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On Hawaiian groups of the infinite dimensional Hawaiian earring

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ABSTRACT

We study Hawaiian groups of the infinite dimensional Hawaiian earring which is the weak join of all finite dimensional Hawaiian earrings. We show that the structure of the 1st Hawaiian groups of the infinite dimensional Hawaiian earring and the one dimensional Hawaiian earring are isomorphic. Also, for $n \geq 2$, we prove that the nth Hawaiian group of the infinite dimensional Hawaiian earring is isomorphic to the direct product of direct sum of a family consisting of quotient of nth homotopy groups of consecutive retractions.

1. Introduction and motivation

M.G. Barratt et al. [3] defined an n-dimensional space, $n \in \mathbb{N}$, as the union of countably infinite number of shrinking n-spheres with a single point in common. In 2000, K. Eda et al. [5] called it the n-dimensional Hawaiian earring, $n = 1, 2, \ldots$, as a generalization of the well-known Hawaiian earring. We denote it by $\mathbb{HE}^n$ and it equals $\bigcup_{k\in\mathbb{N}} \mathbb{S}^n_k$, where $\mathbb{S}^n_k$ is the n-sphere with radius $1/k$ centred at $(1/k, 0, \ldots, 0)$ in $\mathbb{R}^{n+1}$. Here, θ denotes the origin.

U.H. Karimov et al. [8] defined the infinite dimensional Hawaiian earring as the weak join of all finite dimensional Hawaiian earrings. By weak join of a family of spaces $\{(X_i, x_i); i \in I\}$, denoted by $(X, x_*) = \widetilde{\bigvee}_{i\in I}(X_i, x_i)$, we mean the underlying space of wedge space $(\hat{X}, \hat{x}_*) = \bigvee_{i\in I}(X_i, x_i)$ with the weak topology with respect to X_i's, except at the common point. Every open neighbourhood in X at x_* is of the form

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$\bigcup_{i\in F} U_i \cup \bigcup_{i\in I\setminus F} X_i$, where U_i is an open neighbourhood at x_i in $X_i \subset X$ and F is a finite subset of I (see [3, Section 2] and [5, Page 18]).

Karimov et al. [7] defined the nth Hawaiian group of pointed space (X, x_0) as the set of all pointed homotopy classes $[f]$, where $f : (\mathbb{HE}^n, \theta) \to (X, x_0)$ is continuous, with a group operation which comes from the operation of nth homotopy groups. We denote the nth Hawaiian group of a pointed space (X, x_0) by $\mathcal{H}_n(X, x_0)$ which is defined as follows.

Definition 1.1 *([7]).* Let (X, x_0) be a pointed space, $[-]$ denote the class of pointed homotopy, and $n = 1, 2, \ldots$. Then the nth Hawaiian group of (X, x_0) is defined by $\mathcal{H}_n(X, x_0) = \{[f] : \ f : (\mathbb{HE}^n, \theta) \to (X, x_0)\}$ with the multiplication induced by $(f * g)|_{\mathbb{S}^n_k} = f|_{\mathbb{S}^n_k} \cdot g|_{\mathbb{S}^n_k}$ $(k \in \mathbb{N})$ for any $[f], [g] \in \mathcal{H}_n(X, x_0)$, where $\cdot$ denotes the concatenation of n-loops.

One can see that $\mathcal{H}_n : hTop_* \to Groups$ is a covariant functor from the pointed homotopy category, $hTop_*$, to the category of all groups, $Groups$, for $n \geq 1$. The Hawaiian group functor has some advantages over other famous functors such as homotopy, homology and cohomology functors. For instance, there exists a contractible space $C(\mathbb{HE}^1)$, the cone over $\mathbb{HE}^1$, with uncountable 1st Hawaiian group, but trivial homotopy, homology and cohomology groups [7]. Also, this functor can classify some local properties of spaces. For instance, if X has a countable local basis at x_0, then countability of the nth Hawaiian group $\mathcal{H}_n(X, x_0)$ implies locally n-simply connectedness of X at x_0 (see [7, Theorem 2]). Furthermore, there is a converse of the above statement in [2, Corollaries 2.16 and 2.17]: "let X have a countable local basis at x_0, then $\mathcal{H}_n(CX, x)$ is trivial if and only if X is locally n-simply connected at x_0 and it is uncountable otherwise". Therefore, unlike homotopy groups, Hawaiian groups of pointed space (X, x_0) depend on the behaviour of X at x_0, and then their structures depend on the choice of base point. In this regard, there exist some examples of path connected spaces with non-isomorphic Hawaiian groups at different points, such as the n-dimensional Hawaiian earring, where $n \geq 2$ (see [2, Corollary 2.11]).

The following lemma, proven in [2, Lemma 2.2], is a natural way to make a continuous map by the concatenation of a sequence of n-loops. Recall that a sequence of pointed maps $f_k : (X, x_0) \to (Y, y_0)$ $(k \in \mathbb{N})$ is nullconvergent if for each open neighbourhood U of y_0 in Y, there exists $K \in \mathbb{N}$ such that if $k \geq K$, then $im(f_k) \subseteq U$ (see [2, Definition 2.1]).

Lemma 1.2 *([2]). Let $\{f_k : (\mathbb{S}^n, \theta) \to (X, x_0)\}_{k\in\mathbb{N}}$ be a sequence of n-loops. Then $f : (\mathbb{HE}^n, \theta) \to (X, x_0)$ defined by $f|_{\mathbb{S}^n_k} = f_k$ $(k \in \mathbb{N})$ is continuous if and only if $\{f_k\}_{k\in\mathbb{N}}$ is nullconvergent. Moreover, let $\{f'_k\}_{k\in\mathbb{N}}$ be a nullconvergent sequence of n-loops and $f' : (\mathbb{HE}^n, \theta) \to (X, x_0)$ be the continuous map induced by $f'|_{\mathbb{S}^n_k} = f'_k$. If there exists a nullconvergent sequence of homotopy mappings $\{H_k : f_k \simeq f'_k\}_{k\in\mathbb{N}}$, then there is a homotopy map $H : f \simeq f'$ defined by $H|_{\mathbb{S}^n_k \times \mathbb{I}} = H_k$.*

In this paper, we study the structures of the fundamental group and Hawaiian groups of the infinite dimensional Hawaiian earring. We present the fundamental group and the 1st Hawaiian group of $\mathbb{HE}^\infty$ as follows:

$$\pi_1(\mathbb{HE}^\infty) \cong \pi_1(\mathbb{HE}^1) \cong \times^\sigma_{\aleph_0} \mathbb{Z},$$

$$\mathcal{H}_1(\mathbb{HE}^\infty, \theta) \cong \mathcal{H}_1(\mathbb{HE}^1, \theta) \cong \prod_{\aleph_0} \prod_{\aleph_0}^{W} \times^\sigma_{\aleph_0} \mathbb{Z},$$

where, $\prod^W_{i\in I} G_i$ denotes the weak direct product of a family of groups $\{G_i\}_{i\in I}$ and $\times^\sigma_{\aleph_0} \mathbb{Z}$ denotes the free σ-product of countably infinite copies of $\mathbb{Z}$.

