Published for SISSA by Springer

Received: *November 15, 2011*
Accepted: *December 2, 2011*
Published: *December 19, 2011*

Stringy instantons in SU(N) $\mathcal{N} = 2$ non-conformal gauge theories

JHEP12(2011)070

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Abstract: In this paper we explicitly obtain the leading corrections to the SU(N) $\mathcal{N} = 2$ prepotential due to stringy instantons both in flat space-time and in the presence of a non-trivial graviphoton background field. We show that the stringy corrections to the prepotential are expressible in terms of the elementary symmetric polynomials. For $N > 2$ the theory is not conformal; we discuss the introduction of an explicit dependence on the string scale α' in the low-energy effective action through the stringy non-perturbative sector.

Keywords: D-branes, Solitons Monopoles and Instantons, Brane Dynamics in Gauge Theories

ArXiv ePrint: 1111.0842

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doi:10.1007/JHEP12(2011)070

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1 Introduction

In the last fifteen years, the non-perturbative sector of supersymmetric gauge theories [1, 2] has been deeply investigated employing string theory techniques. The seminal papers [3–5] firstly explored the connection between D-branes in string theory with non-perturbative corrections in the corresponding low-energy effective field theories. In fact, it has been shown that the ordinary field theory instanton calculus can be naturally and precisely rephrased in terms of D-brane models; for instance, the ADHM construction yielding a parameterization of the instanton moduli space can be read in terms of D-branes bound states (see for instance [6, 7]).

Several important results and explicit computations within the context of ordinary instanton calculus in gauge theory have been obtained using the string approach. Let us just mention as an instance the detailed analysis and checks of Seiberg-Witten duality, and of string dualities involving D-instantons [8–11]. Moreover, in addition to the ordinary instanton calculus, the string theory framework opens the possibility of investigating new features in non-perturbative physics. This very interesting scenario consists in generalizing

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the ordinary instanton models. Indeed, it is possible to consider D-brane configurations which yield new kinds of non-perturbative corrections to the underlying effective gauge theories [12, 13]. Such generalized instantonic configurations are commonly referred to as *exotic* or *stringy* (for a review see for instance [14]) since their field theoretical interpretation is generally still an open question [15].

The stringy instanton configurations present many peculiar and interesting features which are worth studying carefully. Within the context of D-brane constructions, there is always a D-brane stack whose world-volume contains the four-dimensional spacetime. We refer to these branes as the *gauge* branes. In our specific model, the world-volume of the gauge brane stack coincides with the physical spacetime. As opposed to the gauge branes, the *instanton* branes are by definition point-like from the four-dimensional spacetime perspective.[1]

In a D-brane setup describing an ordinary instanton, the only difference between gauge and instanton branes lies on the fact that while the gauge branes are extended along the 4-dimensional spacetime directions, the instanton branes are here localized: for ordinary configurations the geometrical arrangement and transformation behavior in the internal space is identical between gauge and instanton branes. The hallmark of the exotic character for an instanton configuration is instead represented by a different internal space geometry or symmetry properties between the gauge and the instanton branes. In the D7/D3 systems leading to exotic instantons, the distinction is usually given by different wrappings of the gauge and the instanton branes in the internal space, namely a different geometrical arrangement. In the exotic configurations of our D3/D(-1) model, instead, the internal geometry of the D3 and the D(-1) branes is the same however they behave differently under the orbifold action. As we will see in section 3, the fact that instanton and gauge branes share the same arrangement in the internal space leads to an equal classical action for ordinary and exotic instantons.

In general, one of the main consequences of the different internal behavior between gauge and instanton branes is the absence of charged bosonic moduli associated with open string modes stretching between gauge and instanton branes. For the ordinary instantons, these moduli are associated with the "size" of the instanton itself.[2] In the exotic case under consideration, the charged bosonic moduli are actually projected out by the orbifold action (see [13] for details).

One of the main features of exotic instantons consists in the explicit dependence of their effects on the string scale α'. This implies that we have a signal of stringy aspects within the field theoretical context through the non-perturbative sector of the effective gauge theories. Apart from the intrinsic theoretical interest, this offers a natural introduction of a new scale into the low-energy gauge theory, a feature particularly desirable in a

[1]In the D3/D(-1) model under consideration the D3 branes represent tha gauge branes whereas the D(-1) branes are the instanton branes. In other models, such as the D7/D3 systems, the D7 gauge branes contain the 4-dimensional spacetime and four extra dimensions which are wrapped in the internal space.

[2]Indeed, in some cases it is possible to propose a field theoretical interpretation of the exotic configurations as the "zero size" limit of ordinary instantons (see for instance [15]).

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phenomenological perspective. Actually, stringy effects could lead to an accommodation of some naturalness questions posed by the parameters of phenomenological models such as the see-saw parameters or the hierarchy of the Yukawa couplings involved in GUT models (see for example [14] and references therein). The exotic instanton configurations are able to produce perturbatively prohibited effects like Majorana mass terms and are possible ingredients of SUSY breaking models, particularly because the exotic configurations involve the introduction of features like orientifold planes.

In this paper we generalize the results obtained for SU(2) in [13] to SU(N) gauge theory. In section 2 we provide a brief description of the D3/D(-1) brane setup which we consider with special attention to the moduli content of the model. The D-brane arrangement and the orbifold/orientifold projections lead to a system whose low-energy regime is described by an $\mathcal{N} = 2$ supesymmetric gauge theory with matter transforming in the symmetric representation of the gauge group. As described in section 3, the setup under consideration yields a conformal gauge theory when the number of "colors" is $N = 2$ whereas conformality is lost for $N > 2$. The explicit exotic contributions to the prepotential are computed in section 4 for the lowest values of the instanton topological charge k. In section 5 we study the limit of such corrections in the case of flat background, i.e. vanishing VEV for the graviphoton field. We conclude in section 6 with final remarks and discuss the perspective for future investigation. Eventually, the appendices A, B, C contain usefuls formulæ that have been used throughout the main text.

2 Description of the setup

There are different models presenting exotic instanton configurations (see [14] and references therein). In the present paper we use exactly the same setup and follow the same notation introduced in ref. [13]; there, it has been shown that exploiting Nekrasov's localization techniques [16, 17] one is able to evaluate directly and explicitly the stringy corrections to the low-energy effective action for SU(2) gauge theory. The task we are presently committed to is the elaboration of the same model and its generalization to gauge groups with rank higher than one, namely number of "colors" N bigger that 2.

We focus the attention on a D3/D(-1) model with $\mathbb{Z}_3$ orientifold background along the lines of ref. [13]. Let us here recall the main features of the employed brane model. We consider a system of fractional D-branes, i.e. a set of branes placed at the singularity of the orbifold, which carry different irreducible representations of the orbifold group as firstly discussed in [18]. We choose the Chan-Paton factors of open strings that transform under the orbifold group according to the representations carried by the branes to which the string endpoints are attached. The low-energy limit of the model is described by a gauge theory whose gauge group is encoded in the quiver diagram presented in figure 1.

