. Throughout the paper, all coalgebras are assumed over a field of characteristic zero.

2. The Results

Throughout the paper, C denotes a coalgebra over a field of characteristic zero and I is the identity mapping on C. A *coalgebra* (C, Δ, ε) over a field κ is a κ-vector space C together with the κ-linear maps $\Delta : C \to C \otimes C$ and $\varepsilon : C \to \kappa$, such that $(I_C \otimes \Delta)\Delta = (\Delta \otimes I_C)\Delta$, (coassociativity), and $(I_C \otimes \varepsilon)\Delta = (\varepsilon \otimes I_C)\Delta$, (counitary). The maps Δ and ε are called, respectively, *coproduct* and *counit* of the coalgebra C. A κ-linear map $f : C \to C$ on a κ-coalgebra (C, Δ, ε) is called a *coderivation* if $\Delta f = (I_C \otimes f + f \otimes I_C)\Delta$.

Now we define a new concept, named higher coderivation and then characterize this, but at first we prove some properties, following.

Proposition 2.1. *If f is a coderivation on coalgebra (C, Δ, ε), then we have*

$$\Delta f^n = \sum_{k=0}^{n} \binom{n}{k} (f^k \otimes f^{n-k})\Delta, \tag{2.1}$$

for each nonnegative integer n.

Proof. We use induction on n. For $n = 1$ and $a \in C$ we have

$$\Delta f(a) = \sum a_{(1)} \otimes f(a_{(2)}) + f(a_{(1)}) \otimes a_{(2)},$$

and its true, since f is a coderivation on C. Now suppose that the equality is true for n, then for $n + 1$, in the left side of equality, we have

$$\Delta f^{n+1}(a) = \Delta f^n(f(a)) = \sum_{k=0}^{n} \binom{n}{k} (f^k \otimes f^{n-k})\Delta(f(a)),$$

because of f being a coderivation, we have

$$\begin{aligned}\Delta f^{n+1}(a) &= \sum_{k=0}^{n} \binom{n}{k} (f^k \otimes f^{n-k})(I \otimes f + f \otimes I)\Delta(a) \\ &= \sum_{k=0}^{n} \sum \binom{n}{k} f^k(a_{(1)}) \otimes f^{n+1-k}(a_{(2)}) + f^{k+1}(a_{(1)}) \otimes f^{n-k}(a_{(2)}).\end{aligned}$$

On the other side we have

$$\begin{aligned}&\sum_{k=0}^{n+1} \binom{n+1}{k} (f^k \otimes f^{n+1-k})\Delta(a) \\ &= \sum_{k=0}^{n+1} \sum \binom{n+1}{k} f^k(a_{(1)}) \otimes f^{n+1-k}(a_{(2)}) \\ &= \left[\sum_{k=0}^{n} \sum \left(\binom{n}{k} + \binom{n}{k-1}\right) \left(f^k(a_{(1)}) \otimes f^{n+1-k}(a_{(2)})\right)\right] + f^{n+1}(a_{(1)}) \otimes a_{(2)}\end{aligned}$$

$$
\begin{aligned}
=&\Bigg[\sum_{k=0}^{n}\sum\binom{n}{k}f^k(a_{(1)})\otimes f^{n+1-k}(a_{(2)})\\
&+\sum_{k=-1}^{n-1}\sum\binom{n}{k}\left(f^{k+1}(a_{(1)})\otimes f^{n+1-(k+1)}(a_{(2)})\right)\Bigg]+f^{n+1}(a_{(1)})\otimes a_{(2)}\\
=&\sum_{k=0}^{n}\sum\binom{n}{k}f^k(a_{(1)})\otimes f^{n+1-k}(a_{(2)})+\sum_{k=-1}^{n}\sum\binom{n}{k}f^{k+1}(a_{(1)})\otimes f^{n-k}(a_{(2)}),
\end{aligned}
$$

and we have the result. □

We name the relation (2.1) *general coLiebnitz rule for coderivations.*
If we define a sequence $\{f_n\}$ of linear mappings on C by $f_0 = I$ and $f_n = \frac{\lambda^n}{n!}$, where I is the identity mapping on C, then general coLeibniz rule ensures us that f_n's satisfy the condition

$$\Delta f_n = \sum_{k=0}^{n}(f_k \otimes f_{n-k})\Delta, \tag{2.2}$$

for each nonnegative integer n. This motivates us to consider the sequences $\{f_n\}$ of linear mappings on a coalgebra C satisfying (2.2). We call such a sequence a higher coderivation.