Recall that the infinite dimensional Hawaiian earring is the weak join of all finite dimensional Hawaiian earrings, that is $\mathbb{HE}^\infty = \widetilde{\bigvee}_{n\in\mathbb{N}}\widetilde{\bigvee}_{k\in\mathbb{N}}\mathbb{S}^n_k$. In Lemma 2.1, we present another representation for the infinite dimensional Hawaiian earring as the weak join of spheres of various dimensions according to the sequence

$$1, 2, 1, 2, 3, \dots, 1, \dots, n, 1, \dots, n, n+1, \dots, \tag{1}$$

which we use this homeomorphic copy of the infinite dimensional Hawaiian earring for the rest of the paper.

Let r_i be the ith term of the sequence (1) which can be between 1 to n. Using the above representation of $\mathbb{HE}^\infty$, for $n \geq 2$, we prove the following isomorphism

$$\mathcal{H}_n(\mathbb{HE}^\infty, \theta) \cong \prod_{m\in\mathbb{N}} \bigoplus_{\aleph_0} \frac{\pi_n(\widetilde{\bigvee}_{i\geq m}\mathbb{S}^{r_i}_i, \theta)}{\pi_n(\widetilde{\bigvee}_{i\geq m+1}\mathbb{S}^{r_i}_i, \theta)},$$

where $\oplus$ denotes the direct sum of abelian groups. For proving these isomorphisms we use some properties of spheres such as semilocally strongly contractiblity and locally strongly contractiblity [2, Definition 1.2].

Definition 1.3 *([2]).*

1. A space X is called semilocally strongly contractible at point x_0 if there exists a neighbourhood U of x_0 such that the inclusion map $U \hookrightarrow X$ is nullhomotopic to x_0 [5].
2. A space X is called locally strongly contractible at point x_0 if for each neighbourhood U of x_0 there exists a neighbourhood V of x_0 such that the inclusion map $V \hookrightarrow U$ is nullhomotopic to x_0.

Throughout this paper, all spaces are considered to have a countable local basis at their base points, and also all homotopies are relative to the base point unless stated otherwise.

2. The fundamental group of $\mathbb{HE}^\infty$

In this section, we intend to study the fundamental group of the infinite dimensional Hawaiian earring. Recall that the infinite dimensional Hawaiian earring $\mathbb{HE}^\infty$ is defined as the weak join of all finite dimensional Hawaiian earrings. In the following lemma, we see that the infinite dimensional Hawaiian earring is homeomorphic to a weak join of spheres and we use this homeomorphic copy of $\mathbb{HE}^\infty$ for the rest of the paper.

Lemma 2.1. *The infinite dimensional Hawaiian earring is homeomorphic to the weak join of countably infinite number of spheres with the following sequence of dimensions:*

$$1, 2, 1, 2, 3, \dots, 1, \dots, n, 1, \dots, n, n+1, \dots. \tag{1}$$

Proof. Assume that $X = \widetilde{\bigvee}_{i\in\mathbb{N}}\mathbb{S}^{r_i}_i$ where r_i is the ith term in the sequence (1) and let x_* be the common point. There is a bijection between spheres of the same dimension in $\mathbb{HE}^\infty$ and X. It induces a bijection $g : \mathbb{HE}^\infty \to X$ that maps the kth n-sphere of $\mathbb{HE}^n$ in $\mathbb{HE}^\infty$ onto the n-sphere corresponded to the kth n in the sequence (1). Since g is a bijection, we must prove that g and $h = g^{-1}$ are continuous. Since X is compact and $\mathbb{HE}^\infty$ is Hausdorff, it suffices to show that h is continuous.

Since the restriction of bijection h is continuous on each sphere, we need only to verify the continuity of h at x_*. Let U be an open set containing θ. Then by the topology of weak join in $\mathbb{HE}^\infty$, $U = \bigcup_{1\leq n\leq N} U_n \cup \bigcup_{n>N}\mathbb{HE}^n$ for some N, where U_n is an open set in $\mathbb{HE}^n$, $1 \leq n \leq N$. Again by the topology of weak join in $\mathbb{HE}^n$, for each $1 \leq n \leq N$, $U_n = \bigcup_{1\leq k\leq K_n} V^n_k \cup \bigcup_{k>K_n}\mathbb{S}^n_k$ for some $K_n \in \mathbb{N}$ where V^n_k is an open neighbourhood of θ in $\mathbb{S}^n_k$ for $1 \leq k \leq K_n$. Hence

$$U = \bigcup_{1\le n\le N} \bigcup_{1\le k\le K_n} V_k^n \cup \bigcup_{1\le n\le N} \bigcup_{k>K_n} \mathbb{S}_k^n \cup \bigcup_{n>N} \bigcup_{k\in\mathbb{N}} \mathbb{S}_k^n,$$

where V_k^n is an open set at θ in $\mathbb{S}_k^n$ for $1 \le n \le N$ and $1 \le k \le K_n$. We show that $h^{-1}(U)$ is open. For $h^{-1}(U)$, put $M := \max\big\{\{K_n; 1 \le n \le N\} \cup \{N\}\big\}$. Since $M \ge N$, for $n > M$, $h^{-1}(U)$ contains the whole of all n-spheres in X. Also, since $M \ge K_n$ for all $1 \le n \le N$, $h^{-1}(U)$ contains the whole of all n-spheres after the first M in sequence (1). That is $h^{-1}(U)$ contains the whole of all n-spheres after the first M corresponded to r_i where $i = \frac{M(M+1)}{2}$. Also for the other indices, $h^{-1}(U)$ contains open components. On the other hand, each open set W in X at x_* is of the form $W = \bigcup_{1\le i\le M} W_i \cup \bigcup_{i>M} \mathbb{S}_i^{r_i}$ for some $M \in \mathbb{N}$, where r_i is the ith term of the sequence (1) and W_i is an open set in $\mathbb{S}_i^{r_i}$ at x_*. Hence $h^{-1}(U)$ is open in X. Therefore h is continuous. Since X is compact, $\mathbb{HE}^\infty$ is Hausdorff, and h is a continuous bijection, it is a homeomorphism. □

The homeomorphic copy of the infinite dimensional Hawaiian earring defined in Lemma 2.1 is more familiar than the original one. For instance, it satisfies the hypotheses of [9, theorem 4.1] and [4, Theorem A.1] to obtain its fundamental group. In fact, it is proven that the fundamental group of the weak join of some family of spaces is isomorphic to the free σ-product of the fundamental groups of the given spaces [9,4]. Here, we recall the definition of free σ-product of groups and free σ-group. For more details and proofs see [4].

Definition 2.2 *([4])*. Let $\{G_i\}_{i\in I}$ be an infinite family of groups. We assume that $G_i \cap G_j = \{e\}$ for distinct $i, j \in I$. A σ-word $w : S \to \bigcup_{i\in I} G_i$ is a function, where S is a countable linearly ordered set and $\{\alpha \in S : w(\alpha) \in G_i\}$ is finite for each $i \in I$. We denote the set of all σ-words by $\mathcal{W}^\sigma(G_j : i \in I)$. For finite subset F of I, w_F is the word of finite length obtained by deleting all elements in $\bigcup_{i\notin F} G_i$ from w. In other words, if $S(w, F) = \{\alpha \in S : w(\alpha) \in \bigcup_{i\in F} G_i\}$, then $w_F = w|_{S(w,F)}$. Two σ-words v, w are equivalent, if $v_F = w_F$ holds in the free product group $*_{i\in F} G_i$ for every finite subset F of I. The equivalence class containing w is denoted by $[w]$. A group operation is defined on the set of all equivalence classes of σ-words by the concatenation of presentations of two σ-words. Then $\{[w] :\ w \in \mathcal{W}^\sigma(G_i \colon i \in I)\}$ forms a group which we call the free σ-product of family $\{G_i\}_{i\in I}$, denoted by $\times^\sigma_{i\in I} G_i$. Immediately $*_{i\in I} G_i$ is a subgroup of $\times^\sigma_{i\in I} G_i$. When $G_i \cong \mathbb{Z}$ for all $i \in I$, $\times^\sigma_{i\in I} G_i$ is called the free σ-group on $|I|$ alphabet.