The extended directions of the D3 branes lie along the first four coordinates while the six internal directions are parametrized with three complex variables z^i, $i = 1, 2, 3$. The orbifold acts on the space spanned by each complex variable z^i with one of its irreducible

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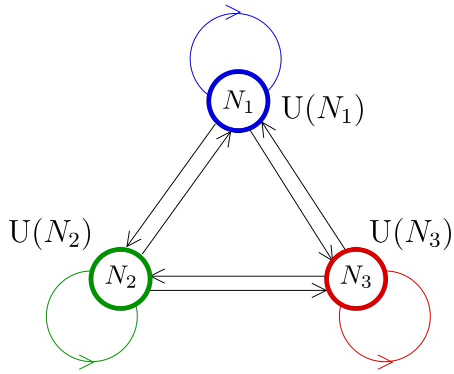

Figure 1. The $\mathbb{C}^3/\mathbb{Z}_3$ orbifold model which corresponds to a configuration of N_1, N_2 and N_3 fractional D3 branes before considering the orientifold projection. The arrows starting and ending on the same node represent $\mathcal{N} = 2$ vector multiplets in the adjoint representation of the $\mathrm{U}(N_i)$ groups. The arrows between different nodes represent bi-fundamental chiral multiplets which pair up into $\mathcal{N} = 2$ hypermultiplets.

representations according to:

$$\begin{pmatrix} z^1 \\ z^2 \\ z^3 \end{pmatrix} \to \begin{pmatrix} \xi\, z^1 \\ \xi^{-1}\, z^2 \\ z^3 \end{pmatrix} , \tag{2.1}$$

where $\xi = e^{\frac{2i\pi}{3}}$. In addition, an O3 plane is added to the background so that its extended directions coincide with the extended directions of the D3 branes. Before implementing the orientifold projection the theory has a hypermultiplet in the bi-fundamental representaion of the gauge groups associated to nodes 2 and 3 of the quiver diagram depicted in figure 1. It is not difficult to show that the orientifolding leads to an identification between these nodes and we end up with a hypermultiplet in the symmetric representation of the gauge group $\mathrm{SU}(N)$ where $N \equiv N_1 = N_2$. With this matter content the one-loop coefficient of the beta-function is $b_1 = 2 - N$. Moreover, orientifolding reduces the instanton group from $\mathrm{U}(k)$ to $\mathrm{SO}(k)$. Note that the representations of the orientifold acting on D-instantons and D3-branes are respectively symmetric and anti-symmetric (see [13]).

We define the following notation in order to account for the D-brane content: $(N_1, N_2) \oplus (k_1, k_2)$ representing N_i D3 and k_i D(-1) branes on the i-th node. Notice that node 3 does not appear because, as just stated, it is identified with node 2. We consider only configurations in which there are no D3-branes on node 1, $N_1 = 0$; according to the results of [13, 18], a configuration presenting both D3 and D(-1) branes on the node 2, $(0, N) \oplus (0, k)$, corresponds to an $\mathrm{SU}(N)$ gauge theory with an ordinary instanton where k represents the instanton number. On the other hand, the configuration $(0, N) \oplus (k, 0)$ is an $\mathrm{SU}(N)$ gauge theory with an exotic k-instanton. Note that the ordinary (exotic) character of the instantonic configurations is associated respectively with the fact that the D3 and the D(-1) branes are (are not) on the same node i.e. are (are not) associate to the same irreducible representation of the orbifold group. The details of the setup

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(Ψ_0, Ψ_1)	$\mathrm{SO}(k)$	$\mathrm{SU}(N)$	$\mathrm{SU}(2) \times \mathrm{SU}(2)'$
(a_μ, M_μ)	$\square\!\square$	$\mathbf{1}$	$(\mathbf{2}, \mathbf{2})$
(λ_c, D_c)	$\begin{smallmatrix}\square\\ \square\end{smallmatrix}$	$\mathbf{1}$	$(\mathbf{1}, \mathbf{3})$
$(\overline{\chi}, \eta)$	$\begin{smallmatrix}\square\\ \square\end{smallmatrix}$	$\mathbf{1}$	$(\mathbf{1}, \mathbf{1})$
(μ, h)	$\square$	$\overline{\mathbf{N}}$	$(\mathbf{1}, \mathbf{1})$
(μ', h')	$\square$	$\overline{\mathbf{N}}$	$(\mathbf{1}, \mathbf{1})$

Table 1. Moduli content of the stringy instanton configuration organized as BRST pairs and their transformation properties under the various symmetry groups.

under considerations are already extensively illustrated in [13], here we only summarize the moduli spectrum in table 1.

3 Explicit dependence on the string scale and renormalization behavior of exotic effects

In the D-brane setups whose low-energy regime yields the ordinary k instanton, the dimension of the integration measure $d\mathcal{M}_k^{(\mathrm{ord})}$ on the instanton moduli space is related to the one-loop coefficient b_1 of the β-function,

$$\left[d\mathcal{M}_k^{(\mathrm{ord})}\right] = (\mathrm{length})^{kb_1} \ . \tag{3.1}$$

In order to have a dimensionless instanton action, we introduce a dimensionful prefactor to compensate (3.1). This dimensionful coefficient in front of the ordinary k instanton moduli integral is given by (see for instance [2]):

$$\mu^{kb_1} e^{-\frac{8\pi^2}{g^2_{\mathrm{YM}}}k} = \Lambda^{kb_1}, \tag{3.2}$$

where μ represents the high-energy renormalization mass scale and $\exp(-8k\pi^2/g^2_{\mathrm{YM}})$ is the classical part of the instanton action. In a stringy context, μ is naturally related to the string scale α',

$$\mu \sim \frac{1}{\sqrt{\alpha'}}. \tag{3.3}$$

Note that the presence of μ introduces then an explicit dependence on the string scale.

Equation (3.2) defines the dynamically generated infrared scale Λ which in non-abelian gauge theory is indeed connected with non-perturbative effects. Furthermore, it is crucial to observe that the dimensionful factor (3.2) associated with the ordinary instanton moduli integral can be expressed in terms of Λ alone, that is to say, the dependence on α' disappears. In other terms, the dependence on the high-energy renormalization scale is transmuted into the low-energy scale Λ which is completely interpretable in terms of the underlying effective field theory description.

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As already mentioned, stringy instantons present different features with respect to the ordinary cases. According to ref. [13], the dimension of the moduli space in the $(0, N) \oplus (k, 0)$ exotic configuration is

$$\left[d\mathcal{M}_k^{(\mathrm{ex})}\right] = (\mathrm{length})^{k(2-N)} = (\mathrm{length})^{-kb_1}. \tag{3.4}$$

In ref. [13] we specialized the analysis to the peculiar situation with SU(2) gauge theory possessing the one-loop coefficient of the beta function b_1 equal to zero and being therefore conformal. In the conformal case, even the exotic contributions lead to dimensionless non-perturbative corrections which do not introduce explicitly the string scale α' in the low-energy effective theory.

Notice that in (3.4) the exponent has a different sign if compared with (3.1). As a consequence, the dimensionful prefactor in front of the exotic moduli integral will be as follows:

$$\mu^{-kb_1} e^{-\frac{8\pi^2}{g^2_{\mathrm{YM}}}k}. \tag{3.5}$$

Two comments are in due here: The first observation is related to the classical part of the action which, in our model, is the same for both ordinary and exotic configurations. The reason why this happens is described in detail in the end of this section. The second point is that the dimensionful factor (3.5) in front of the exotic instanton moduli integral cannot be expressed in terms of Λ alone.

Using the exponentiated renormalization group equation (3.2) together with (3.3), we have that the dimensionful exotic prefactor is proportional to

$$(\alpha')^{kb_1} \Lambda^{kb_1}. \tag{3.6}$$

As anticipated, the string scale is manifestly present in the exotic corrections to the pre-potential and couplings of the underlying low-energy gauge theory.

We noted that the classical contribution of ordinary and exotic instantons within our model is the same. This fact may seem unexpected but becomes natural as we observe that, in contrast to other possible brane models, in our setup the gauge and the instanton branes have the same geometry in the internal space. If this would not be the case, it is in general possible to have a different classical contribution for exotic and ordinary configurations (see [14]). Let us underline once more that in the model under consideration the feature marking the difference between ordinary and the exotic instanton setups lies in the different transformation properties of the CP factors under the action of the orbifold.

4 Stringy corrections to the prepotential

4.1 Preliminaries

Nekrasov's localization technique[3] plays a crucial rôle in performing the explicit evaluation of the partition function of the model under analysis. The extended $\mathcal{N} = 2$ supersymmetry

[3]For details about Nekrasov's localization technique and its applications see e.g. [16, 17].