Definition 2.1. Let C be a coalgebra. We define a sequence $\{f_n\}$ of linear mappings on C a *higher coderivation* if $\Delta f_n(a) = \sum_{k=0}^{n}(f_k \otimes f_{n-k})\Delta(a)$ for each $a \in C$ and each nonnegative integer n.

Though, if $\lambda : C \to C$ is a coderivation then $f_n = \frac{\lambda^n}{n!}$ is a higher coderivation. We name this kind of higher coderivation an *ordinary higher coderivation.*

Proposition 2.2. *Let $\{f_n\}$ be a higher coderivation on a coalgebra C with $f_0 = I$. Then there is a sequence $\{\lambda_n\}$ of coderivations on C such that*

$$(n+1)f_{n+1} = \sum_{k=0}^{n} f_{n-k}\lambda_{k+1},$$

for each nonnegative integer n.

Proof. We use induction on n. Because of $\{f_n\}$ being a higher coderivation, for $n = 0$ we have

$$
\begin{aligned}
\Delta f_1(a) &= [(f_0 \otimes f_1) + (f_1 \otimes f_0)]\Delta(a)\\
&= \sum f_0(a_{(1)}) \otimes f_1(a_{(2)}) + f_1(a_{(1)}) \otimes f_0(a_{(2)})\\
&= \sum a_{(1)} \otimes f_1(a_{(2)}) + f_1(a_{(1)}) \otimes a_{(2)}.
\end{aligned}
$$

Thus, if $\lambda_0 = I$ and $\lambda_1 = f_1$, then λ_1 is a coderivation on $\mathcal{A}$ and

$$\Delta(f_0\lambda_1)(a) = \Delta(\lambda_1(a)) = \sum \lambda_0(a_{(1)}) \otimes \lambda_1(a_{(2)}) + \lambda_1(a_{(1)}) \otimes \lambda_0(a_{(2)}).$$

Now suppose that λ_k it is defined and is a coderivation for $k \leq n$. Putting $\lambda_{n+1} = (n+1)f_{n+1} - \sum_{k=0}^{n-1} f_{n-k}\lambda_{k+1}$, we show that the well-defined mapping λ_{n+1} is a coderivation on C. For $a \in C$, since $\{f_n\}$ is a higher coderivation and $\lambda_1, \dots, \lambda_n$ are coderivations, we have

$$\begin{aligned}
\Delta\lambda_{n+1}(a) = (n+1)\Delta f_{n+1}(a) - &\sum_{k=0}^{n-1} \Delta(f_{n-k}\lambda_{k+1})(a) \\
=&(n+1)\Delta f_{n+1}(a) \\
&- \sum_{k=0}^{n-1}\sum_{l=0}^{n-k}\sum (f_l \otimes f_{n-k-l})\left(a_{(1)} \otimes \lambda_{k+1}(a_{(2)}) + \lambda_{k+1}(a_{(1)}) \otimes a_{(2)}\right) \\
=&(n+1)\sum_{k=0}^{n+1}(f_k \otimes f_{n+1-k})\Delta(a) \\
&- \sum_{k=0}^{n-1}\sum_{l=0}^{n-k}\sum (f_l \otimes f_{n-k-l})\left(a_{(1)} \otimes \lambda_{k+1}(a_{(2)}) + \lambda_{k+1}(a_{(1)}) \otimes a_{(2)}\right) \\
=&(n+1)\sum_{k=0}^{n+1}\sum f_k(a_{(1)}) \otimes f_{n+1-k}(a_{(2)}) \\
&- \sum_{k=0}^{n-1}\sum_{l=0}^{n-k}\sum (f_l \otimes f_{n-k-l})\left(a_{(1)} \otimes \lambda_{k+1}(a_{(2)}) + \lambda_{k+1}(a_{(1)}) \otimes a_{(2)}\right) \\
=&(n+1)\sum_{k=0}^{n+1}\sum f_k(a_{(1)}) \otimes f_{n+1-k}(a_{(2)}) \\
&- \sum_{k=0}^{n-1}\sum_{l=0}^{n-k}\sum f_l(a_{(1)}) \otimes f_{n-k-l}(\lambda_{k+1}(a_{(2)})) + f_l(\lambda_{k+1}(a_{(1)})) \otimes f_{n-k-l}(a_{(2)}).
\end{aligned}$$