The notion of free σ-group was introduced when the structure of the fundamental group of one dimensional Hawaiian earring, $\pi_1(\mathbb{HE}^1)$ was studied. More generally, for spaces made by the weak join of some family of spaces, the following theorem holds [4, Theorem A.1].

Theorem 2.3 *([4]). Let X_i be locally 1-simply connected at x_i for each $i \in \mathbb{N}$ and $(X, x_*) = \widetilde{\bigvee}_{i\in\mathbb{N}}(X_i, x_i)$. Then*

$$\pi_1(X, x_*) \cong \times^\sigma_{i\in\mathbb{N}} \pi_1(X_i, x_i).$$

In the following lemma, we use Theorem 2.3, Lemma 2.1 and [9, Theorem 4.1] to obtain the structure of the fundamental group of the infinite dimensional Hawaiian earring, $\pi_1(\mathbb{HE}^\infty)$.

Theorem 2.4. *The inclusion map $i : \mathbb{HE}^1 \to \mathbb{HE}^\infty$ induces an isomorphism on the fundamental groups. That is $\pi_1(\mathbb{HE}^\infty) \cong \pi_1(\mathbb{HE}^1) \cong \times^\sigma_{\aleph_0} \mathbb{Z}$.*

Proof. Since $\mathbb{HE}^1$ is a retract of $\mathbb{HE}^\infty$, the inclusion map induces a monomorphism. To prove surjectivity, we recall the structure of $\pi_1(\mathbb{HE}^\infty)$. The infinite dimensional Hawaiian earring, as represented in Lemma 2.1, has properties of [9, Theorem 4.1] and Theorem 2.3. Thus, there is a monomorphism from $\pi_1(\mathbb{HE}^\infty)$ into the group $\lim_{\leftarrow m} *_{i\le m}\pi_1(\mathbb{S}^{r_i})$ with image consisting of all countably infinite tuples of words such that the number of components including letter of type j is finite for all $j \in \mathbb{N}$ by [9, Theorem 4.1]. By Theorem 2.3, $\pi_1(\mathbb{HE}^\infty) \cong \times^{\sigma}_{i\in\mathbb{N}}\pi_1(\mathbb{S}^{r_i})$ which is induced by the natural retractions. Since for $r_i > 1$, the group $\pi_1(\mathbb{S}^{r_i})$ is trivial, $\pi_1(\mathbb{HE}^\infty) \cong \times^{\sigma}_{\aleph_0}\pi_1(\mathbb{S}^1)$. Hence any map $\alpha : \mathbb{S}^1 \to \mathbb{HE}^\infty$ is homotopic to a map with image contained in $i(\mathbb{HE}^1)$. Therefore $\pi_1(i)$ is onto as desired. $\square$

Note that the inclusion map in Theorem 2.4 does not induce an isomorphism for higher dimensions.

Remark 2.5.

1. Let $n \ge 2$. If one considers $\pi_n(i) : \pi_n(\mathbb{HE}^1) \to \pi_n(\mathbb{HE}^\infty)$, then $\pi_n(i)$ is not an isomorphism, because $\pi_n(\mathbb{HE}^1)$ is trivial and $\pi_n(\mathbb{HE}^\infty)$ is not. In fact, $\pi_n(\mathbb{HE}^\infty)$ contains $\pi_n(\mathbb{HE}^n)$ as a subgroup which is isomorphic to the direct product of infinite cyclic groups $\prod_{\aleph_0}\mathbb{Z}$ [5, Corollary 1.2].
2. Moreover, let $n \ge 2$ and $j : \mathbb{HE}^n \to \mathbb{HE}^\infty$ be the inclusion map. Then $\pi_n(j) : \pi_n(\mathbb{HE}^n) \to \pi_n(\mathbb{HE}^\infty)$ is not an isomorphism, because it is not onto. For example, let α be the non-trivial simple n-loop in $\mathbb{S}^n_1$. Then there is no n-loop in $\mathbb{S}^n_1$ which is mapped to the n-loop $(\alpha, \alpha, 0, 0, \dots)$ in $\mathbb{S}^1_1 \vee \mathbb{S}^n_1$ by the inclusion map j. Recall that $\pi_n(\mathbb{S}^1_1 \vee \mathbb{S}^n_1) \cong \bigoplus_{\aleph_0}\mathbb{Z}$ [6, Page 364, Example 4.27].

3. The first Hawaiian group of $\mathbb{HE}^\infty$

In this section, we study the 1st Hawaiian group of the infinite dimensional Hawaiian earring, $\mathcal{H}_1(\mathbb{HE}^\infty, \theta)$ by the inclusion map $i : \mathbb{HE}^1 \to \mathbb{HE}^\infty$ which induces isomorphism on the fundamental groups as proven in Theorem 2.4. However, there are maps inducing isomorphisms on homotopy groups and not inducing isomorphisms on Hawaiian groups (see [7, Remark 1]). In the following theorem, we find some conditions under which a given map induces isomorphism on Hawaiian groups.

Theorem 3.1. *Let $F : (X, x_0) \to (Y, y_0)$ be a continuous map which induces isomorphisms $\pi_n(F) : \pi_n(X, x_0) \cong \pi_n(Y, y_0)$ and $\pi_n(F_{U_m}) : \pi_n(U_m, x_0) \cong \pi_n(V_m, x_0)$, $m \in \mathbb{N}$, for two local bases $\{U_m\}_{m\in\mathbb{N}}$ in X at x_0 and $\{V_m\}_{m\in\mathbb{N}}$ in Y at y_0. Then*