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of the system allows us to define a "singlet" linear combination of the supercharges

$$Q' \equiv Q_{\dot{\alpha}\dot{\beta}}\epsilon^{\dot{\alpha}\dot{\beta}}\,, \tag{4.1}$$

representing a fermionic operator which can be shown to be nilpotent and interpretable as a BRST operator. Indeed, the action is Q' exact i.e. it can be written as the Q' variation of an appropriate fermionic expression Ξ usually referred to as the *gauge fermion*,

$$S = Q'\,\Xi\,. \tag{4.2}$$

The nilpotency of Q' is consistent with the fact that the action S is supersymmetric. The BRST structure of the field content of the model allows us to rescale the fields and consider a particular limit known as the *localization limit* in which the action becomes quadratic in the fields. The limit resembles formally a saddle-point approximation but it can be proven that, because of the BRST structure of the model, it leads to exact results.

We are interested in computing the stringy non-perturbative partition functions corresponding to the lowest values of the instanton charge k. The total partition function is given by summing over all k-instanton partition functions as the following

$$Z = \sum_{k=0}^{\infty} q^k Z_k\,, \tag{4.3}$$

where the dimensionful parameter q is defined as

$$q \equiv \mu^{b_1} e^{2\pi \mathrm{i}\tau} \tag{4.4}$$

with b_1 the one-loop coefficient of the β-function. Notice that b_1 as seen in (3.4) grows as the number of "colors" N increases. Only for the case $N = 2$ the coefficient b_1 vanishes and we deal with a conformal theory. In other cases, as discussed in section 3 the partition function Z_k has dimension μ^{-kb_1} which is compensated by the dimension of the prefactors q^k yielding a dimensionless partition function Z. The total prepotential is related to the partition function being proportional the logarithm of the function Z (see e.g. [9]),

$$F^{(n.p.)} = \mathcal{E}\log Z \tag{4.5}$$

where $\mathcal{E}$ as will be seen later, is the product of the eigenvalues of the graviphoton field. Expanding the logarithm, the total prepotential is analogously obtained by adding up the contributions from all instanton charges,

$$F^{(n.p.)} = \sum_{k=1}^{\infty} q^k F_k. \tag{4.6}$$

The coefficients F_k's are recursively obtained using equations (4.3) and (4.5). The three leading coefficients which are of interest in this paper are given by

$$F_1 = \mathcal{E}Z_1, \tag{4.7a}$$

$$F_2 = \mathcal{E}Z_2 - \frac{F_1^2}{2\mathcal{E}}, \tag{4.7b}$$

$$F_3 = \mathcal{E}Z_3 - \frac{F_1F_2}{\mathcal{E}} - \frac{F_1^3}{6\mathcal{E}^2}. \tag{4.7c}$$

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Exploiting the localization techniques, the partition function Z_k can be expressed as follows:

$$Z_k = \mathcal{N}_k \int \frac{d\chi}{2\pi \mathrm{i}} \frac{\mathcal{P}(\chi)\mathcal{R}(\chi)}{\mathcal{Q}(\chi)} \ . \tag{4.8}$$

The functions $\mathcal{P}(\chi)$, $\mathcal{R}(\chi)$ and $\mathcal{Q}(\chi)$ are given by

$$\mathcal{P}(\chi) \equiv \mathrm{Pf}_{\left(\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array}{}_{,\mathbf{1},(\mathbf{1},\mathbf{3})'}\right)}(Q'^2), \tag{4.9a}$$

$$\mathcal{R}(\chi) \equiv \det{}_{\left(\square_{,\bar{\mathbf{N}},(\mathbf{1},\mathbf{1})}\right)}(Q'^2), \tag{4.9b}$$

$$\mathcal{Q}(\chi) \equiv \det{}^{1/2}_{\left(\begin{array}{|c|c|}\hline \ & \ \\ \hline\end{array}{}_{,\mathbf{1},(\mathbf{1},\mathbf{2})}\right)}(Q'^2). \tag{4.9c}$$

Here we follow the same notation and conventions introduced in [13] to which we refer for further details. We simply recall that the subscripts in equations (4.9) denote the representations of the instanton group, gauge group and Lorentz group respectively. The action of the BRST charge squared Q'^2 on a generic modulus $\bullet$ is given by:

$$Q'^2 \bullet = T_{\mathrm{SO}(k)}(\chi)\bullet - T_{\mathrm{SU}(N)}(\phi)\bullet + T_{\mathrm{SU}(2)\times\mathrm{SU}(2)'}(\mathcal{F})\bullet\,, \tag{4.10}$$

where the T's are the infinitesimal transformations of the groups SO(k), SU(N) and SU(2)$\times$SU(2)$'$ in the appropriate representations. The matrices $\chi_{N\times N}$, $\phi_{k\times k}$ and $\mathcal{F}_{4\times 4}$ are respectively the gauge parameters of the groups above. Let us remind that ϕ is the scalar field belonging to the gauge sector (i.e. the open string states with both ends on the D3's), χ is the instanton D$(-1)/D(-1)$ modulus we have already introduced and $\mathcal{F}$ is the close string graviphoton field. The integral in (4.8) is clearly singular in correspondence to the zeros of the denominator $\mathcal{Q}(\chi)$. Moreover, the integrand becomes one in the limit $\chi \to \infty$; this singularity is spurious and can be cured as described in [19]. In the integral (4.8) the modulus χ has been assumed real. By shifting the zeros of $\mathcal{Q}$ by a small positive value along the imaginary axis, say iϵ, the integration over χ can be treated as a contour integral in the complex upper half-plane. This as well regularizes the divergence at infinity. Nevertheless, the direct evaluation of (4.8) is possible only by brute-force for just small instanton numbers and small gauge group ranks; we can however use all the symmetries of the theory in order to shorten the length of the calculations and making it possible to extend the approach also to higher instanton numbers [13] and higher gauge group ranks.

The partition function given in (4.8) is invariant under three symmetry groups: the instanton group SO(k), the gauge group SU(N) and the Lorentz group SU(2) $\times$ SU(2)$'$. One can take advantage of these symmetries to express all the fields in the Cartan basis in which the integration becomes simpler. Changing the basis for the fields ϕ and $\mathcal{F}$ associated with the gauge group and the Lorentz group respectively, causes no change in the integrand of (4.8), however expressing χ into the Cartan subalgebra is possible at a price of introducing a Vandermonde determinant, because the modulus χ appears in the integration measure;

$$\chi \to \overrightarrow{\chi} \cdot \overrightarrow{H}_{\mathrm{SO}(k)} = \chi_i H^i_{\mathrm{SO}(k)} \tag{4.11}$$

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$H^i_{\mathrm{SO}(k)}$ being the Cartan generators of SO(k) group and $i = 1, \ldots, \mathrm{rank}\ \mathrm{SO}(k)$. The partition function then takes the following form

$$Z_k = \mathcal{N}_k \int \prod_i \frac{d\chi_i}{2\pi \mathrm{i}} \Delta(\vec{\chi}) \frac{\mathcal{P}(\vec{\chi})\mathcal{R}(\vec{\chi})}{\mathcal{Q}(\vec{\chi})}\,, \tag{4.12}$$

where $\Delta(\chi)$ denotes the Vandermonde determinant. The functions in the integrand (4.12) are related to a determinant or a Pfaffian of the operator Q'^2 in the appropriate representation to which the moduli belong. The values of such functions is given by the product of all eigenvalues of Q'^2 in the corresponding representations. Following [9], we exploit the weights of the representations to rewrite the functions appearing in the integrand (4.12). The Vandermonde determinant is given by

$$\Delta\,(\vec{\chi}) = \prod_{\vec{\rho}\,\in\,\mathrm{adj}\neq 0} \vec{\chi}.\vec{\rho}, \tag{4.13}$$