Now, by properties of tensor product, we have

$$\begin{aligned}
\Delta\lambda_{n+1}(a) =&\sum_{k=0}^{n+1}\sum (k+n+1-k)\left(f_k(a_{(1)}) \otimes f_{n+1-k}(a_{(2)})\right) \\
&- \sum_{k=0}^{n-1}\sum_{l=0}^{n-k}\sum (f_l \otimes f_{n-k-l})\left(a_{(1)} \otimes \lambda_{k+1}(a_{(2)}) + \lambda_{k+1}(a_{(1)}) \otimes a_{(2)}\right) \\
=&\sum_{k=0}^{n+1}\sum k f_k(a_{(1)}) \otimes f_{n+1-k}(a_{(2)}) + f_k(a_{(1)}) \otimes (n+1-k) f_{n+1-k}(a_{(2)}) \\
&- \sum_{k=0}^{n-1}\sum_{l=0}^{n-k}\sum (f_l \otimes f_{n-k-l})\left(a_{(1)} \otimes \lambda_{k+1}(a_{(2)}) + \lambda_{k+1}(a_{(1)}) \otimes a_{(2)}\right).
\end{aligned}$$

Writing

$$K = \sum_{k=0}^{n+1}\sum k f_k(a_{(1)}) \otimes f_{n+1-k}(a_{(2)}) - \sum_{k=0}^{n-1}\sum_{\ell=0}^{n-k}\sum f_\ell\lambda_{k+1}(a_{(1)}) \otimes f_{n-k-\ell}(a_{(2)}),$$

$$L = \sum_{k=0}^{n+1} \sum f_k(a_{(1)}) \otimes (n+1-k) f_{n+1-k}(a_{(2)})$$
$$- \sum_{k=0}^{n-1} \sum_{\ell=0}^{n-k} \sum f_\ell(a_{(1)}) \otimes f_{n-k-\ell}\lambda_{k+1}(a_{(2)}),$$

we have $\Delta\lambda_{n+1}(a) = K + L$. Let us compute K and L. In the summation $\sum_{k=0}^{n-1} \sum_{\ell=0}^{n-k}$, we have $0 \leq k+\ell \leq n$ and $k \neq n$. Thus, if we put $r = k+\ell$ then we can write it as the form $\sum_{r=0}^{n} \sum_{k+\ell=r, k\neq n}$. Putting $\ell = r-k$ we indeed have

$$K = \sum_{k=0}^{n+1} \sum k f_k(a_{(1)}) \otimes f_{n+1-k}(a_{(2)})$$
$$- \sum_{r=0}^{n} \sum_{0\leq k\leq r, k\neq n} \sum f_{r-k}\lambda_{k+1}(a_{(1)}) \otimes f_{n-r}(a_{(2)})$$
$$= \sum_{k=0}^{n+1} \sum k f_k(a_{(1)}) \otimes f_{n+1-k}(a_{(2)})$$
$$- \sum \left(\sum_{r=0}^{n-1} \sum_{k=0}^{r} f_{r-k}\lambda_{k+1}(a_{(1)}) \otimes f_{n-r}(a_{(2)} \right) - \sum_{k=0}^{n-1} f_{n-k}\lambda_{k+1}(a_{(1)}) \otimes a_{(2)}.$$

Putting $r+1$ instead of k in the first summation we have

$$K + \sum_{k=0}^{n-1} \sum f_{n-k}\lambda_{k+1}(a_{(1)}) \otimes a_{(2)}$$
$$= \sum_{r=0}^{n} \sum (r+1) f_{r+1}(a_{(1)}) \otimes f_{n-r}(a_{(2)}) - \sum_{r=0}^{n-1} \sum_{k=0}^{r} \sum f_{r-k}\lambda_{k+1}(a_{(1)}) \otimes f_{n-r}(a_{(2)})$$
$$= \sum \left(\sum_{r=0}^{n-1} \left[(r+1) f_{r+1}(a_{(1)}) - \sum_{k=0}^{r} f_{r-k}\lambda_{k+1}(a_{(1)}) \right] \otimes f_{n-r}(a_{(2)} \right)$$
$$+ (n+1) f_{n+1}(a_{(1)}) \otimes a_{(2)}).$$