$$\mathcal{H}_n(F) : \mathcal{H}_n(X, x_0) \cong \mathcal{H}_n(Y, y_0).$$

Proof. Let $F_* := \mathcal{H}_n(F) : \mathcal{H}_n(X, x_0) \to \mathcal{H}_n(Y, y_0)$ be the induced homomorphism on the nth Hawaiian group. We prove that F_* is injective and surjective. Let $[g], [h] \in \mathcal{H}_n(X, x_0)$, and $F_*([g]) = F_*([h])$. Thus $[F \circ g] = [F \circ h]$ or equivalently $F \circ g \simeq F \circ h$ rel $\{\theta\}$. Hence $F \circ g|_{\mathbb{S}^n_k} \simeq F \circ h|_{\mathbb{S}^n_k}$ rel $\{\theta\}$ for $k \in \mathbb{N}$. By the isomorphism $\pi_n(F) : \pi_n(X, x_0) \to \pi_n(Y, y_0)$, we can conclude that $g|_{\mathbb{S}^n_k} \simeq h|_{\mathbb{S}^n_k}$ rel $\{\theta\}$ for $k \in \mathbb{N}$. Moreover, for each $m \in \mathbb{N}$, there is $K_m \in \mathbb{N}$ such that if $k \ge K_m$, then $im(g|_{\mathbb{S}^n_k}), im(h|_{\mathbb{S}^n_k}) \subseteq U_m$. Also $F_{U_m} : U_m \to V_m$ induces an isomorphism on homotopy groups for each $m \in \mathbb{N}$. It implies that there is a homotopy map $g|_{\mathbb{S}^n_k} \simeq h|_{\mathbb{S}^n_k}$ rel $\{\theta\}$ in U_m for $K_m \le k < K_{m+1}$. Since $\{U_m\}$ is a local basis at x_0, by Lemma 1.2 one can compile homotopies to make a homotopy $g \simeq h$ rel $\{\theta\}$, and hence F_* is injective. Now let $[h] \in \mathcal{H}_n(Y, y_0)$. There is a sequence $\{K_m\}_{m\in\mathbb{N}}$ such that if $K_m \le k < K_{m+1}$, then $im(h|_{\mathbb{S}^n_k}) \subseteq V_m$. By isomorphism $\pi_n(F_{U_m})$, for $m \in \mathbb{N}$, there is a sequence of maps $\{g_k\}_{k\in\mathbb{N}}$ such that $[g_k] \in \pi_n(U_m, x_0)$ and $\pi_n(F)[g_k] = [h|_{\mathbb{S}^n_k}]$ for $K_m \le k < K_{m+1}$. Define $g : (\mathbb{HE}^n, \theta) \to (X, x_0)$ by $g|_{\mathbb{S}^n_k} = g_k$. Since $\{U_m\}_{m\in\mathbb{N}}$ is a local basis at x_0, g is continuous. Also since $\pi_n(F)([g_k]) = [h|_{\mathbb{S}^n_k}]$, where we define $g|_{\mathbb{S}^n_k} = g_k$, then $F_*([g]) = [h]$. Therefore, F_* is surjective. $\square$

Theorem 3.1 helps us to study the structure of $\mathcal{H}_1(\mathbb{HE}^\infty, \theta)$ by the structure of $\mathcal{H}_1(\mathbb{HE}^1, \theta)$. We recall the structure of $\mathcal{H}_1(\mathbb{HE}^1, \theta)$ in the following lemma which was proven in [1, Lemma 4.1].

Lemma 3.2 *([1]). Let $\mathcal{B}$ be the subgroup of $\prod_{\aleph_0} \times^{\sigma}_{\aleph_0} \mathbb{Z}$ consisting of all countably infinite tuples of reduced σ-words such that the number of components including letter of type m is finite, for all $m \in \mathbb{N}$. Then*

$$\mathcal{H}_1(\mathbb{HE}^1, \theta) \cong \mathcal{B}.$$

The group $\mathcal{B}$ is a subgroup of $\prod_{\aleph_0} \times^{\sigma}_{\aleph_0} \mathbb{Z}$ and Lemma 3.2 does not determine the exact structure of $\mathcal{H}_1(\mathbb{HE}^1, \theta)$. In the following theorem, we establish the structure of group $\mathcal{B}$ in a more generalized case.

Theorem 3.3. *Let X_i be locally strongly contractible at x_i for each $i \in I$ and $(X, x_*) = \widetilde{\bigvee}_{i\in\mathbb{N}}(X_i, x_i)$. Then*

$$\mathcal{H}_1(X, x_*) \cong \prod_{m\in\mathbb{N}} \prod_{\aleph_0}^{W} \times^{\sigma}_{i\geq m} \pi_1(X_i, x_i).$$

Proof. First, by generalizing the proof of [1, Lemma 4.1] for (X, x_*), one can see that $\mathcal{H}_1(X, x_*)$ is isomorphic to the subgroup $\mathcal{A}$ of $\prod_{\aleph_0} \times^{\sigma}_{i\in\mathbb{N}} \pi_1(X_i, x_i)$ consisting of all countably infinite tuples of reduced σ-words such that the number of components including letter of group $G_m = \pi_1(X_m, x_m)$ is finite, for all $m \in \mathbb{N}$. Now we show that this group is isomorphic to $\prod_{m\in\mathbb{N}} \prod^{W}_{\aleph_0} \times^{\sigma}_{i\geq m} G_i$. Note that since $\aleph_0 = \aleph_0 \times \aleph_0$, there is a natural isomorphism $\prod_{\aleph_0} \times^{\sigma}_{\aleph_0} G_i \cong \prod_{\aleph_0} \prod_{\aleph_0} \times^{\sigma}_{\aleph_0} G_i$. Now we can consider $\mathcal{A}$ as a subgroup of $\prod_{\aleph_0} \prod_{\aleph_0} \times^{\sigma}_{\aleph_0} G_i$. Hence it suffices to verify the equality $\mathcal{A} = \prod_{m\in\mathbb{N}} \prod^{W}_{\aleph_0} \times^{\sigma}_{i\geq m} G_i$ as two subgroups of $\prod_{\aleph_0} \prod_{\aleph_0} \times^{\sigma}_{\aleph_0} G_i$. Let $\{\{w^m_k\}_{k\in\mathbb{N}}\}_{m\in\mathbb{N}} \in \mathcal{A}$. Then w^m_k's are reduced σ-words such that the number of ones including some letter of group $G_m = \pi_1(X_m, x_m)$ is finite, for all $m \in \mathbb{N}$. One can consider the number of σ-words of $\{\{w^m_k\}_{k\in\mathbb{N}}\}_{m\in\mathbb{N}}$ including letter of group G_m and not including letter of group G_l for $l < m$ equals K_m. That is $\{w^m_k\}_{k\in\mathbb{N}} \in \prod^{W}_{\aleph_0} \times^{\sigma}_{i\geq m} G_i$ for each $m \in \mathbb{N}$. Hence $\{\{w^m_k\}_{k\in\mathbb{N}}\}_{m\in\mathbb{N}} \in \prod_{m\in\mathbb{N}} \prod^{W}_{\aleph_0} \times^{\sigma}_{i\geq m} G_i$.

Conversely, let $\{\{w^m_k\}_{k\in\mathbb{N}}\}_{m\in\mathbb{N}} \in \prod_{m\in\mathbb{N}} \prod^{W}_{\aleph_0} \times^{\sigma}_{i\geq m} G_i$. Then for each $m \in \mathbb{N}$, the number of σ-words of $\{w^m_k\}_{k\in\mathbb{N}}$ including letter of group $\times^{\sigma}_{i\geq m} G_i$ equals the natural number K_m. That is for each $m \in \mathbb{N}$, the number of σ-words including letter of group G_m is less than or equal to $K_1 + \ldots + K_m$ which is finite. Therefore $\{\{w^m_k\}_{k\in\mathbb{N}}\}_{m\in\mathbb{N}} \in \mathcal{A}$. Hence $\mathcal{A} = \prod_{m\in\mathbb{N}} \prod^{W}_{\aleph_0} \times^{\sigma}_{i\geq m} G_i$ as required. □

Using Theorems 3.1 and 3.3 we are in a position to give a structure for the 1st Hawaiian group of the infinite dimensional Hawaiian earring as follows.

Corollary 3.4.