where $\vec{\rho}$ is the *root vector* of SO(k). The VEV of the chiral superfield ϕ, expressed in the Cartan basis with $H^u_{\mathrm{SU}(N)}$ the generators of the Cartan subalgebra of the SU(N) group, is given by

$$\phi \to \vec{\phi} \cdot \vec{H}_{\mathrm{SU}(N)} = \sum_u \phi_u H^u_{\mathrm{SU}(N)} \tag{4.14}$$

in which $u = 1, \ldots, \mathrm{rank}\ \mathrm{SU}(N)$. The function $\mathcal{R}(\chi)$ in the numerator of the integrand (4.12) is expressed as

$$\mathcal{R}\,(\vec{\chi}) = \prod_{\vec{\pi}\in \boldsymbol{k}} \prod_{\vec{\gamma}\in N} \left(\vec{\chi}\cdot\vec{\pi} - \vec{\phi}\cdot\vec{\gamma}\right) \tag{4.15}$$

where $\vec{\pi}$ and $\vec{\gamma}$ are the *weight vectors*, respectively, in the vector and fundamental representations of SO(k) and SU(N). Similarly,

$$\mathcal{P}\,(\vec{\chi}) = \prod_{\vec{\rho}\,\in\,\mathrm{adj}} \prod_{\vec{\alpha}}^{(+)} \left(\vec{\chi}\cdot\vec{\rho} - \vec{f}\cdot\vec{\alpha}\right) = \prod_{\vec{\rho}\,\in\,\mathrm{adj}} \prod_{a} (\vec{\chi}\cdot\vec{\rho} - f_a)\;. \tag{4.16}$$

Here $\vec{\rho}$ is again the weight vector (root vector) in the adjoint representation of SO(k) and $\vec{\alpha}$ as explained in [9], is the positive weight vector of the auxiliary SO(3) group associated with the anti-selfdual/anti-chiral part of the Lorentz group (i.e. the group rotating the λ_c's). Therefore a in (4.16) takes only the value $a = 1$. The graviphoton field which belongs to the Lorentz group SU(2)×SU(2)′ along the Cartan directions takes the following form

$$\mathcal{F} = \begin{pmatrix} E_1\sigma_2 & 0 \\ 0 & E_2\sigma_2 \end{pmatrix}\;. \tag{4.17}$$

The function $Q(\vec{\chi})$ in the denominator of the integrand (4.8) is expressed as

$$\mathcal{Q}\,(\vec{\chi}) = \prod_{\vec{\sigma}\,\in\,\mathrm{sym}} \prod_{\vec{\beta}}^{(+)} \left(\vec{\chi}.\vec{\sigma} - \vec{f}.\vec{\beta}\right) = \prod_{\vec{\sigma}\,\in\,\mathrm{sym}} \prod_{A=1}^{2} (\vec{\chi}.\vec{\sigma} - E_A) \tag{4.18}$$

where $\vec{\sigma}$ denotes the weight vector in the symmetric representation of the instanton group SO(k) and $\vec{\beta}$ is the positive weight of the Lorentz group SU(2)×SU(2)′ which can be read immediately from (4.17).

4.2 1-instanton contribution

In the $k = 1$ case, the instanton group is SO(1) so the Cartan basis is 1-dimensional and $\vec{\chi}$ has only one component χ. Following equations (4.13), (4.16), (4.15), (4.18), the Vandermonde determinant $\Delta(\chi)$ and the functions $\mathcal{P}(\chi)$, $\mathcal{R}(\chi)$, $\mathcal{Q}(\chi)$ are trivially given by

$$\Delta(\chi) = 1\,, \tag{4.19}$$

$$\mathcal{P}(\chi) = 1\,, \tag{4.20}$$

$$\mathcal{R}(\chi) = \prod_{\vec{\gamma}\in N}\left(-\vec{\Phi}\cdot\vec{\gamma}\right)\,, \tag{4.21}$$

$$\mathcal{Q}(\chi) = E_1E_2 = \mathcal{E}\;. \tag{4.22}$$

The partition function (4.12) for $k = 1$ then becomes

$$Z_1 = \frac{\mathcal{N}_1}{\mathcal{E}}\prod_{\vec{\gamma}\in N}\left(-\vec{\Phi}.\vec{\gamma}\right)\,, \tag{4.23}$$

where the $\vec{\gamma}$'s are the weights of SU(N) in the fundamental representation. Using the explicit expression for the weights of the SU(N) group, one obtains (see appendix A)

$$\prod_{\vec{\gamma}\in N}\left(-\vec{\Phi}.\vec{\gamma}\right) = (-1)^N\det(\Phi)\;. \tag{4.24}$$

The exotic 1-instanton partition function for SU(N) gauge theory is therefore

$$Z_1 = \frac{\mathcal{N}_1}{\mathcal{E}}(-1)^N\det(\Phi)\,, \tag{4.25}$$

where $\mathcal{N}_1$ is an overall normalization coefficient.

1-instanton prepotential F_1. The exotic 1-instanton contribution to the prepotential is:

$$F_1 = \mathcal{E}Z_1\,, \tag{4.26}$$

therefore, using (4.25), we obtain

$$F_1 = \mathcal{N}_1(-1)^N\det(\Phi)\;. \tag{4.27}$$

4.3 2-instanton contribution

The $k = 2$ partition function integral reads:

$$Z_2 = (-1)^{N+1}\frac{\mathcal{N}_2 f}{\mathcal{E}}\int\frac{d\chi}{2\pi\mathrm{i}}\frac{\prod_{i=1}^{N}\left[\chi^2-(\Phi\cdot\gamma_i)^2\right]}{\prod_{A=1}^{2}(2\chi-E_A)(-2\chi-E_A)} \tag{4.28}$$

Choosing the contour of integration to be in the upper half-plane, there are only two positive poles, $\chi = \frac{E1}{2}$ and $\frac{E2}{2}$, enclosed in the integration path. The result of the integration is simply given by the sum of the corresponding residues, namely

$$Z_2 = \frac{(-1)^{N+1}\mathcal{N}_2}{4\mathcal{E}\,(E_1-E_2)}\left(\frac{\prod_{i=1}^{N}\left[\left(\frac{E_1}{2}\right)^2-\left(\vec{\phi}\cdot\vec{\gamma_i}\right)^2\right]}{E_1}-\frac{\prod_{i=1}^{N}\left[\left(\frac{E_2}{2}\right)^2-\left(\vec{\phi}\cdot\vec{\gamma_i}\right)^2\right]}{E_2}\right)\,, \tag{4.29}$$

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where we have set $f = E_1 + E_2$. However, this is not the desired form for the result because in the limit of flat background, i.e. $E_1, E_2 \to 0$, the prepotential is seemingly divergent. Moreover, we do not have yet a concrete answer in terms of the physical invariant quantities, namely $\mathrm{tr}\Phi^2$ and $\det\Phi$. The numerators in equation (4.29) need therefore to be further elaborated. In fact, it turns out that the expansions of the polynomials in the numerators of (4.29) are just monomials where the coefficients are given in terms of the elementary symmetric polynomials (see appendix B). Expanding equation (4.29) we obtain

$$Z_2 = \frac{\mathcal{N}_2}{8\mathcal{E}\left(E_1 - E_2\right)} \left\{ \sum_{j=0}^{N-2} (-1)^{N+j+1} e_j\left(X_1, \ldots, X_N\right) \left[\left(\frac{E_1}{2}\right)^{2(N-j)-1} - \left(\frac{E_2}{2}\right)^{2(N-j)-1} \right] \right\} + \frac{\mathcal{N}_2}{16\mathcal{E}} e_{N-1}\left(X_1, \ldots, X_N\right) + \frac{\mathcal{N}_2}{4\mathcal{E}^2} e_N\left(X_1, \ldots, X_N\right) \,, \tag{4.30}$$