By our assumption

$$(r+1) f_{r+1}(a) = \sum_{k=0}^{r} (f_{r-k}\lambda_{k+1})(a),$$

for $r = 0, \ldots, n-1$. We can therefore deduce that

$$K = \sum \left[(n+1) f_{n+1}(a_{(1)}) - \sum_{k=0}^{n-1} f_{n-k}\lambda_{k+1}(a_{(1)}) \right] \otimes a_{(2)} = \sum \lambda_{n+1}(a_{(1)}) \otimes a_{(2)}.$$

By a similar argument we have

$$L = \sum a_{(1)} \otimes \left[(n+1) f_{n+1}(a_{(2)}) - \sum_{k=0}^{n-1} f_{n-k}\lambda_{k+1}(a_{(2)}) \right] = \sum a_{(1)} \otimes \lambda_{n+1}(a_{(2)}).$$

Thus,

$$\Delta\lambda_{n+1}(a) = K + L = (I \otimes \lambda_{n+1} + \lambda_{n+1} \otimes I)\Delta(a),$$

whence λ_{n+1} is a coderivation on C. □

To illustrate the recursive relation mentioned in Proposition 2.2, let us compute some terms of $\{d_n\}$.

Example 2.1. Using Proposition 2.2, the first five terms of $\{f_n\}$ are

$$\begin{aligned}
f_0 &= I,\\
f_1(a) &= f_0(\lambda_1(a)) = \lambda_1(a) \to f_1 = \lambda_1,\\
2f_2(a) &= f_1(\lambda_1(a)) + f_0(\lambda_2(a)) = {\lambda_1}^2(a) + \lambda_2(a) \to 2f_2 = \lambda_1^2 + \lambda_2,\\
f_2 &= \frac{1}{2}\lambda_1^2 + \frac{1}{2}\lambda_2,\\
3f_3 &= f_2\lambda_1 + f_1\lambda_2 + f_0\lambda_3 = \left(\frac{1}{2}\lambda_1^2 + \frac{1}{2}\lambda_2\right)\lambda_1 + \lambda_1\lambda_2 + \lambda_3,\\
f_3 &= \frac{1}{6}\lambda_1^3 + \frac{1}{6}\lambda_2\lambda_1 + \frac{1}{3}\lambda_1\lambda_2 + \frac{1}{3}\lambda_3,\\
4f_4 &= f_3\lambda_1 + f_2\lambda_2 + f_1\lambda_3 + f_0\lambda_4\\
&= \left(\frac{1}{6}\lambda_1^3 + \frac{1}{6}\lambda_2\lambda_1 + \frac{1}{3}\lambda_1\lambda_2 + \frac{1}{3}\lambda_3\right)\lambda_1 + \left(\frac{1}{2}\lambda_1^2 + \frac{1}{2}\lambda_2\right)\lambda_2 + \lambda_1\lambda_3 + \lambda_4,\\
f_4 &= \frac{1}{24}\lambda_1^4 + \frac{1}{24}\lambda_2\lambda_1^2 + \frac{1}{12}\lambda_1\lambda_2\lambda_1 + \frac{1}{12}\lambda_3\lambda_1 + \frac{1}{8}\lambda_1^2\lambda_2 + \frac{1}{8}\lambda_2^2 + \frac{1}{4}\lambda_1\lambda_3 + \frac{1}{4}\lambda_4.
\end{aligned}$$

Theorem 2.1. *Let $\{f_n\}$ be a higher coderivation on a coalgebra C with $f_0 = I$. Then there is a sequence $\{\lambda_n\}$ of coderivations on C such that*

$$(n+1)f_{n+1} = \sum_{i=2}^{n+1}\left(\sum_{\sum_{j=1}^{i} r_j = n}\left(\prod_{j=1}^{i}\frac{1}{r_i + \cdots + r_j}\right)\lambda_{r_i}\cdots\lambda_{r_1}\right),$$

where the inner summation is taken over all positive integers r_j, with $\sum_{j=1}^{i} r_j = n$.