1. *Let $\mathbb{HE}^\infty$ be the infinite dimensional Hawaiian earring and θ be the common point. Then*

$$\mathcal{H}_1(\mathbb{HE}^\infty, \theta) \cong \mathcal{H}_1(\mathbb{HE}^1, \theta) \cong \prod_{\aleph_0} \prod_{\aleph_0}^{W} \times^{\sigma}_{\aleph_0} \mathbb{Z}.$$

2. *Also, if $a \in \mathbb{HE}^\infty$ and $a \neq \theta$, then*

$$\mathcal{H}_1(\mathbb{HE}^\infty, a) \cong \prod_{\aleph_0}^{W} \pi_1(\mathbb{HE}^\infty) \cong \prod_{\aleph_0}^{W} \times^{\sigma}_{\aleph_0} \mathbb{Z}.$$

Proof. 1. The inclusion map $\mathbb{HE}^1 \hookrightarrow \mathbb{HE}^\infty$ induces isomorphism on the fundamental groups by Theorem 2.4. Similar to the proof of Theorem 2.4, one can prove that the isomorphisms hold for appropriate countable local bases, because generators are mapping to generators isomorphically by [9, Theorem 4.1]. Thus, $i : \mathbb{HE}^1 \to \mathbb{HE}^\infty$ has properties of Theorem 3.1, and then the isomorphism $\mathcal{H}_1(\mathbb{HE}^\infty, \theta) \cong \mathcal{H}_1(\mathbb{HE}^1, \theta)$ holds by Theorem 3.1. Now by Theorem 3.3, we have

$$\mathcal{H}_1(\mathbb{HE}^1,\theta) \cong \prod_{m\in\mathbb{N}} \prod_{\aleph_0}^{W} \times^{\sigma}_{i\geq m} \pi_1(\mathbb{S}^1_i,\theta).$$

Since $\pi_1(\mathbb{S}^1_i,\theta) \cong \mathbb{Z}$, we have $\times^{\sigma}_{i\geq m}\pi_1(\mathbb{S}^1_i,\theta) \cong \times^{\sigma}_{i\geq m}\mathbb{Z}$. Also since $card\{i \geq m\} = \aleph_0$ and $card\{m \in \mathbb{N}\} = \aleph_0$, the required isomorphism holds.

2. Since $\mathbb{HE}^\infty$ is semilocally strongly contractible at a, by [2, Theorem 2.5], $\mathcal{H}_1(\mathbb{HE}^\infty, a) \cong \prod^W_{\aleph_0} \pi_1(\mathbb{HE}^\infty)$. By Theorem 2.4, $\pi_1(\mathbb{HE}^\infty) \cong \times^{\sigma}_{\aleph_0}\mathbb{Z}$, and the result holds. □

Note that Corollary 3.4 shows that the group $\mathcal{B}$, which was defined descriptively as a subgroup of $\prod_{\aleph_0} \times^{\sigma}_{\aleph_0}\mathbb{Z}$ in Lemma 3.2, is isomorphic to the well-known group $\prod_{\aleph_0}\prod^W_{\aleph_0} \times^{\sigma}_{\aleph_0}\mathbb{Z}$.

4. Higher Hawaiian groups of $\mathbb{HE}^\infty$

In this section, we intend to study Hawaiian groups of weak join of some family of spaces such as Hawaiian earrings. We present Hawaiian groups of weak join as the direct product of direct sum of a family of quotient of homotopy groups of consecutive retractions. As a consequence, we see that higher Hawaiian groups of the infinite dimensional Hawaiian earring is isomorphic to the direct product of a family of quotient groups.

Theorem 4.1. *Let X_i be locally strongly contractible at point x_i. Also let $(X, x_*) = \widetilde{\bigvee}_{i\in\mathbb{N}}(X_i, x_i)$ be the weak join space of the family $\{(X_i, x_i)\}_{i\in\mathbb{N}}$. Then for $n \geq 2$*

$$\mathcal{H}_n(X, x_*) \cong \prod_{m\in\mathbb{N}} \bigoplus_{\aleph_0} \frac{\pi_n(\widetilde{\bigvee}_{i\geq m}X_i, x_*)}{\pi_n(\widetilde{\bigvee}_{i\geq m+1}X_i, x_*)}. \tag{2}$$

Proof. First we define some notations to simplify the formulas. Put $Y_m := \widetilde{\bigvee}_{i\geq m}X_i$, and $P_m := \pi_n(Y_m, x_*)$ for $m \in \mathbb{N}$. Consider

$$\psi : \mathcal{H}_n(X, x_*) \to \prod_{m\in\mathbb{N}}\prod_{k\in\mathbb{N}} \frac{P_m}{P_{m+1}}, \text{ by } \psi([f]) = \big\{\{[R_m \circ f|_{\mathbb{S}^n_k}]P_{m+1}\}_{k\in\mathbb{N}}\big\}_{m\in\mathbb{N}},$$

where $R_m : X \to Y_m$ is the identity on X_i for $i \geq m$ and sends X_i to the x_* for $i < m$. Since Y_{m+1} is a retract of Y_m by the natural retraction, P_{m+1} is a subgroup of P_m, and thus the correspondence makes sense. Moreover, ψ is a composition of homomorphisms, and so is itself a homomorphism. Since X_i is locally strongly contractible at x_i, for $i \in \mathbb{N}$, there exists a nested local basis $\{V^i_j\}_{j\in\mathbb{N}}$ at x_i such that the inclusion mapping $V^i_j \hookrightarrow V^i_{j-1}$ is homotopic to the constant map at x_i in V^i_{j-1} by a homotopy, H^i_j. Let $\{U_m\}_{m\in\mathbb{N}}$ be the local basis at x_* obtained by $U_m = (\bigvee_{i<m}V^i_m) \vee (Y_m)$. This family is a local basis at point x_*, for if $U \subseteq X$ is an open set, then there exist $K \in \mathbb{N}$ and natural numbers $j_1, \cdots, j_{K-1}$ such that $X_i \subseteq U$ (for $i \geq K$) and $V^i_{j_i} \subseteq U \cap X_i$ (for $i < K$), because $\{V^i_j\}_{j\in\mathbb{N}}$ is a local basis for X_i and $U \cap X_i$ is an open set in X_i for all $i \in \mathbb{N}$. Now putting $m = \max\{K, j_1, \ldots, j_{K-1}\}$ implies that $U_m \subseteq U$. Also for $m \geq 2$, the inclusion $U_m \hookrightarrow U_{m-1}$ is homotopic to $R_m|_{U_m}$ in U_{m-1} by a homotopy obtained by joining homotopies H^i_m's for $i < m$, and identity for the others.

i) The homomorphism ψ is an injection.