where the elementary symmetric polynomials $e_j(X_1, \ldots, X_N)$ with $X_i = \left(\vec{\Phi} \cdot \vec{\gamma}_i\right)^2$ have been defined in appendix B. The advantage of using elementary symmetric polynomials is that they can be translated into power sums which in turn are essentially polynomials in $\mathrm{tr}\Phi^2$. From (B.13), (B.12) and (A.6) one has

$$e_j\left(\left(\vec{\phi} \cdot \vec{\gamma_1}\right)^2, \ldots, \left(\vec{\phi} \cdot \vec{\gamma_N}\right)^2\right) = \frac{1}{j!} \begin{vmatrix} \mathrm{tr}\Phi^2 & 1 & 0 & \cdots & \\ \mathrm{tr}\Phi^4 & \mathrm{tr}\Phi^2 & 2 & 0 & \cdots \\ \vdots & & \ddots & \ddots & \\ \mathrm{tr}\Phi^{2(j-1)} & \mathrm{tr}\Phi^{2(j-2)} & \cdots & \mathrm{tr}\Phi^2 & j-1 \\ \mathrm{tr}\Phi^{2j} & \mathrm{tr}\Phi^{2(j-1)} & \cdots & \mathrm{tr}\Phi^4 & \mathrm{tr}\Phi^2 \end{vmatrix} \tag{4.31}$$

The expansion of the square brackets in equation (4.30) making use of (C.7) leads to

$$\left(\frac{E_1}{2}\right)^{2(N-j)-1} - \left(\frac{E_2}{2}\right)^{2(N-j)-1} = \left(\frac{1}{2}\right)^{2(N-j)} \left(E_1 - E_2\right) \sum_{i=1}^{N-j-1} \mathrm{tr}\mathcal{F}^{2(N-i-j)} \mathcal{E}^{i-1} + \left(\frac{1}{2}\right)^{2(N-j)-1} \left(E_1 - E_2\right) \mathcal{E}^{N-j-1} \tag{4.32}$$

where $\mathcal{E} = E_1 E_2$. From equation (4.17) the trace of the powers of the graviphoton field $\mathcal{F}$ reads

$$\mathrm{tr}(\mathcal{F}^{2n}) = 2\left(E_1^{2n} + E_2^{2n}\right) \,. \tag{4.33}$$

Performing some algebraic passages exploiting (4.32) and (4.33) one obtains the general partition function of the exotic 2-instanton in $\mathrm{SU}(N)$ gauge theory in terms of the elementary symmetric polynomials e_j:

$$Z_2 = \frac{\mathcal{N}_2}{4\mathcal{E}^2} e_N + \frac{\mathcal{N}_2}{16\mathcal{E}} e_{N-1} + \sum_{j=0}^{N-2} \frac{\mathcal{N}_2 (-1)^{N+j+1}}{2^{2(N-j)+3}} e_j \left(\sum_{i=1}^{N-j-1} \mathrm{tr}\mathcal{F}^{2(N-i-j)} \mathcal{E}^{i-2} + 2\mathcal{E}^{N-j-2} \right) \tag{4.34}$$

where e_j's are given by (4.31).

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2-instanton prepotential F_2. The $k = 2$ exotic contribution to the prepotential is given by

$$F_2 = \mathcal{E} Z_2 - \frac{F_1}{2\mathcal{E}}\,, \tag{4.35}$$

where the 1-instanton prepotential correction F_1 has been given in (4.27). Substituting F_1 and Z_2 in (4.35) we obtain

$$\begin{aligned} F_2 =& \,\mathcal{N}_2 \sum_{j=0}^{N-2} \frac{(-1)^{N+j+1}}{2^{2(N-j)+2}} e_j \left(\sum_{i=1}^{N-j-1} \mathrm{tr}\mathcal{F}^{2(N-i-j)} \mathcal{E}^{i-1} + 2\mathcal{E}^{N-j-1} \right) \\ &+ \frac{\mathcal{N}_2}{4\mathcal{E}} e_N - \frac{\mathcal{N}_1^2}{2\mathcal{E}} \det \Phi^2 + \frac{\mathcal{N}_2}{16} e_{N-1} \end{aligned} \tag{4.36}$$

It can be seen easily that none of the terms appearing in the summation (4.36) is singular. The only divergence arises in the second line and it is of order $\mathcal{E}^{-1}$,

$$F_2 = \frac{1}{\mathcal{E}} \left(\frac{\mathcal{N}_2}{4} - \frac{\mathcal{N}_1^2}{2} \right) \det \Phi^2 + \dots \tag{4.37}$$

where the dots indicate the non-divergent terms and we have used $e_N = \det \Phi^2$. So, in order to have no divergency, the free parameter $\mathcal{N}_2$ must satisfy the condition

$$\mathcal{N}_2 = 2\,\mathcal{N}_1^2\,. \tag{4.38}$$

Eventually, the prepotential for a 2-instanton in the SU(N) gauge theory is the following

$$F_2 = \frac{\mathcal{N}_1^2}{8} e_{N-1} + \mathcal{N}_1^2 \sum_{j=0}^{N-2} \frac{(-1)^{N+j+1}}{2^{2(N-j+1)}} e_j \left(\sum_{i=1}^{N-j-1} \mathrm{tr}\mathcal{F}^{2(N-i-j)} \mathcal{E}^{i-1} + 2\mathcal{E}^{N-j-1} \right) \tag{4.39}$$

For the special case of SU(2) gauge group, the exotic $k = 2$ prepotential reads:

$$F_2 = -\frac{\mathcal{N}_1^2}{64} \left(\mathrm{tr}\mathcal{F}^2 + 2\mathcal{E} \right) + \frac{\mathcal{N}_1^2}{8} e_1\,. \tag{4.40}$$

The gauge field Φ in the Cartan basis of SU(2) is given by the matrix

$$\Phi = \frac{1}{2} \begin{pmatrix} \phi & 0 \\ 0 & -\phi \end{pmatrix}, \tag{4.41}$$

therefore $e_1 = \mathrm{tr}\Phi^2$ is equivalent to $e_1 = -2 \det \Phi$. The $k = 2$ prepotential correction we read from the generic formula (4.39) in the special case $N = 2$ is therefore in complete agreement with the expression already obtained explicitly in [13].

4.4 3-instanton contribution

The evaluation of the exotic 3-instanton contribution to the prepotential is still more involved than the 1 and 2-instanton cases investigated in sections 4.2 and 4.3. Indeed in this case there are more poles in the path integral which make its evaluation lengthier. Some identities provided in appendix C help to shorten the calculations.

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The functions in the integrand of (4.8) can be given by (4.15), (4.16) and (4.18) for $k = 3$. The partition function then reads

$$Z_3 = (-1)^N \frac{\mathcal{N}_3}{\mathcal{E}^2} \det \Phi \times I \,, \tag{4.42}$$

where

$$I = \int \frac{d\chi}{2\pi \mathrm{i}} \frac{f\chi^2 (\chi - f)(\chi + f) \prod_{i=1}^N \left[\chi^2 - \left(\vec{\phi} \cdot \vec{\gamma_i}\right)^2\right]}{\prod_{A=1}^2 (2\chi - E_A)(-2\chi - E_A)(\chi - E_A)(-\chi - E_A)} \tag{4.43}$$