Proof. We show that if f_n is of the above form then it satisfies the recursive relation of Proposition 2.2. Since the solution of the recursive relation is unique, this proves the theorem. Simplifying the notation we put $a_{r_i,\ldots,r_1} = \prod_{j=1}^{i}\frac{1}{r_i+\cdots+r_j}$. Note that if $r_1 + \cdots + r_i = n+1$ then $(n+1)a_{r_i,\ldots,r_1} = a_{r_i,\ldots,r_2}$. Moreover, $a_{n+1} = \frac{1}{n+1}$. Now we have

$$\begin{aligned}
(n+1)f_{n+1} &= \sum_{i=2}^{n+1}\left(\sum_{\sum_{j=1}^{i} r_j = n+1} a_{r_i,\ldots,r_1}(n+1)\lambda_{r_i}\cdots\lambda_{r_1}\right) + \lambda_{n+1}\\
&= \sum_{i=2}^{n+1}\left(\sum_{r_1=1}^{n+2-i}\sum_{\sum_{j=2}^{i} r_j = n+1-r_1} a_{r_i,\ldots,r_2}\lambda_{r_i}\cdots\lambda_{r_2}\right)\lambda_{r_1} + \lambda_{n+1}\\
&= \sum_{r_1=1}^{n}\sum_{i=2}^{n-(r_1-1)}\left(\sum_{\sum_{j=2}^{i} r_j = n-(r_1-1)} a_{r_i,\ldots,r_2}\lambda_{r_i}\cdots\lambda_{r_2}\right)\lambda_{r_1} + \lambda_{n+1}
\end{aligned}$$

$$= \sum_{r_1=1}^{n} f_{n-(r_1-1)}\lambda_{r_1} + \lambda_{n+1}$$
$$= \sum_{k=0}^{n} f_{n-k}\lambda_{k+1}. \qquad \square$$

Example 2.2. We evaluate the coefficients $a_{r_i,\ldots,r_1}$ for the case $n = 4$.

For $n = 4$ we can write

$$4 = 1+3 = 3+1 = 2+2 = 1+1+2 = 1+2+1 = 2+1+1 = 1+1+1+1.$$

By the definition of $a_{r_i,\ldots,r_1}$ we have

$$\begin{aligned}
a_4 &= \frac{1}{4},\\
a_{1,3} &= \frac{1}{1+3}\cdot\frac{1}{3} = \frac{1}{12},\\
a_{3,1} &= \frac{1}{3+1}\cdot\frac{1}{1} = \frac{1}{4},\\
a_{2,2} &= \frac{1}{2+2}\cdot\frac{1}{2} = \frac{1}{8},\\
a_{1,1,2} &= \frac{1}{1+1+2}\cdot\frac{1}{1+2}\cdot\frac{1}{2} = \frac{1}{24},\\
a_{1,2,1} &= \frac{1}{1+2+1}\cdot\frac{1}{2+1}\cdot\frac{1}{1} = \frac{1}{12},\\
a_{2,1,1} &= \frac{1}{2+1+1}\cdot\frac{1}{1+1}\cdot\frac{1}{1} = \frac{1}{8},\\
a_{1,1,1,1} &= \frac{1}{1+1+1+1}\cdot\frac{1}{1+1+1}\cdot\frac{1}{1+1}\cdot\frac{1}{1} = \frac{1}{24}.
\end{aligned}$$

We can therefore deduce that

$$f_4 = \frac{1}{4}\lambda_4 + \frac{1}{12}\lambda_3\lambda_1 + \frac{1}{4}\lambda_1\lambda_3 + \frac{1}{8}\lambda_2\lambda_2 + \frac{1}{24}\lambda_2\lambda_1\lambda_1 + \frac{1}{12}\lambda_1\lambda_2\lambda_1 + \frac{1}{8}\lambda_1\lambda_1\lambda_2 + \frac{1}{24}\lambda_1\lambda_1\lambda_1\lambda_1.$$

Theorem 2.2. *Let C be a coalgebra, F be the set of all higher coderivations $\{f_n\}_{n=0,1,\ldots}$ on C with $f_0 = I$ and Λ be the set of all sequences $\{\lambda_n\}_{n=0,1,\ldots}$ of coderivations on C with $\lambda_0 = 0$. Then there is a one to one correspondence between F and Λ.*

Proof. Let $\{\lambda_n\} \in \Lambda$. Define $f_n : C \to C$ by $f_0 = I$ and

$$f_n = \sum_{i=1}^{n}\left(\sum_{\sum_{j=1}^{i} r_j = n}\left(\prod_{j=1}^{i}\frac{1}{r_i+\cdots+r_j}\right)\lambda_{r_i}\cdots\lambda_{r_1}\right).$$

We show that $\{f_n\} \in F$. By Theorem 2.1, $\{f_n\}$ satisfies the recursive relation

$$(n+1)f_{n+1} = \sum_{k=0}^{n} f_{n-k}\lambda_{k+1}.$$

To show that $\{f_n\}$ is a higher coderivation, we use induction on n. For $n=0$ we have

$$\Delta f_0(a)=\Delta(a)=\sum a_{(1)}\otimes a_{(2)}=\sum f_0(a_{(1)})\otimes f_0(a_{(2)})=\sum (f_0(a))_{(1)}\otimes (f_0(a))_{(2)}.$$