Let $[f]$ be an element of $\mathcal{H}_n(X, x_*)$ so that $\psi([f]) = e$. Then $[R_m \circ f|_{\mathbb{S}^n_k}] \in P_{m+1}$ for every $m, k \in \mathbb{N}$ or equivalently $R_m \circ f|_{\mathbb{S}^n_k}$ is homotopic, in Y_m, to a mapping with image included in Y_{m+1}. Since $f : (\mathbb{HE}^n, \theta) \to (X, x_*)$ is continuous, for $m \in \mathbb{N}$, there exists the minimum natural number K_m such that if $k \geq K_m$, then $im(f|_{\mathbb{S}^n_k}) \subseteq U_{m+1}$. Note that since $U_{m+1} \subseteq U_m$ for each $m \in \mathbb{N}$, the sequence $\{K_m\}_{m\in\mathbb{N}}$ is an increasing sequence. Fix $k \geq K_m$. So $f|_{\mathbb{S}^n_k}$ is homotopic to the corresponding contracted map on the space Y_{m+1} in U_m because $[R_m \circ f|_{\mathbb{S}^n_k}] \in \pi_n(Y_{m+1})$. By induction, for $l \in \mathbb{N}$, let $f|_{\mathbb{S}^n_k}$ in U_m is homotopic to n-loop g_l whose image is contained in Y_{m+l-1}. Since $[R_{m+l-1} \circ f|_{\mathbb{S}^n_k}] \in P_{m+l}$ and $[f|_{\mathbb{S}^n_k}] = [g_l]$ in U_m, we have $[R_{m+l-1} \circ g_l] \in P_{m+l}$. Thus there exist n-loop g_{l+1} in Y_{m+l}, and homotopy mapping G_l such that $G_l : g_l \simeq g_{l+1}$ in Y_{m+l-1}. Note that by transitivity of the relation of homotopy, $g_{l+1} \simeq f|_{\mathbb{S}^n_k}$ in U_m. By gluing homotopies, G_l's, we prove that $f|_{\mathbb{S}^n_k}$ is nullhomotopic in U_m as follows. Define $G : \mathbb{S}^n_k \times \mathbb{I} \to U_m$ by the rule

$$\begin{cases} G|_{\mathbb{S}^n_k \times [1-\frac{1}{2^{l-1}}, 1-\frac{1}{2^l}]} = G_l(-, h_l) \\ G|_{\mathbb{S}^n_k \times \{1\}} = x_*, \end{cases}$$

where l is a natural number and h_l is the linear mapping taking the interval $[1-\frac{1}{2^{l-1}}, 1-\frac{1}{2^l}]$ to $\mathbb{I} = [0,1]$. By gluing lemma, G is continuous on all points of its domain except the case that the second component equals 1. Then it suffices to verify continuity at the points whose second components equal 1. Since $\mathbb{S}^n_k \times \mathbb{I}$ and X are first countable spaces, we can verify the continuity of G by convergent sequences. Let $\{(a_s, t_s)\}_{s\in\mathbb{N}}$ be a convergent sequence with $t_s \to 1$ and $a_s \to a$, where $a \in \mathbb{S}^n_k$ is arbitrary. For each $s \in \mathbb{N}$, there exists $l_s \in \mathbb{N}$ such that $t_s \in [1 - \frac{1}{2^{l_s-1}}, 1 - \frac{1}{2^{l_s}}]$. Since $t_s \to 1$, then $l_s \to \infty$. We know that $im(G_{l_s}) \subseteq Y_{m+l_s-1}$. Thus $G(a_s, t_s) = G_{l_s}(a_s, t_s) \to x_*$ and so we are done. Hence G is continuous and also

$$\begin{aligned} G(a,0) &= G|_{\mathbb{S}^n_k \times [0,\frac{1}{2}]}(a,0) = G_1 \circ h_1(a,0) = f|_{\mathbb{S}^n_k}(a), \\ G(a,1) &= G|_{\mathbb{S}^n_k \times \{1\}}(a,1) = x_*, \\ G(\theta,t) &= G_l(\theta, h_l(t)) = x_*. \end{aligned}$$

Thus $G : f|_{\mathbb{S}^n_k} \simeq c_{x_*}$ in U_m. Note that $U_1 = Y_1 = X$. Thus $K_1 = 1$ and then $\{K_m\}_{m\in\mathbb{N}}$ is an increasing sequence starting at $K_1 = 1$. Also, for each k with $K_m \leq k < K_{m+1}$, $f|_{\mathbb{S}^n_k}$ is nullhomotopic in U_m. Therefore all $f|_{\mathbb{S}^n_k}$'s are nullhomotopic with a sequence of nullconvergent homotopies. Now by Lemma 1.2, f is nullhomotopic and hence ψ is injective.

ii) The image of ψ equals the direct product, that is $\psi(\mathcal{H}_n(X, x_*)) = \prod_{m\in\mathbb{N}} F_m$ where $F_m = \bigoplus_{\aleph_0} \frac{P_m}{P_{m+1}}$.

Let $\big\{\{[f^m_k]P_{m+1}\}_{k\in\mathbb{N}}\big\}_{m\in\mathbb{N}} \in \psi\big(\mathcal{H}_n(X, x_*)\big)$. Then there is $f : (\mathbb{HE}^n, \theta) \to (X, x_*)$ such that $f^m_k = R_m \circ f|_{\mathbb{S}^n_k}$. Since f is continuous, for each $m \in \mathbb{N}$, there exists $K_m \in \mathbb{N}$ such that if $k \geq K_m$ then $im(f|_{\mathbb{S}^n_k}) \subseteq U_m$. Fix $m \in \mathbb{N}$. We want to show that $[f|_{\mathbb{S}^n_k}] \in P_{m+1}$ for $k \geq K_{m+2}$. Since $k \geq K_{m+2}$, there exists n-loop g in U_{m+2} such that $i \circ g = f|_{\mathbb{S}^n_k}$. Let $i : U_{m+2} \to X$, $j : U_{m+2} \to U_{m+1}$ and $k : U_{m+1} \to X$ be the inclusion maps. Since j is homotopic to $R_{m+2}|_{U_{m+2}}$ and $i = k \circ j$, $f|_{\mathbb{S}^n_k} = i \circ g = k \circ j \circ g \simeq k \circ R_{m+2}|_{U_{m+2}} \circ g$. Since $im(R_{m+2}) \subseteq Y_{m+2} \subseteq Y_{m+1}$, $f|_{\mathbb{S}^n_k} = k \circ R_{m+2}|_{U_{m+2}} \circ g$ is homotopic to an n-loop whose image is contained in Y_{m+1}. That is $[f|_{\mathbb{S}^n_k}]$ belongs to $\pi_n(Y_{m+1}) = P_{m+1}$ as desired. Then $[f^m_k] = [R_m \circ f|_{\mathbb{S}^n_k}] \in P_{m+1}$ for $k \geq K_{m+2}$, which implies that $[f^m_k]P_{m+1}$ is the identity except possibly for $k < K_{m+2}$. That is $\{[f^m_k]P_{m+1}\}_{k\in\mathbb{N}} \in \bigoplus_{\aleph_0} \frac{P_m}{P_{m+1}} = F_m$. Since m is arbitrary, $\{[f^m_k]P_{m+1}\}_{k\in\mathbb{N}} \in \bigoplus_{\aleph_0} \frac{P_m}{P_{m+1}} = F_m$ for each $m \in \mathbb{N}$. Thus $\big\{\{[f^m_k]P_{m+1}\}_{k\in\mathbb{N}}\big\}_{m\in\mathbb{N}} \in \prod_{m\in\mathbb{N}} F_m$. Therefore, $\psi\big(\mathcal{H}_n(X, x_*)\big) \subseteq \prod_{m\in\mathbb{N}} F_m$.