Among the 8 poles in (4.43) there are four positive poles, $\chi = E_1, E_2, E_1/2, E_2/2$, that place inside the contour of integration encompassing the complex upper half-plane. The result of the integration is hence the sum of all the residues corresponding to the singularities inside the contour. We have

$$I = I_{E_1} + I_{E_2} + I_{E_1/2} + I_{E_2/2} \,, \tag{4.44}$$

where $I_{E_1}, I_{E_2}, I_{E_1/2}, I_{E_2/2}$ are the residues at each pole given by:

$$I_{E_1} = \frac{2E_2 \prod_{i=1}^N \left[E_1^2 - \left(\vec{\phi} \cdot \vec{\gamma_i}\right)^2\right]}{12E_1 (E_2 - 2E_1)(E_1 - E_2)} \tag{4.45}$$

$$I_{E_2} = \frac{2E_1 \prod_{i=1}^N \left[E_2^2 - \left(\vec{\phi} \cdot \vec{\gamma_i}\right)^2\right]}{12E_2 (2E_2 - E_1)(E_1 - E_2)} \tag{4.46}$$

$$I_{E_1/2} = \frac{(3E_1 + 2E_2) \prod_{i=1}^N \left[\left(\frac{E_1}{2}\right)^2 - \left(\vec{\phi} \cdot \vec{\gamma_i}\right)^2\right]}{12E_1 (E_1 - E_2)(E_1 - 2E_2)} \tag{4.47}$$

$$I_{E_2/2} = \frac{(3E_2 + 2E_1) \prod_{i=1}^N \left[\left(\frac{E_2}{2}\right)^2 - \left(\vec{\phi} \cdot \vec{\gamma_i}\right)^2\right]}{12E_2 (E_1 - E_2)(2E_1 - E_2)} \tag{4.48}$$

To obtain the residues above we have again set $f = E_1 + E_2$. Taking advantage of the identity

$$\prod_{i=1}^N \left[\alpha^2 - \left(\vec{\phi} . \vec{\gamma_i}\right)^2\right] = \sum_{j=0}^N (-1)^j \, e_j \alpha^{2(N-j)} \,, \tag{4.49}$$

and performing some algebraic passages, one obtains the following expression for the integral (4.43):

$$I = \frac{\sum_{j=0}^N (-1)^j \, e_j \left\{(E_1 - 2E_2) L_{N-j} + (E_2 - 2E_1) G_{N-j}\right\}}{12E_1E_2 (2E_1 - E_2)(E_1 - E_2)(E_1 - 2E_2)} \,, \tag{4.50}$$

where L_k and G_k are defined as

$$L_k = 2E_2^2 E_1^{2k} - 2E_1^2 \frac{E_2^{2k}}{2^{2k}} - 3E_1E_2 \frac{E_2^{2k}}{2^{2k}} \tag{4.51}$$

$$G_k = 3E_1E_2 \frac{E_1^{2k}}{2^{2k}} + 2E_2^2 \frac{E_1^{2k}}{2^{2k}} - 2E_1^2 E_2^{2k} \,. \tag{4.52}$$

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In the presence of the graviphoton background field, the partition function Z_k is expected to have the following form

$$Z_k = \mathcal{E}^{-k} f_k(\mathrm{tr}\Phi^2, E_1, E_2) + \mathcal{E}^{-k+1} f_{k-1}(\mathrm{tr}\Phi^2, E_1, E_2) + \ldots + f_1(\mathrm{tr}\Phi^2, E_1, E_2)\,, \qquad (4.53)$$

where the f_i's are regular polynomial functions of $\mathrm{tr}\Phi^2$ and E_1, E_2. Again, it is clear that (4.50) is not yet in the desired form. To go on further one should use the identities given in (C.3)–(C.6). Employing these identities, the denominator of (4.50) cancels out an equal factor in the numerator.

After having expanded the numerator of (4.50) and having exploited (C.7), the partition function turns out to be:

$$\begin{aligned}
Z_3 = {}& \frac{\mathcal{N}_3}{12\mathcal{E}^3}\det\Phi e_N + \frac{\mathcal{N}_3}{16\mathcal{E}^2}\det\Phi e_{N-1} - \frac{\mathcal{N}_3}{192\mathcal{E}^2}\det\Phi e_{N-2}\left(\frac{3}{2}\mathrm{tr}\mathcal{F}^2 - 5\mathcal{E}\right) \\
& - \frac{\mathcal{N}_3}{\mathcal{E}^2}\det\Phi \sum_{j=0}^{N-3}\frac{(-1)^{N+j}e_j}{2^{2(N-j)+3}}\left(\sum_{k=1}^{N-j-1}\mathrm{tr}\mathcal{F}^{2(N-j-k)}\mathcal{E}^{k-1} + 2\mathcal{E}^{N-j-1}\right) \\
& + \frac{\mathcal{N}_3}{3\mathcal{E}^2}\det\Phi \sum_{j=0}^{N-3}(-1)^{N+j}e_j\sum_{i=1}^{N-j-2}\frac{1}{2^{i+2}}\left(\sum_{k=1}^{N-j-i-1}\mathrm{tr}\mathcal{F}^{2(N-j-i-k)}\mathcal{E}^{k+i-1} + 2\mathcal{E}^{N-j-1}\right) \\
& - \frac{\mathcal{N}_3}{3\mathcal{E}^2}\det\Phi \sum_{j=0}^{N-3}(-1)^{N+j}e_j\sum_{i=N-j+1}^{2(N-j-1)}\frac{1}{2^{i+2}} \\
& \times\left(\sum_{k=1}^{i-N+j}\mathrm{tr}\mathcal{F}^{2(i-N+j-k+1)}\mathcal{E}^{2(N-j-1)+k-i} + 2\mathcal{E}^{N-j-1}\right) \\
& + \frac{\mathcal{N}_3}{3\mathcal{E}^2}\det\Phi \sum_{j=0}^{N-3}(-1)^{N+j}e_j\frac{1}{2^{N-j+1}}\mathcal{E}^{N-j-1} \qquad (4.54)
\end{aligned}$$

where the elementary symmetric polynomials e_j have been defined in (4.31) and $\mathrm{tr}\mathcal{F}^{2n}$ is given in (4.17).

3-instanton prepotential F_3. The prepotential for the $k=3$ exotic instanton is given by

$$F_3 = \mathcal{E}Z_3 - \frac{F_2F_1}{\mathcal{E}} - \frac{F_1^3}{6\mathcal{E}^2}\,. \qquad (4.55)$$

Plugging the partition function (4.54), the 2 and 1-instanton prepotentials (4.35) and (4.27) into (4.55) we obtain

$$\begin{aligned}
F_3 = {}& \left(\frac{\mathcal{N}_3}{12\mathcal{E}^2} - \frac{\mathcal{N}_1^3}{6\mathcal{E}^2}\right)\det\Phi^3 + \left(\frac{\mathcal{N}_3}{16\mathcal{E}} - \frac{\mathcal{N}_1^3}{8\mathcal{E}}\right)\det\Phi e_{N-1} \\
& - \left(\frac{\mathcal{N}_3}{192\mathcal{E}} - \frac{\mathcal{N}_1^3}{64\mathcal{E}}\right)\det\Phi e_{N-2}\mathrm{tr}\mathcal{F}^2 \\
& + \left(\frac{\mathcal{N}_3}{2\mathcal{E}} - \frac{\mathcal{N}_1^3}{\mathcal{E}}\right)\det\Phi\sum_{j=0}^{N-3}\frac{(-1)^{N+j+1}}{2^{2(N-j)+1}}e_j\mathcal{E}^{N-j-1} \\
& + \left(\frac{\mathcal{N}_3}{2\mathcal{E}} - \frac{\mathcal{N}_1^3}{\mathcal{E}}\right)\det\Phi\sum_{j=0}^{N-3}\frac{(-1)^{N+j+1}}{2^{2(N-j+1)}}e_j\sum_{i=1}^{N-j-1}\mathrm{tr}\mathcal{F}^{2(N-i-j)}\mathcal{E}^{i-1} + \ldots \qquad (4.56)
\end{aligned}$$