Let us assume that $\Delta f_k(a)=\sum_{i=0}^{k}(f_i\otimes f_{k-i})\Delta(a)$ for $k\leq n$. Thus, we have

$$\begin{aligned}(n+1)\Delta f_{n+1}(a)=&\sum_{k=0}^{n}\Delta f_{n-k}\lambda_{k+1}(a)\\
=&\sum_{k=0}^{n}\sum_{i=0}^{n-k}(f_i\otimes f_{n-k-i})\Delta\lambda_{k+1}(a)\\
=&\sum_{k=0}^{n}\sum_{i=0}^{n-k}(f_i\otimes f_{n-k-i})(I\otimes\lambda_{k+1}+\lambda_{k+1}\otimes I)\Delta(a)\\
=&\sum_{k=0}^{n}\sum_{i=0}^{n-k}\sum(f_i\otimes f_{n-k-i})\left(\sum a_{(1)}\otimes\lambda_{k+1}(a_{(2)})\otimes\lambda_{k+1}(a_{(1)})\otimes a_{(2)}\right)\\
=&\sum_{k=0}^{n}\sum_{i=0}^{n-k}\sum f_i(a_{(1)})\otimes f_{n-k-i}(\lambda_{k+1}(a_{(2)}))\\
&+f_i(\lambda_{k+1}(a_{(1)})\otimes f_{n-k-i}(a_{(2)}).\end{aligned}$$

Using our assumption, we can write

$$\begin{aligned}(n+1)\Delta f_{n+1}(a)=&\sum_{i=0}^{n}\sum f_i(a_{(1)})\otimes(n-i+1)f_{n-i+1}(a_{(2)})\\
&+\sum_{i=0}^{n}\sum(n-i+1)\left(f_{n-i+1}(a_{(1)})\otimes f_i(a_{(2)})\right)\\
=&\sum_{i=0}^{n}\sum(n+1-i)f_i(a_{(1)})\otimes f_{n+1-i}(a_{(2)})\\
&+\sum_{i=1}^{n+1}\sum i(f_i(a_{(1)})\otimes f_{n+1-i}(a_{(2)})\\
=&(n+1)\sum_{k=0}^{n+1}\sum f_k(a_{(1)})\otimes f_{n+1-k}(a_{(2)})\\
=&(n+1)\sum_{k=0}^{n+1}(f_k\otimes f_{n+1-k})\Delta(a).\end{aligned}$$

Thus, $\{f_n\}\in F$.

Conversely, suppose that $\{f_n\}\in F$. Define $\lambda_n:C\to C$ by $\lambda_0=0$ and

$$\lambda_n=nf_n-\sum_{k=0}^{n-2}f_{n-1-k}\lambda_{k+1}.$$

Then Proposition 2.2 ensures us that $\{\lambda_n\} \in \Lambda$. Now define $\varphi : \Lambda \to F$ by $\varphi(\{\lambda_n\}) = \{f_n\}$, where

$$f_n = \sum_{i=1}^{n} \left(\sum_{\sum_{j=1}^{i} r_j = n} \left(\prod_{j=1}^{i} \frac{1}{r_i + \cdots + r_j} \right) \lambda_{r_i} \cdots \lambda_{r_1} \right).$$

Now φ is clearly a one to one correspondence. □

Recall that a higher coderivation $\{f_n\}$ is called ordinary if there is a coderivation λ such that $f_n = \frac{\lambda^n}{n!}$ for all n.

Corollary 2.1. *A higher coderivation $\{f_n\} = \varphi(\{\lambda_n\})$ on a coalgebra C is ordinary if and only if $\lambda_n = 0$ for $n \geq 2$. In this case $f_n = \frac{f_1^n}{n!}$.*

3. Conclusion

In this paper proving an equality for a coderivation on a coalgebra C, named general coLiebnitz rule for coderivations, we defined higher coderivations on a coalgebra C and then we characterized them in terms of the coderivations on C. Indeed, we showed that each higher coderivation is a combination of compositions of coderivations. Finally we proved there is a one to one correspondence between the set of all higher coderivations on C and all sequences of coderivations on C. As a corollary we characterize all higher coderivations which are ordinary.

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