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Reconstructing Image From Noisy Radon Projections Using Shearlet Transform

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Abstract—The Radon transform on the two-dimensional space, called X-ray transform, is reviewed. Some methods to reconstruct an X-ray image from its sinogram are investigated. Important details are lost when the shearlet transform is used to retrieve an image from its noisy Rdadon data. We present a new theorem to relate the shearlet and Radon transforms to come up with this challenge. An optimum threshold to use is obtained. The algorithm of fast finite shearlet transform is improved, and numerical results show the notable superiority of the proposed method over other existing algorithms.

Index Terms—X-ray transform, shearlet transform, image reconstruction, noisy data

I. Introduction

Almost after a half of a century later, Soviet and American writers [Stein (1972); Vainshtein and Orlove (1972);] West and Cormac (1973) cite Radon's paper in 1917 as a basis for re-constructing projections[4]. Fritz John then revived the subject in important papers during the 1930s and found significant applications in differential equations. Now, the Radon transform in Euclidean space is a suitable method for a large group of reconstruction problems. The X-ray transform is a special case for Radon transform, see [1], [2]. The X-ray images, called CT-scan show the interior parts of a body in different shades of black and white. X-rays are emitted to objects from an under control source with known initial energy. During the process, part of the radiation is absorbed and Different tissues absorb different amounts of the radiation. The main problem is to determine attenuation function from a noisy Radon dataset. In recent years, useful mathematical tools such as wavelet, curvelet and shearlet transform are presented so some new ideas in inverse problems, noise removing and approximation theory are implemeneted. If a ray satisfies the Beer-Lambert conditions and passes a distance x through a homogeneous tissue, in which case $\frac{I}{I_0} = e^{-\mu \cdot x}$, where μ is the desired attenuation coefficient of the tissue. Using this formula for a beam which passes throught the path L, we have

$$p = (R\mu)(\rho, \phi) = -\log(\frac{I}{I_0}) = \int_L \mu(x, y)ds, \qquad (1)$$

where ρ is the distance of the line to the origin,ϕ is the specified angle and R denotes the Radon transform. In (1), p is called a projection. Varying the value of ϕ make a profile matrix that can be shown as a sinogram. The Fourier-Slice Theorem plays a key role in the reconstructing an image from its corresponding sinogram. Let $f \in L^1(\mathbb{R}^2)$. For each $\omega \in \mathbb{R}$ and $\theta \in [0, 2\pi]$ then

$$\mathfrak{F}_2 f(\omega\cos\theta, \omega\sin\theta) = \mathfrak{F}_1(Rf)(\omega, \theta), \qquad (2)$$

where $\mathfrak{F}_2$ is the two-dimensional Fourier transform and $\mathfrak{F}_1$ is the Fourier transform corresponding to the variable of s. Let B be the back projection operator then for the function $Rf(t, \theta) \in L^1([0, 2\pi] \times \mathbb{R})$ we have

$$BRf(x, y) = \frac{1}{\pi}\int_0^{\pi} Rf(x\cos\theta + y\sin\theta, \theta)d\theta, \qquad (3)$$

The following theorem is the bedrock for the method of F.B.P. .

theorem 1: Let $f \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ then

$$f(x, y) = \frac{1}{2}B\{\mathfrak{F}_1^{-1}[|\omega|\mathfrak{F}_1(Rf)(\omega, \theta)]\}(x, y). \qquad (4)$$

In (4), retrieving an image f just by applying the back projection operator on acquired data is impossible due to $|\omega|$. However it acts as a special filter for Rf . Suppose that ϕ is a function with the limited frequency band, assuming $g = \mathfrak{F}_1\phi$ then

$$f(x, y) \approx \frac{1}{2}B(\mathfrak{F}_1^{-1}g * Rf)(x, y).$$

General rule for the the most of low-pass filters of g is $g(\omega) = |\omega| \cdot \mathfrak{F}(\omega)\pi_L(\omega)$, where π_L is the indicator function

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of the interval $[-L, L]$. One can see the definition of the shearlet transform in detail in [4]. The following theorem is the basis for the proposed algorithm.

theorem 2: Let $f \in L^2(\mathbb{R}^2)$ then

$$SH_\psi f(a,s,b) = \int_0^\pi \int_{-\infty}^{\infty} \mathfrak{F}_1(Rf(\omega,\theta)) \overline{\hat{\psi}_{a,s,b}(\omega\cos\theta, \omega\sin\theta)}|\omega| d\omega d\theta. \quad (5)$$

II. Discretization of Transforms

When $2M+1$ parallel and equal distance (at a distance of d) beams are emitted at in p different directions of θ_k then

$$\theta_k = \{\frac{k\pi}{p};\ 0 \le k \le p-1\}, \quad t_j = \{jd := -M \le j \le M\}.$$