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where the dots indicate the non-singular terms. We see that one single condition on the only free parameter $\mathcal{N}_3$ can entirely remove all the singularities; assuming that

$$\mathcal{N}_3 = 2\mathcal{N}_1^3 \tag{4.57}$$

the 3-instanton prepotential in the SU(N) gauge theory under consideration is

$$\begin{aligned} F_3 = & \frac{\mathcal{N}_1^3}{12} \det \Phi \, e_{N-2} \\ & + \frac{\mathcal{N}_1^3}{3} \det \Phi \sum_{j=0}^{N-3} (-1)^{N+j} \, e_j \mathcal{E}^{N-j-2} \left(1 + \frac{1}{2^{2(N-j-1)}} - \frac{1}{2^{N-j-2}}\right) \\ & + \frac{\mathcal{N}_1^3}{3} \det \Phi \sum_{j=0}^{N-3} (-1)^{N+j} \, e_j \sum_{i=1}^{N-j-2} \frac{1}{2^{i+1}} \mathcal{E}^{i-1} \sum_{k=1}^{N-j-i-1} \mathrm{tr} \mathcal{F}^{2(N-j-i-k)} \mathcal{E}^{k-1} \\ & - \frac{\mathcal{N}_1^3}{3} \det \Phi \sum_{j=0}^{N-3} (-1)^{N+j} \, e_j \sum_{i=N-j+1}^{2(N-j-1)} \frac{1}{2^{i+1}} \mathcal{E}^{2(N-j)-i-2} \sum_{k=1}^{i-N+j} \mathrm{tr} \mathcal{F}^{2(i-N+j-k+1)} \mathcal{E}^{k-1} . \end{aligned} \tag{4.58}$$

As already done for lower instanton numbers, we can check the $k = 3$ general formula against the special $N = 2$ case explicitly obtained in [13]. The F_3 prepotential for the SU(2) gauge theory is given by only the first line of (4.58) because the second and the third lines are summations beginning from $N = 3$, so

$$F_3 = \frac{\mathcal{N}_1^3}{12} \det \Phi \, , \tag{4.59}$$

where we have used $e_0 = 1$. This coincides exactly with the result for F_3 provided in [13].

5 Prepotential in flat background

The RR graviphoton background field we considered was a necessary tool in order to calculate the prepotential using Nekrasov's approach. It indeed regularizes the divergence due to the infinity coming from the integration over the moduli describing the position of the instanton [9]. Applying the localization method, one can retain just the quadratic terms in the action being therefore able to perform explicitly the integration with no difficulty. In addition, after having obtained the instanton contributions to the prepotential in the presence of the graviphoton background field one may be interested in the same contributions but in a flat background. To this end it is enough to let

$$E_1, E_2 \to 0 \qquad \text{or equivalently} \qquad \mathcal{F} \to 0. \tag{5.1}$$

In this limit the 1-instanton contribution remains unchanged because it is actually independent of the background and the rank of the gauge group up to an overall sign. So in the flat background case we still have:

$$F_1 = (-1)^N \mathcal{N}_1 \det \Phi \ . \tag{5.2}$$

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For the 2-instanton case the only non-vanishing term in (4.39) in the limit (5.1) turns out to be for $j = N - 1$ and $i = 1$. Therefore

$$F_2 = \frac{\mathcal{N}_1^2}{8} e_{N-1}. \tag{5.3}$$

Similarly the 3-instanton prepotential in flat background results in

$$F_3 = \frac{\mathcal{N}_1^3}{12} \det \Phi \, e_{N-2}, \tag{5.4}$$

where e_j is given by (4.31) and $\det \Phi$ has been introduced in terms of the weight vectors in appendix A.

6 Final remarks and conclusion

In the present paper we have found explicitly the non-perturbative stringy corrections to the $\mathcal{N} = 2$ prepotential in $\mathrm{SU}(N)$ gauge theory for generic N, thus generalizing the results already obtained in [13] for the special case of $\mathrm{SU}(2)$ gauge group. The setup of the model is the same as the one described in [13], i.e. we take some set of D3-branes along with D-instantons (D(-1)-branes) in the orbifold $\mathbb{C}^2/\mathbb{Z}_3$ and O3-plane backgrounds. The stringy effects arise when D-instantons and D3-branes transform under different representations of the orbifold group. In other words, we deal with a triangular quiver diagram with the $(k, 0) \oplus (0, N)$ arrangement of branes appropriately chosen in order to obtain the desired low-energy effective theory. So the exotic character of the instantons stems from the fact that the D3's and the D(-1)'s occupy different nodes of the quiver diagram.

For $N > 2$ the one-loop coefficient b_1 of the β-function, takes non-zero values and the corresponding gauge theory is therefore not conformal. It was argued in section 3 that, in the model considered in this paper, the exotic instanton moduli integral has dimension $(\text{length})^{k(2-N)}$. The partition function Z_k of the k-instanton becomes therefore dimensionful. However, the total partition function Z which is the sum over all instanton contributions as seen in (4.3), remains dimensionless because of the introduction of a dimensionful factor in q defined in (4.4).

The prefactor q^k's in (4.6) compensate appropriately also the dimension of the k-instanton prepotential leading to the total prepotential with dimension $(\text{length})^2$. It is worth noting that only for the special case $N = 2$ all k-instanton prepotentials have the same dimension therefore they may be summed up to a logarithmic closed form as proposed in [13]. For cases $N > 2$ this does not happen anymore because all prepotentials associated to different instanton charges in the series (4.5) have different dimensions, hence they are no more additive.

One can check explicitly that the dimensionful quantities in (4.27), (4.39) and (4.58) compensate the dimension of the prefactor yielding the correct dimension $(\text{length})^2$ for the prepotential. Let us elaborate more this point by looking for instance at the $k = 2$ result (4.39). The dimensionful quantities appearing in F_2 are the elementary symmetric polynomials e_j's, the trace of the graviphoton field $\mathrm{tr}\mathcal{F}$, and $\mathcal{E}$. Noting that E_1 and E_2

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have the dimension of length, the $\mathcal{E}$ and tr$\mathcal{F}$ both have dimension (length)^2, furthermore from (4.31) can be seen that e_j has dimension (length)^{2j}. Considering all dimensionful terms in (4.39), it turns out that the dimension of F_2 equals (length)$^{2(N-1)}$ which combining with the dimension of the prefactor (mass)$^{2(N-2)}$ leads to the expected dimension (length)^2. The case $k = 3$ can be checked similarly.

In order to compute the non-perturbative corrections due to the stringy instanton charges up to $k = 3$, we take advantage of the properties of the elementary symmetric polynomials and their relations with the power sums. The general results that we find are perfectly in agreement with the special case $N = 2$ provided in [13]. In the limit of flat background (vanishing graviphoton background), we have shown that the results are simplified significantly because they are expressible using only the elementary symmetric polynomials. Dimensional analysis of the measure of the moduli integral shows that for the brane arrangement in the quiver diagram considered in this paper, the conformality of the gauge theory occures only if the gauge group is taken to be SU(2). For higher gauge group ranks a dimensionful prefactor in front of the moduli integral is required in order to compensate the dimension of the moduli measure. The stringy character of the corrections is manifest in the prefactor since it depends explicitly on the string scale α'. It may be interesting to investigate models in which we can look for stringy instantons in conformal theories for arbitrary gauge group ranks. The stringy corrections in the prepotential for such models would be independent of the string scale. One therefore may expect to have once again a logarithmic behavior for the total prepotential analogous to the logarithmic function proposed in [13].

Acknowledgments

We would like to thank A. Lerda for his essential guidance and precious advice in (but not only) the preparation of this paper. We would like to thank also Marco Billó, Igor Pesando, Marialuisa Frau for several fruitful discussions. H.G. is also grateful for the hospitality of the institute of theoretical physics in Vienna (Austria) university of technology where the last stages of this work were done.