To prove the converse inclusion, let $\big\{\{[f^m_k]P_{m+1}\}_{k\in\mathbb{N}}\big\}_{m\in\mathbb{N}} \in \prod_{m\in\mathbb{N}} F_m$. Define $f : \mathbb{HE}^n \to X$ by

$$f|_{\mathbb{S}^n_k} = f_k = f^{m_k}_k \cdot R_{m_k+1} \circ \left(f^{m_k}_k\right)^{-1} \cdot f^{m_k+1}_k \cdot R_{m_k+2} \circ \left(f^{m_k+1}_k\right)^{-1} \cdot \ldots, \tag{3}$$

where $m_k \in \mathbb{N}$ is the minimum number for which $[f^{m_k}_k]P_{m_k+1}$ is not the identity element. Note that to simplify the equalities, the inclusion maps are omitted. If there is not such a natural number, define $m_k = \infty$ and $f|_{\mathbb{S}^n_k}$ as the constant map at the base point. Recall that $\cdot$ denotes the concatenation of n-loops. For an infinite number of n-loops, one can divide the unit n-cube into infinitely many n-cubes whose diameters tend to zero (see [5]). Since $[f_k^m] \in \pi_n(Y_m, x_*)$, the definition makes sense. That is f_k is a well-defined continuous n-loop at x_* because whenever m becomes greater, $im(f_k^m) \subseteq Y_m$ becomes smaller. To prove continuity of f, by Lemma 1.2, it suffices to show that for each $\{\{[f_k^m]P_{m+1}\}_{k\in\mathbb{N}}\}_{m\in\mathbb{N}} \in \prod_{m\in\mathbb{N}} F_m$, the sequence $\{f_k\}_{k\in\mathbb{N}}$ defined in equation (3), is nullconvergent. Equivalently, we must show that the sequence $\{m_k\}_{k\in\mathbb{N}}$ converges to infinity. That is for every $M \in \mathbb{N}$, there is $K \in \mathbb{N}$ such that if $k \geq K$, then $m_k > M$. For each $M \in \mathbb{N}$, put $K = \max\{k;\ m_k \leq M\} + 1$. Since the set $\{k;\ m_k \leq M\}$ is finite, the natural number K exists. If $\{k;\ m_k \leq M\}$ is infinite, then for some m, $1 \leq m \leq M$, $[f_k^m]P_{m+1}$ is not the identity for an infinite subset A of $\{k \in \mathbb{N}\}$. This is a contradiction because $\{[f_k^m]P_{m+1}\}_{k\in\mathbb{N}} \in F_m = \bigoplus_{\aleph_0} \frac{P_m}{P_{m+1}}$ for each $m \in \mathbb{N}$ and it implies that $[f_k^m]P_{m+1}$ is the identity element for each $k \in \mathbb{N}$ except a finite number. Therefore the set $\{k;\ m_k \leq M\}$ is finite. Now if $k \geq K$, then $k > \max\{k;\ m_k \leq M\}$. That is $m_k > M$. Hence for $M \in \mathbb{N}$, there exists $K \in \mathbb{N}$ such that if $k \geq K$ then $m_k > M$. Therefore $\{m_k\}_{k\in\mathbb{N}}$ converges to infinity. To verify $\psi([f]) = \{\{[f_k^m]P_{m+1}\}_{k\in\mathbb{N}}\}_{m\in\mathbb{N}}$, by the definition of homomorphism ψ, we must check the equality $[R_m \circ f|_{\mathbb{S}_k^n}]P_{m+1} = [f_k^m]P_{m+1}$ for each $m, k \in \mathbb{N}$. Fix $k \in \mathbb{N}$ and $f_k = f|_{\mathbb{S}_k^n}$. First assume that $m < m_k$. Since $m + 1 \leq m_k$, $Y_{m_k} \subseteq Y_{m+1}$, and then

$$im(f_k) = im(f_k^{m_k} \cdot R_{m_k+1} \circ \left(f_k^{m_k}\right)^{-1} \cdot f_k^{m_k+1} \cdot R_{m_k+2} \circ \left(f_k^{m_k+1}\right)^{-1} \cdot \ldots) \subseteq Y_{m_k} \subseteq Y_{m+1}.$$

Thus $[f_k]$ can be considered as an element of $P_{m+1} = \pi_n(Y_{m+1}, x_*)$, and then $[R_m \circ f_k]P_{m+1}$ equals the identity element. Moreover, $[f_k^m]P_{m+1}$ equals the identity for $m < m_k$ because m_k is the minimum number for which $[f_k^m]P_{m+1}$ is not the identity. If $m = m_k$, then

$$\begin{aligned}
[R_m \circ f_k] &= [R_{m_k} \circ f_k] \\
&= [R_{m_k} \circ \left(f_k^{m_k} \cdot R_{m_k+1} \circ \left(f_k^{m_k}\right)^{-1} \cdot f_k^{m_k+1} \cdot R_{m_k+2} \circ \left(f_k^{m_k+1}\right)^{-1} \cdot \ldots\right)] \\
&= [R_{m_k} \circ f_k^{m_k} \cdot R_{m_k} \circ R_{m_k+1} \circ \left(f_k^{m_k}\right)^{-1} \cdot R_{m_k} \circ f_k^{m_k+1} \cdot \ldots] \\
&= [R_{m_k} \circ f_k^{m_k}][R_{m_k} \circ R_{m_k+1} \circ \left(f_k^{m_k}\right)^{-1} \cdot R_{m_k} \circ f_k^{m_k+1} \cdot \ldots] \\
&= [R_{m_k} \circ f_k^{m_k}][R_{m_k+1} \circ \left(f_k^{m_k}\right)^{-1} \cdot f_k^{m_k+1} \cdot R_{m_k+2} \circ \left(f_k^{m_k+1}\right)^{-1} \cdot \ldots],
\end{aligned}$$

and thus

$$\begin{aligned}
[R_m \circ f_k]P_{m+1} &= [R_{m_k} \circ f_k]P_{m_k+1} \\
&= [R_{m_k} \circ f_k^{m_k}][R_{m_k+1} \circ \left(f_k^{m_k}\right)^{-1} \cdot f_k^{m_k+1} \ldots]P_{m_k+1} \\
&= [R_{m_k} \circ f_k^{m_k}]P_{m_k+1} \\
&= [f_k^m]P_{m+1},
\end{aligned}$$

as desired. Now let $m = m_k + l$ for some $l = 1, 2, \ldots$. Since $R_{m_k+l} \circ R_{m_k+b} = R_{m_k+l}$ for $b \leq l$, we have

$$\begin{aligned}
[R_m \circ f_k] &= [R_{m_k+l} \circ f_k] \\
&= [R_{m_k+l} \circ \left(f_k^{m_k} \cdot R_{m_k+1} \circ \left(f_k^{m_k}\right)^{-1} \cdot f_k^{m_k+1} \cdot R_{m_k+2} \circ \left(f_k^{m_k+1}\right)^{-1} \cdot \ldots\right)] \\
&= [R_{m_k+l} \circ f_k^{m_k} \cdot R_{m_k+l} \circ R_{m_k+1} \circ \left(f_k^{m_k}\right)^{-1} \cdot R_{m_k+l} \circ f_k^{m_k+1} \cdot \\
&\quad R_{m_k+l} \circ R_{m_k+2} \circ \left(f_k^{m_k+1}\right)^{-1} \cdot \ldots \cdot R_{m_k+l} \circ f_k^{m_k+l} \cdot \\
&\quad R_{m_k+l} \circ R_{m_k+l+1} \circ \left(f_k^{m_k+l}\right)^{-1} \cdot \ldots]
\end{aligned}$$