Assuming the function $f \in L^1(\mathbb{R}^2)$ and $p, M \in \mathbb{N}$, the discrete Radon transform of f along the lines l_{t_j,θ_k} is defined as $R_D f[j,k] = Rf(t_j, \theta_k)$. In Fig. 1, a phantom image and its matrix data $[R_D f[j,k]]$ has been shown. This image is used as a standard image to test various algorithms in the field of medical image processing, see [5]. For a discrete N - periodic function g, the discrete form for Fourier transform and its inverse see [3], [4]. To get the discrete form for the back projection operator, in (3)-we use sigma instead of the integral sign and $0 \le k \le p-1$ instead of $0 \le \theta \le \pi$ and we replace $d\theta$, by $\frac{k\pi}{p}$. The discrete function g is considered. The discrete back projection operator in (3) is defined as follows

$$Bg(x,y) = \frac{1}{N}\sum_{k=0}^{N-1} g(x\cos\frac{k\pi}{N} + y\sin\frac{k\pi}{N}, \frac{k\pi}{N})$$

Finally, the reconstruction formula $f(x,y) \simeq \frac{1}{2}B(\mathfrak{F}_1^{-1}g * Rf)(x,y)$ is discretized as

$$f(x,y) \approx \frac{1}{2N}\sum_{k=0}^{N-1} h(x\cos\frac{k\pi}{N} + y\sin\frac{k\pi}{N}, \frac{k\pi}{N}),$$

where $h = \mathfrak{F}_1 g * Rf$. Fig. 2 shows the result of retrieving the phantom with and without the filtering. In 2016 Mu et al. have modified the F. B. P. method to retrieve an image from its noisy data, see [6]. In theorem 1, the sets Θ and Ω result from the discretization of the intervals $[0, \pi]$ and $(-\infty, +\infty)$ Then (5) can be written as

$$SH_\psi f(a,s,b) = \Sigma_{\theta\in\Theta}\Sigma_{\omega\in\Omega}\mathfrak{F}_1(Rf(\omega,\theta)) \overline{\hat{\psi}_{a,s,b}(\omega\cos\theta, \omega\sin\theta)}|\omega|.$$

The inverted discrete shearlet transform is applied to recover a phantom image from its noisy Radon data.

III. Numerical Result

We have data about Rf only in the polar coordinate systems, so to find the value of f at the points (x_m, y_n) we need interpolation. In table II, the results of combining different types of filters and interpolation methods are presented.

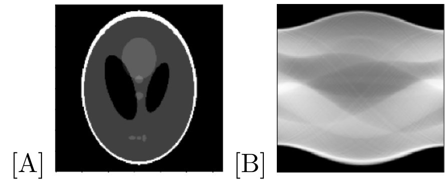

Fig. 1. (A) Phantom image and (B) sinogram

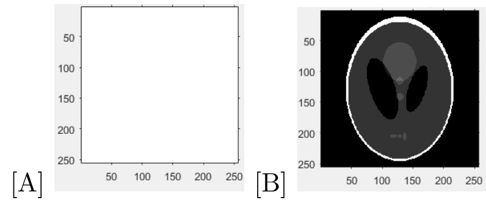

Fig. 2. (A) Retrieving phantom without filtering and (B) reconstructed sinogram with filtering and interpolation

The results of the comparision of proposed algorithm with the other three investigated methods are shown in Table I. We applied all algorithms to reconstruct a phantom image from its Radon data in the presence of the Gaussian noise with mean $\mu = 0$ and variance $\sigma^2 = 0.0025$. the results show that the modified method of F. B. P. by Mu and his colleagues, M. F. B. P., is more accurate and faster in performance than the F. B. P. method. Although the F. F. SH. method has better results than M. F. B. P. but our proposed method-based on the theorem 1 is the best among them see table I.

IV. Concolusion

In table II the numerical results of using some different filters and interpolation methods in the B. P. method indicates that promoting the recovery accuracy is not significant by changing the method of interpolation or the type of filters. So one should think to modify the method. Three different ways to reconstruct an image from noisy Radon data are investigated. Numerical results show that our proposed method based on the new theorem 1 is the best among them see table I.

TABLE I
Comparision of the proposed method with other algorithms

Reconstructing image	Corr.	MSE	PSNR
F. B. P.	0.98059	0.00183	68
M. F. B. P.	0.98042	0.00184	73
F. F. SH.	0.97066	0.00276	81
P. F. F. SH.	0.95357	0.00244	84

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TABLE II
Filter types and interpolation methods in F. B. P.

Interpolation Method	Linear		Spline	
Filter Type	Corr.	MSE	Corr.	MSE
Ram-Lak	0.9805	0.0018	0.9804	0.0018
Shepp-Logan	0.9842	0.0014	0.9816	0.0017
Cosine	0.9833	0.0015	0.9808	0.0017
Haan	0.9829	0.0015	0.9804	0.0018

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