A Weights of SU(N) fundamental representation

The rank of the group SU(N) i.e. the dimension of the maximally commuting subalgebra of $\mathfrak{su}(N)$ is $N-1$. This means among all N^2-1 traceless hermitian generators, $N-1$ generators (constructing Cartan subalgebra) are diagonalized. We denote the $N-1$ Cartan generators of SU(N) group by H_m where $m = 1, \ldots, N-1$. These are $N \times N$ hermitian and traceless diagonal real matrices that can be chosen to be

$$[H_m]_{ij} = \frac{1}{\sqrt{2m\,(m+1)}} \left(\sum_{k=1}^{m} \delta_{ik}\delta_{jk} - m\,\delta_{i,m+1}\delta_{j,m+1} \right) \tag{A.1}$$

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A state in the fundamental representation of SU(N) group is N dimensional. There are then equal number of $(N-1)$-dimensional weight vectors defined by

$$[\gamma_i]_m = [H_m]_{ii} = \frac{1}{\sqrt{2m\,(m+1)}} \left(\sum_{k=1}^{m} \delta_{ik} - m\delta_{i,m+1} \right) \tag{A.2}$$

where i runs from 1 to N and denotes the ith weight. The index m denotes the mth entry of the $(N-1)$-dimensional vector. The VEV of the super chiral field Φ as mentioned in the text can be brought into the Cartan basis. It is therefore the linear combination of the Cartan generators:

$$\Phi = \sum_m \phi_m H_m \tag{A.3}$$

now

$$\det\Phi = \det\left(\sum_m \phi_m H_m\right) = \prod_i \sum_m \phi_m\,[H_m]_{ii} = \prod_i \sum_m \phi_m\,[\gamma_i]_m = \prod_i \vec{\Phi}.\vec{\gamma}_i \tag{A.4}$$

this proves (4.24). We also have

$$\mathrm{tr}\Phi = \mathrm{tr}\left(\sum_m \phi_m H_m\right) = \sum_i \left(\sum_m \phi_m\,[H_m]_{ii}\right) = \sum_i \left(\sum_m \phi_m\,[\gamma_i]_m\right) = \sum_i \vec{\Phi}.\vec{\gamma}_i \tag{A.5}$$

which implies

$$\mathrm{tr}\Phi^k = \sum_i \left(\vec{\Phi}.\vec{\gamma}_i\right)^k \tag{A.6}$$

B Elementary symmetric polynomials and power sums

The numerator in equations (4.29) and (4.45)–(4.48) can be expanded into a monic polynomial; in general we have

$$\begin{aligned}\prod_{j=1}^{N} (\lambda - X_j) = {} & \lambda^N - e_1\,(X_1,\ldots,X_N)\,\lambda^{N-1} + e_2\,(X_1,\ldots,X_N)\,\lambda^{N-2} - \ldots \\ & + (-1)^N e_N\,(X_1,\ldots,X_N)\end{aligned} \tag{B.1}$$

where $e_k(X_1,\ldots,X_N)$ is called Elementary Symmetric Polynomial of order k and is defined by

$$e_k\,(X_1,\ldots,X_N) = \sum_{1\le j_1\le j_2\le\ldots\le j_N\le N} X_{j_1}X_{j_2}\ldots X_{j_N} \tag{B.2}$$

which means

$$e_0\,(X_1,\ldots,X_N) = 1 \tag{B.3}$$

$$e_1\,(X_1,\ldots,X_N) = \sum_{1\le j\le N} X_j \tag{B.4}$$

$$e_2\,(X_1,\ldots,X_N) = \sum_{1\le j_1\le j_2\le N} X_{j_1}X_{j_2} \tag{B.5}$$

$$\vdots$$

$$e_N\,(X_1,\ldots,X_N) = X_1\ldots X_N. \tag{B.6}$$

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When $X_1, \ldots, X_N$ are entries of a diagonalized matrix X (which is so in this paper), then

$$e_N (X_1, \ldots, X_N) = \det X. \tag{B.7}$$

For example for $N = 3$

$$\begin{aligned}
e_0 (X_1, X_2, X_3) &= 1 && \text{(B.8)}\\
e_1 (X_1, X_2, X_3) &= X_1 + X_2 + X_3 && \text{(B.9)}\\
e_2 (X_1, X_2, X_3) &= X_1 X_2 + X_2 X_3 + X_2 X_3 && \text{(B.10)}\\
e_3 (X_1, \ldots, X_3) &= X_1 X_2 X_3 && \text{(B.11)}
\end{aligned}$$

On the other hand the Power Sum of order k, $p_k(X_1, \ldots, X_N)$ is defined by

$$p_k (X_1, \ldots, X_N) = \sum_{i=1}^{N} X_i^k = X_1^k + \ldots + X_N^k \tag{B.12}$$

The relation between the Power Sum and the Elementary Symmetric Polynomials is given by

$$e_k = \frac{1}{k!} \begin{vmatrix} p_1 & 1 & 0 & \cdots & \\ p_2 & p_1 & 2 & 0 & \cdots \\ \vdots & & \ddots & \ddots & \\ p_{k-1} & p_{k-2} & \cdots & p_1 & k-1 \\ p_k & p_{k-1} & \cdots & p_2 & p_1 \end{vmatrix} \tag{B.13}$$

C Useful identities

The functions L_k and G_k

$$\begin{aligned}
L_k &= 2E_2^2 E_1^{2k} - 2E_1^2 \frac{E_2^{2k}}{2^{2k}} - 3E_1 E_2 \frac{E_2^{2k}}{2^{2k}} && \text{(C.1)}\\
G_k &= 3E_1 E_2 \frac{E_1^{2k}}{2^{2k}} + 2E_2^2 \frac{E_1^{2k}}{2^{2k}} - 2E_1^2 E_2^{2k} && \text{(C.2)}
\end{aligned}$$

introduced in (4.51) and (4.52) satisfy the following identities:

$$\begin{aligned}
(E_1 - 2E_2) L_0 + (E_2 - 2E_1) G_0 &= (E_1 - 2E_2)(E_1 - E_2)(2E_1 - E_2) && \text{(C.3)}\\
(E_1 - 2E_2) L_1 + (E_2 - 2E_1) G_1 &= -\frac{3}{4} E_1 E_2 (E_1 - 2E_2)(E_1 - E_2)(2E_1 - E_2) && \text{(C.4)}\\
(E_1 - 2E_2) L_2 + (E_2 - 2E_1) G_2 &= -\frac{1}{16} E_1 E_2 (E_1 - 2E_2)(E_1 - E_2)(2E_1 - E_2)\\
&\quad \times \left[3\left(E_1^2 + E_2^2\right) - 5E_1 E_2\right] && \text{(C.5)}
\end{aligned}$$

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and for $N-j\geq 3$ we have

$$\begin{aligned}
&(E_1-2E_2)\,L_{N-j}+(E_2-2E_1)\,G_{N-j} = \\
&\frac{3}{2^{2(N-j)}}E_1E_2\,(2E_1-E_2)\,(2E_2-E_1)\left[E_1^{2(N-j)-1}-E_2^{2(N-j)-1}\right] \\
&+\sum_{i=1}^{N-j-2}\frac{1}{2^{i-1}}\,(E_1E_2)^{i+1}\,(2E_1-E_2)\,(E_1-2E_2)\left[E_1^{2(N-j-i)-1}-E_2^{2(N-j-i)-1}\right] \\
&-\sum_{i=N-j+1}^{2(N-j-1)}\frac{1}{2^{i-1}}\,(E_1E_2)^{2(N-j)-i}\,(2E_1-E_2)\,(E_1-2E_2)\left[E_1^{2(i-N+j)+1}-E_2^{2(i-N+j)+1}\right] \\
&+\frac{1}{2^{N-j-1}}\,(E_1E_2)^{(N-j)}\,(2E_1-E_2)\,(E_1-2E_2)\,(E_1-E_2)
\end{aligned} \tag{C.6}$$

We have also made use repeatedly of the following identity

$$x^{2n-1}-y^{2n-1}=(x-y)\sum_{i=1}^{n-1}\left(x^{2(n-i)}+y^{2(n-i)}\right)(xy)^{i-1}+(x-y)\,(xy)^{n-1} \tag{C.7}$$

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