$$= [R_{m_k+l} \circ f_k^{m_k} \cdot R_{m_k+l} \circ \left(f_k^{m_k}\right)^{-1} \cdot R_{m_k+l} \circ f_k^{m_k+1} \cdot R_{m_k+l} \circ \left(f_k^{m_k+1}\right)^{-1} \cdot$$
$$R_{m_k+l} \circ f_k^{m_k+2} \cdot \ldots \cdot R_{m_k+l} \circ f_k^{m_k+l} \cdot R_{m_k+l+1} \circ \left(f_k^{m_k+l}\right)^{-1} \cdot \ldots]$$
$$= [R_{m_k+l} \circ f_k^{m_k+l}][R_{m_k+l+1} \circ \left(f_k^{m_k+l}\right)^{-1} \cdot \ldots],$$

and thus,

$$\begin{aligned}
[R_m \circ f_k]P_{m+1} &= [R_{m_k+l} \circ f_k]P_{m_k+l+1} \\
&= [R_{m_k+l} \circ f_k^{m_k+l}][R_{m_k+l+1} \circ \left(f_k^{m_k+l}\right)^{-1} \cdot \ldots]P_{m_k+l+1} \\
&= [R_{m_k+l} \circ f_k^{m_k+l}]P_{m_k+l+1} \\
&= [R_m \circ f_k^m]P_{m+1} \\
&= [f_k^m]P_{m+1},
\end{aligned}$$

and we are done. That is for each $m, k \in \mathbb{N}$, $[R_m \circ f_k]P_{m+1} = [f_k^m]P_{m+1}$. Therefore $\psi([f]) = \big\{\{[f_k^m]P_{m+1}\}_{k\in\mathbb{N}}\big\}_{m\in\mathbb{N}}$. Hence $\prod_{m\in\mathbb{N}} F_m \subseteq \psi\big(\mathcal{H}_n(X, x_*)\big)$ and then $\prod_{m\in\mathbb{N}} F_m = \psi\big(\mathcal{H}_n(X, x_*)\big)$.

Consequently, we have isomorphism $\psi : \mathcal{H}_n(X, x_*) \to \prod_{m\in\mathbb{N}} F_m$, where $F_m = \bigoplus_{\aleph_0} \frac{\pi_n(\widetilde{\bigvee}_{i\geq m} X_i, x_*)}{\pi_n(\widetilde{\bigvee}_{i\geq m+1} X_i, x_*)}$, and then $\mathcal{H}_n(X, x_*) \cong \prod_{m\in\mathbb{N}} \bigoplus_{\aleph_0} \frac{\pi_n(\widetilde{\bigvee}_{i\geq m} X_i, x_*)}{\pi_n(\widetilde{\bigvee}_{i\geq m+1} X_i, x_*)}$. □

Note that Theorem 4.1 is a generalization of [2, Theorem 2.10] as follows.

Theorem 4.2 *([2]). Let $n \geq 2$. Suppose that X_i is an $(n-1)$-connected Tikhonov space which is locally strongly contractible at the base point x_i for $i \in \mathbb{N}$. If $(X, x_*) = \widetilde{\bigvee}_{i\in\mathbb{N}}(X_i, x_i)$, then*

$$\mathcal{H}_n(X, x_*) \cong \prod_{i\in\mathbb{N}} \bigoplus_{\aleph_0} \pi_n(X_i, x_i).$$

By [5, Theorem 1.1], $\pi_n(\widetilde{\bigvee}_{i\geq m} X_i, x_*) \cong \prod_{i\geq m} \pi_n(X_i, x_i)$ for all $m \in \mathbb{N}$, and also $\frac{\pi_n(\widetilde{\bigvee}_{i\geq m} X_i, x_*)}{\pi_n(\widetilde{\bigvee}_{i\geq m+1} X_i, x_*)} \cong \pi_n(X_m, x_m)$. Therefore

$$\mathcal{H}_n(X, x_*) \cong \prod_{m\in\mathbb{N}} \bigoplus_{\aleph_0} \frac{\pi_n(\widetilde{\bigvee}_{i\geq m} X_i, x_*)}{\pi_n(\widetilde{\bigvee}_{i\geq m+1} X_i, x_*)} \cong \prod_{m\in\mathbb{N}} \bigoplus_{\aleph_0} \pi_n(X_m, x_m).$$

In [2, Corollary 2.11], as a consequence of Theorem 4.2, we establish the nth Hawaiian group of the m-dimensional Hawaiian earring up to isomorphism as follows. For $m, n \geq 2$, if $n < m$, then $\mathcal{H}_n(\mathbb{HE}^m, x)$ is trivial for arbitrary point $x \in \mathbb{HE}^m$, and if $\theta \neq a \in \mathbb{HE}^m$, then

$$\mathcal{H}_m(\mathbb{HE}^m, \theta) \cong \prod_{\aleph_0} \bigoplus_{\aleph_0} \mathbb{Z}, \qquad \mathcal{H}_m(\mathbb{HE}^m, a) \cong \bigoplus_{\aleph_0} \prod_{\aleph_0} \mathbb{Z}.$$

In the following corollary we study the case $n > m$. Since $\mathbb{HE}^m = \widetilde{\bigvee}_{i\in\mathbb{N}} \mathbb{S}_i^m$ satisfies the hypotheses of Theorem 4.1, Corollary 4.3 is an immediate consequence of Theorem 4.1, when we replace X_i with $\mathbb{S}_k^m$ in the isomorphism (2).

Corollary 4.3. *Let $m, n \in \mathbb{N}$ and $n > m$. Then for $n \geq 2$*

$$\mathcal{H}_n(\mathbb{HE}^m, \theta) \cong \prod_{i\in\mathbb{N}} \bigoplus_{\aleph_0} \frac{\pi_n(\widetilde{\bigvee}_{k\geq i}\mathbb{S}^m_k, \theta)}{\pi_n(\widetilde{\bigvee}_{k\geq i+1}\mathbb{S}^m_k, \theta)}.$$

Some homotopy and Hawaiian groups of finite dimensional Hawaiian earrings were studied in [4] and [2] respectively. These techniques use $(n-1)$-connectedness of n-spheres of the n-dimensional Hawaiian earring. But since dimension of spheres of the infinite dimensional Hawaiian earring is not bounded, $(n-1)$-connectedness does not hold except for $n = 1$, that is the first homotopy and Hawaiian groups which were investigated in previous sections. Now by Lemma 2.1, $\mathbb{HE}^\infty \simeq \widetilde{\bigvee}_{i\in\mathbb{N}}\mathbb{S}^{r_i}_i$ where r_i is the ith term of sequence (1), and then $\mathbb{HE}^\infty$ satisfies the hypotheses of Theorem 4.1. Therefore, replace X_i with $\mathbb{S}^{r_i}_i$ in isomorphism (2) and obtain the corollary stated below.

Corollary 4.4. *Let r_i be the ith term of sequence*

$$1, 2, 1, 2, 3, \ldots, 1, \ldots, n, 1, \ldots, n, n+1, \ldots, \tag{1}$$

and $n \geq 2$. Then

$$\mathcal{H}_n(\mathbb{HE}^\infty, \theta) \cong \prod_{m\in\mathbb{N}} \bigoplus_{\aleph_0} \frac{\pi_n(\widetilde{\bigvee}_{i\geq m}\mathbb{S}^{r_i}_i, \theta)}{\pi_n(\widetilde{\bigvee}_{i\geq m+1}\mathbb{S}^{r_i}_i, \theta)}.$$

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